Minimizers of a class of constrained vectorial variational problems: Part I.

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Abstract. In this paper, we prove the existence of minimizers of a class of multi-constrained variational problems. We consider systems involving a nonlinearity that does not satisfy compactness, monotonicity, neither symmetry properties. Our approach hinges on the concentration-compactness approach. In the second part, we will treat orthogonal constrained problems for another class of integrands using density matrices method.

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1. Introduction

Let $c > 0$ a prescribed constant, we consider the following minimization problem

$$\mathcal{I}_c = \inf\{\mathcal{J}(\vec{u}), \qquad \vec{u} \in \mathcal{S}_c\}, \tag{$\mathcal{I}$}$$

where $\mathcal{J}$ models an energy functional given as follows

$$\mathcal{J}(\vec{u}) = \int_{\mathbb{R}^N} \left(\frac{1}{2}\, |\nabla \vec{u}|^2 - F(x, \vec{u})\right) dx,$$

with $\vec{u} = (u_1, ..., u_m) \in \times^m H^1(\mathbb{R}^N) := \vec{H}^1$ for all integer $m \geq 1$ and a Carathéodory function F satisfying few assumptions listed below. The set $\mathcal{S}_c$ is given by

$$\mathcal{S}_c = \{\vec{u} = (u_1, ..., u_m) \in \times^m H^1, \qquad \sum_{i=1}^m \int_{\mathbb{R}^N} u_i^2 = c^2\}.$$

Formally (rigourously under some regularity assumptions on F), solutions of ($\mathcal{I}$) satisfy

$$\begin{cases} \Delta u_1 + \partial_1 F(x, u_1, ..., u_m) + \lambda u_1 = 0, \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \Delta u_m + \partial_m F(x, u_1, ..., u_m) + \lambda u_m = 0. \end{cases}$$

with λ being the Lagrange multiplier associated to the mass constraint and $\partial_i := \partial_{u_i}$. In particular, when $\partial_i F(x, u_1, ..., u_m) = \partial_i F(x, |u_1|, ..., |u_m|)$, solutions of $(\mathcal{I})$ can also be viewed as standing waves of the following non-linear Schrödinger system

$$\begin{cases} i\partial_t \phi_1(t,x) + \Delta_{xx}\phi_1 + \partial_1 F(x, |\phi_1|, .., |\phi_m|)\, \phi_1 = 0, \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ i\partial_t \phi_m(t,x) + \Delta_{xx}\phi_m + \partial_m F(x, |\phi_1|, ..., |\phi_m|)\, \phi_m = 0, \\ \\ \phi_i(0,x) = \phi_i^0(x) \quad 1 \le i \le m. \end{cases}$$

To our knowledge, the literature is completly silent about $(\mathcal{I})$ when $m \geq 2$ and the non-linearity F does not satisfy the standard convexity, compactness, symmetry or monotonicity properties. Such a problem appears in many areas, rational mechanics and engineering for instance and especially in non-linear optics, [10, 11].

In this contribution, our purpose is to prove the existence of minimizers to the problem $(\mathcal{I})$ for a given function $F : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ satisfying $F \in \mathcal{D}(\mathbb{R}^N \times \mathbb{R}^m)$ and

$\mathcal{A}_0$: For all $x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}^m$, there exist $A, B > 0$ and $0 < \ell < \frac{4}{N}$ such that for all $1 \leq i \leq m$, the function F satisfies

$$0 \leq F(x, \vec{s}) \leq A(|\vec{s}|^2 + |\vec{s}|^{\ell+2}) \text{ and } \partial_i F(x, \vec{s}) \leq B(|s| + |\vec{s}|^{\ell+1}).$$

$\mathcal{A}_1$: There exist $\Delta > 0,\ S > 0,\ R > 0, \alpha_1, ..., \alpha_m > 0, t \in [0, 2)$ such that for all $|x| \geq R$ and $|\vec{s}| < S$ with $t < N\left(1 - \frac{\alpha}{2}\right) + 2$ and $\alpha = \sum_{i=1}^{m} \alpha_i$, it holds

$$F(x, \vec{s}) > \Delta |x|^{-t} |s_1|^{\alpha_1} ... |s_m|^{\alpha_m}.$$

$\mathcal{A}_2$: For all $x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}^m$ and $\theta \geq 1$, we have

$$F(x, \theta\vec{s}) \geq \theta^2 F(x, \vec{s}).$$

Moreover, we assume that there exists a periodic function $F^\infty(x, \vec{s})$, that is there exists $z \in \mathbb{Z}^N$ such that $F^\infty(x + z, \vec{s}) = F^\infty(x, \vec{s})$ for all $\vec{s} \in \mathbb{R}^N$ and $\vec{s} \in \mathbb{R}^m$, satisfying $\mathcal{A}_1$ and

$\mathcal{A}_3$: There exists $0 < \alpha < \frac{4}{N}$ such that it holds uniformly for all $\vec{s} \in \mathbb{R}^m$

$$\lim_{|x| \to +\infty} \frac{F(x, \vec{s}) - F^\infty(x, \vec{s})}{|\vec{s}|^2 + |\vec{s}|^{\alpha+2}} = 0.$$

$\mathcal{A}_4$: For all $x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}^m$, there exist $A', B' > 0$ and $0 < \beta < \ell < \frac{4}{N}$ such that for all $1 \leq i \leq m$, the function F^∞ satisfies

$$0 \leq F^\infty(x, \vec{s}) \leq A'(|\vec{s}|^{\beta+2} + |\vec{s}|^{\ell+2}) \text{ and } \partial_i F^\infty(x, \vec{s}) \leq B'(|s|^{\beta+1} + |\vec{s}|^{\ell+1}).$$

$\mathcal{A}_5$: There exists $\sigma \in \left[0, \frac{4}{N}\right)$ such that for all $x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}^m$ and $\theta \geq 1$, it holds

$$F^\infty(x, \theta\vec{s}) \geq \theta^{\sigma+2} F^\infty(x, \vec{s}).$$

$\mathcal{A}_6$: For all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}^m$, we have $F^\infty(x,\vec{s}) \leq F(x,\vec{s})$ with strict inequality in a measurable set having a positive Lebesgue measure.

The class of nonlinearities satisfying $\mathcal{A}_0 - \mathcal{A}_6$ is certainly not empty. Actually, it contains physical cases. For the sake of simplicity we shall here state the following example in the setting $m = 2$ which can be extended to $m > 2$. Let $k \in \mathbb{N}^\star$ and for all $1 \leq i \leq k$, let the reals $l_{1,j}, l_{2,j} > 0$ such that $l_{1,j} + l_{2,j} < \frac{4}{N}$ the function

$$F(r,\vec{s}) = p(r)\,|\vec{s}|^2 + q(r)\sum_{i=1}^{k} |s_1|^{l_{1,j}+1}\,|s_2|^{l_{2,j}+1},$$

where $p, q : [0,+\infty) \mapsto \mathbb{R}_+$ being two bounded mapping satisfying $p(r) \xrightarrow[r\to+\infty]{} 0$ and $p(r) \xrightarrow[r\to+\infty]{} q_\infty$ with $q_\infty \leq q(r)$ for almost all r and $q_\infty < q(r)$ in a set with measure greater than 0.

To our knowledge, all existing results addressed the nonlinearity of the type $F(r,\vec{s}) = \frac{1}{2p}\,s_1^{2p} + \frac{1}{2p}\,s_2^{2p} + \frac{\beta}{p}\,s_1^p\,s_2^p$. It is known that single mode optical fibers are not unimodal but bimodal due to the presence of birefringence which heavily influences the way of propagation along the fiber. F is related to the index of refraction of the media in which the wave propagates. By Snell's law, it is not reasonable to assume that F has such a form although in some situations, it provides with a good approximation of the index of refraction. We refer the reader Refs. [1, 2, 3, 5, 8, 9] for more detail concerning applications. Let us also mention that the example we gave above describes also the Kerr-like photorefractive media in optics. It appears in the binary mixture of Bose-Einstein condensates in two different hyperfine states.

Our main result is the following

Theorem 1.1. *Let $\mathcal{A}_1 - \mathcal{A}_6$ hold true, then there exists $\vec{u}_c \in \mathcal{S}_c$ such that $\mathcal{J}(\vec{u}_c) = \mathcal{I}_c$.*

Also, we have the following intermediate result

Theorem 1.2. *If $\mathcal{A}_1$ holds true for F^∞, $\mathcal{A}_4$ and $\mathcal{A}_5$ are satisfied, then there exists $\vec{u} \in \mathcal{S}_c$ such that $\mathcal{J}^\infty(\vec{u}_c) = \mathcal{I}_c^\infty$ where*

$$\mathcal{J}^\infty(\vec{u}) = \int_{\mathbb{R}^N}\left(\frac{1}{2}\,|\nabla\vec{u}|^2 - F^\infty(x,\vec{u})\right)dx,$$

$$\mathcal{I}_c^\infty = \inf\{J^\infty(\vec{u}), \qquad \vec{u} \in \mathcal{S}_c\}. \qquad (\mathcal{I}_\infty)$$

Our proofs of Theorems 1.1 and 1.2 are based on the breakthrough concentration-compactness principle, [6, 7]. Such a principle states in the one-constrained setting for

$$(\imath) \qquad \imath_c = \inf\{j(u), \qquad \int_{\mathbb{R}^N} u^2 = c^2\},$$

where $j(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \int_{\mathbb{R}^N} f(x,u(x))\,dx$, that if $\{u_n\}_{n\in\mathbb{N}}$ is a minimizing sequence of the problem $(\imath)$, then only one of the three following scenarios can occur.

- *Vanishing*: $\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} u_n^2(x)dx = 0.$
- *Dichotomy*: There exists $a\in(0,c)$ such that $\forall\,\varepsilon>0, \exists\, n_0\in\mathbb{N}$ and two bounded sequences in $H^1(\mathbb{R}^N)$, $\{u_{n,1}\}_{n\in\mathbb{N}}$ and $\{u_{n,2}\}_{n\in\mathbb{N}}$ (all depending on ε) such that for every $n\geq n_0$, it holds

$$|\int_{\mathbb{R}^N} u_{n,1}^2\,dx - a^2| < \varepsilon; \quad |\int_{\mathbb{R}^N} u_{n,2}^2\,dx - (c^2-a^2)| < \varepsilon$$

  with $\lim_{n\to+\infty}$ *dist supp* $(u_{n,1},u_{n,2}) = +\infty$.
- *Compactness*: There exists a sequence $\{y_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$ such that, for all $\varepsilon>0$, there exists $R(\varepsilon)$ such that for all $n\in\mathbb{N}$

$$\int_{B(y_n,R(\varepsilon))} u_n^2(x)dx \geq c^2-\varepsilon.$$

The seminal work of P.L. Lions states a general line of attack to exclude the two first alternatives. When one knows that compactness is the only possible case, $(\imath)$ becomes much more easier to handle. Indeed, to rule out vanishing the main ingredient is to get a strict sign of the value of $\imath_c$ (let us say $\imath_c<0$ without loss of generality). This can be obtained by dilatation arguments or test functions techniques . The more delicate point is to prove that dichotomy cannot occur. For that purpose, Lions suggested a heuristic approach based on the strict subadditivity inequality

$$\imath_c < \imath_a + \imath_{c-a}^{\infty} \quad \forall\, a\in(0,c), \tag{1.1}$$

where

$$\imath_c^{\infty} = \inf\{j^{\infty}(u), \qquad \int_{\mathbb{R}^N} |u|^2dx = c^2\},$$

$$j^{\infty}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \int_{\mathbb{R}^N} f^{\infty}(x,u(x))\,dx,$$

and f^{∞} is defined as in $\mathcal{A}_3$. On the other hand, we should establish suitable assumptions on f for which $j(u_n)\geq j(u_{n,1})+j^{\infty}(u_{n,2})-g(\delta)$ where $g(\delta)\to 0$ as $\delta\to 0$. This fact requires a deep study of the functionals j and j^{∞}. The continuity of $\imath_c$ and $\imath_c^{\infty}$ also plays a crucial role to show that dichotomy cannot occur. These issues do not seem to be discussed in the seminal paper of Lions.

When one knows that compactness is the only plausible alternative, the strict inequality $\imath_c < \imath_c^{\infty}$ is very helpful to prove that $(\imath)$ admits a solution. Let us mention that (1.1) and $\imath_c<\imath_c^{\infty}$ seem to be inescapable to rule out the dichotomy in Lions method. In the most interesting cases $(\imath_c^{\infty}\neq 0)$. In order to get $\imath_c<\imath_c^{\infty}$, we need first to apply the concentration-compactness method to the problem at infinity. This problem is less complicated than the original one since it has translation invariance properties. The key tool to prove that $\imath_c^\infty$ is achieved, that is

$$\exists\, u_\infty \in S_c \quad \text{such that} \quad j^\infty(u_\infty) = \imath_c^\infty, \tag{1.2}$$

is the strict subadditivity inequality $\imath_c^\infty < \imath_a^\infty + \imath_{c-a}^\infty$. On the other hand, it is quite easy to establish assumptions on f such that for all $u \in H^1(\mathbb{R}^N)$, we get $j(u) < j^\infty(u)$.Therefore, $\imath_c \leq \imath_c^\infty$. Thus, with (1.2), we get $\imath_c < \imath_c^\infty$. Hence to obtain (1.1), it suffices to prove that $\imath_c \leq \imath_a + \imath_{c-a}$ which can be immediately derived from the following property

$$f(x, \theta s) \geq \theta^2 f(x, s) \quad \forall\, s \in \mathbb{R}_+, x \in \mathbb{R} \text{ and } \theta > 1.$$

To study the multi-constrained variational problem $(\mathcal{I})$, we will follow the same line of attack described in details above. Let us first emphasize that even for $m = 1$, it does not seems to us that the discussion presented in Ref. [6, 7] contains all the details and some steps are only stated heuristically. Also, to our knowledge, there are no previous results dealing with $(\mathcal{I})$ when $m \geq 2$ and the non-linearity F does not satisfy the classical convexity, compactness and monotonicity properties. Quite recently, in Ref. [4], the author was able to generalize and extend previous results addressed to $(\mathcal{I})$ when F is radial and supermodular (i.e $\partial_i \partial_j F \geq 0 \ \forall\, 1 \leq i \neq j \leq m$ when F is smooth).

In the vectorial context, the equivalent of (1.1) is

$$\mathcal{I}_c < \mathcal{I}_a + \mathcal{I}_{c-a}^\infty \quad \forall\, 0 < a < c. \tag{1.3}$$

We will first prove that $\mathcal{I}_c < 0$ in Lemma 2.3. This property together with $\mathcal{A}_2$ will permit us to infer

$$\mathcal{I}_c \leq \mathcal{I}_a + \mathcal{I}_{c-a} \quad \forall\, 0 < a < c. \tag{1.4}$$

Following the same approach detailed for the scalar case, we will then study $(\mathcal{I}_\infty)$ and prove that this variational problem has a minimum. That is, there exists $\vec{u}_c^\infty \in \mathcal{S}_c$ such that

$$\mathcal{J}^\infty(\vec{u}_c^\infty) = \mathcal{I}_c^\infty. \tag{1.5}$$

This equality is obtained thanks to the subadditivity condition

$$\mathcal{I}_c^\infty < \mathcal{I}_a^\infty + \mathcal{I}_{c-a}^\infty \quad \forall\, 0 < a < c,$$

which is proved in part b) of Lemma 2.5. On the other hand, $\mathcal{A}_6$ tells us that for all $\vec{u} \in H^1(\mathbb{R}^N)$, we have

$$\mathcal{J}(\vec{u}) < \mathcal{J}^\infty(\vec{u}).$$

Therefore, with (1.5) we get

$$\mathcal{I}_c < \mathcal{I}_c^\infty. \tag{1.6}$$

Now, (1.4) and (1.6) lead to (1.3). Then using the properties of the splitting sequences $\vec{v}_n$ and $\vec{w}_n$ (see appendix) and those of the functionals $\mathcal{J}$ and $\mathcal{J}^\infty$ (Lemma 2.1), we prove that any minimizing sequence of $(\mathcal{I})$ is such that

$$\mathcal{J}(\vec{u}_n) \geq \mathcal{J}(\vec{v}_n) + \mathcal{J}^\infty(\vec{w}_n) - \delta \quad \delta \to 0,$$

or

$$\mathcal{J}(\vec{u}_n) \geq \mathcal{J}^\infty(\vec{v}_n) + \mathcal{J}(\vec{w}_n) - \delta.$$

This leads to a contradiction with (1.3). Therefore compactness occurs and we can conclude that Theorem 1.1 holds true using (1.6).

For the convenience of the reader, we summarize our approach (inspired by Lions principle) into the following steps

i) Obtain useful properties about the functionals $\mathcal{J}$ and $\mathcal{J}^\infty$ (Lemma 2.1).
ii) Prove that $\mathcal{I}_c < 0$ and $\mathcal{I}_c^\infty < 0$ (Lemma 2.3).
iii) Show that $\mathcal{I}_c \leq \mathcal{I}_a + \mathcal{I}_{c-a}$ (Lemma 2.5).
iv) Prove that $(\mathcal{I}_\infty)$ is achieved thanks to the strict inequality

$$\mathcal{I}_c^\infty < \mathcal{I}_a^\infty + \mathcal{I}_{c-a}^\infty.$$

v) Prove that $I_c < I_c^\infty$ (Lemma 2.6).
vi) The inequality $\mathcal{I}_c < \mathcal{I}_a + \mathcal{I}_{c-a}^\infty$ follows from Step *iii* and Step *v*.
vii) Only compactness can occur. In fact Step *ii* permits us to rule out vanishing. Step *i* and step *vi* will be crucial to eliminate dichotomy.

From now on, $|\vec{s}|$ will denote the modulus of the vector $\vec{s} = (s_1, ..., s_m)$ where $s_i \in \mathbb{R}$ for all $1 \leq i \leq m$ and $m \in \mathbb{N}^\star$. The critical sobolev exponent will be denoted $2^\star = \frac{2N}{N-2}$. Also, if $\vec{u} = (u_1, ..., u_m) \in \times^m L^p(\mathbb{R}^N) := \vec{L}^p$, then $\|\vec{u}\|_{\vec{L}_p} := \sum_{i=1}^m \|u_i\|_p$ and equivalently for all functional spaces. The notation $\|\cdot\|_p$ stands for the $L^p(\mathbb{R}^N)$ norm and we shall write L^p instead of $L^p(\mathbb{R}^N)$, H^1 instead of $H^1(\mathbb{R}^N)$ etc. Moreover, we shall use implicitly the obvious estimate $\|\vec{u}\|_{\vec{L}_p} \leq c_p \sum_{i=1}^m \|u_i\|_p$ and c_p the associated universal constant.

2. A few technical Lemmata

We start by collecting some useful Lemmas. First of all, we claim

Lemma 2.1. *Let F satisfies $\mathcal{A}_0$. Then*

i) a) *$\mathcal{J} \in C^1(\vec{H}^1, \mathbb{R})$ and there exists a constant $E > 0$ such that for all $\vec{u} \in \vec{H}^1$, it holds*

$$\|\mathcal{J}'(\vec{u})\|_{\vec{H}^{-1}} \leq E\left(\|\vec{u}\|_{\vec{H}^1} + \|\vec{u}\|_{\vec{H}^1}^{1+\frac{4}{N}}\right).$$

b) *$\mathcal{J}^\infty \in C^1(\vec{H}^1, \mathbb{R})$ and there exists a constant $E_\infty > 0$ such that for all $\vec{u} \in \vec{H}^1$, it holds*

$$\|\mathcal{J}^{\infty'}(\vec{u})\|_{\vec{H}^{-1}} \leq E_\infty\left(\|\vec{u}\|_{\vec{H}^1} + \|\vec{u}\|_{\vec{H}^1}^{1+\frac{4}{N}}\right).$$

ii) There exist constants $A_i, B_i > 0$ such that for all $\vec{u} \in S_c$, we have (with σ, σ_1 and q, q_1 defined in the proof below)

$$\mathcal{J}(\vec{u}) \geq A_1\|\nabla\vec{u}\|_2^2 - A_2 c^2 - A_3 c^{(1-\sigma)(\ell+2)q},$$

$$\mathcal{J}^\infty(\vec{u}) \geq B_1\|\nabla\vec{u}\|_2^2 - B_2 c^{(1-\sigma_1)(\beta+2)q_1} - B_3 c^{(1-\sigma)(\ell+2)q}.$$

iii) a) *$\mathcal{I}_c > -\infty$ and any minimzing sequence of $(\mathcal{I})$ is bounded in $\vec{H}^1$.*

b) $\mathcal{I}_c^\infty > -\infty$ *and any minimizing sequence of* $(\mathcal{I}_\infty)$ *is bounded in* $\vec{H}^1$.

iv) a) *The mapping* $c \mapsto \mathcal{I}_c$ *is continuous on* $(0, +\infty)$.
b) *The mapping* $c \mapsto \mathcal{I}_c^\infty$ *is continuous on* $(0, +\infty)$.

Proof. We prove the first assertion. For that purpose, we introduce a cutoff function $\varphi : \mathbb{R}^m \to \mathbb{R}$ such that $\varphi(\vec{s}) = 1$ if $|\vec{s}| \le 1$, $\varphi(\vec{s}) = -|\vec{s}| + 2$ if $1 \le |\vec{s}|$ and $\varphi(\vec{s}) = 0$ otherwise. Now, for all $1 \le i \le m$, we introduce

$$\partial_i^1 F(x, \vec{s}) = \varphi(\vec{s})\partial_i F(x, \vec{s}), \qquad |\partial_i^1 F(x, \vec{s})| \le B(1 + 2^{\ell+1})|\vec{s}|,$$
$$\partial_i^2 F(x, \vec{s}) = (1 - \varphi(\vec{s}))\partial_i F(x, \vec{s}), \qquad |\partial_i^2 F(x, \vec{s})| \le 2B|\vec{s}|^{1+\frac{4}{N}}.$$

Also, we let $p = \frac{2N}{N+2}$ if $N \ge 3$, $p = \frac{4}{3}$ if $N \le 2$ and $q = p\left(1 + \frac{4}{N}\right)$. The definitions above imply that $\partial_i^1 F(x, .) \in C(\vec{L}^2, L^2)$, $\partial_i^2 F(x, .) \in C(\vec{L}^q, L^p)$ and there exists a constant $K > 0$ such that

$$\|\partial_i^1 F(x, \vec{u})\|_2 \le K \|\vec{u}\|_2, \qquad \forall \vec{u} \in \vec{L}^2,$$
$$\|\partial_i^2 F(x, \vec{u})\|_p \le K \|\vec{u}\|_q^{1+\frac{4}{N}}, \qquad \forall\, \vec{u} \in \vec{L}^q.$$

Noticing that $\vec{H}^1$ is continuously embedded in $\vec{L}^q$ since $q \in [2, 2^\star]$ for $N \ge 3$ and $q \in [2, +\infty)$ for $N \le 2$, and $\vec{L}^p$ is continuously embedded in $\vec{H}^{-1}$ since $p' \in [2, 2^\star]$ for $N \ge 3$ and $p' \in [2, +\infty)$ for $N \le 2$. We can assert that $\partial_i F(x, .) + \partial_i^2 F(x, .) \in C(\vec{H}^1, \vec{H}^{-1})$ and there exists a constant $C > 0$ such that for all $\vec{u} \in \vec{H}^1$, it holds

$$\|\partial_i F(x, \vec{u})\|_{\vec{H}^{-1}} \le C \left\{ \|\vec{u}\|_{\vec{H}^1} + \|\vec{u}\|_{\vec{H}^1}^{1+\frac{4}{N}} \right\}.$$

On the other hand

$$\int_{\mathbb{R}^N} F(x, \vec{u})\, dx \le A(\|\vec{u}\|_2^2 + \|\vec{u}\|_{\ell+2}^{\ell+2}) \le C \left(\|\vec{u}\|_{\vec{H}^1}^2 + \|\vec{u}\|_{\vec{H}^1}^{\ell+2} \right)$$

which implies that $\mathcal{J} \in C^1(\vec{H}^1, \mathbb{R})$ by standard arguments of differential calculus . Thus

$$\mathcal{J}'(\vec{u})\vec{v} = \int_{\mathbb{R}^N} \left(\sum_{i=1}^m \nabla u_i\, \nabla v_i - \partial_i F(x, \vec{u}) v_i \right) dx, \quad \forall\, \vec{u}, \vec{v} \in \vec{H}^1.$$

Therefore,

$$\|\mathcal{J}'(\vec{u})\|_{\vec{H}^{-1}} \le C \left\{ \|\vec{u}\|_{\vec{H}^1} + \|\vec{u}\|_{\vec{H}^1}^{1+\frac{4}{N}} \right\}, \quad \forall\, \vec{u} \in \vec{H}^1.$$

The assertion concerning the functional $\mathcal{J}^\infty$ can be proved similarly and we skip the proof for the sake of shortness. Now, we turn to the proof of the second assertion. Let $\vec{u} := (u_1, ..., u_m) \in S_c$. Using $\mathcal{A}_0$, we have

$$\int_{\mathbb{R}^N} F(x, \vec{u})\, dx \le Ac^2 + A \sum_{i=1}^m \int_{\mathbb{R}^N} |u_i(x)|^{\ell+2}\, dx.$$

Now, let $\sigma = \frac{N}{2}\frac{\ell}{\ell+2}$, then for all $1 \leq i \leq m$, thanks to the Gagliardo-Nirenberg inequality, we have

$$\|u_i\|_{\ell+2}^{\ell+2} \leq A"\|u_i\|_2^{(1-\sigma)(\ell+2)}\|\nabla u_i\|_2^{\sigma(\ell+2)}. \tag{2.1}$$

Next, letting $\varepsilon > 0$, $p = \frac{4}{N\ell}$ and q such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then Young's inequality leads to

$$\|u_i\|_{\ell+2}^{\ell+2} \leq \left\{\frac{A"}{\varepsilon}\|u_i\|_2^{(1-\sigma)(\ell+2)}\right\}^q \frac{1}{q} + \frac{N\ell}{4}\{\varepsilon^{\frac{4}{N\ell}}\|\nabla u_i\|_2^2\}.$$

Consequently,

$$\mathcal{J}(\vec{u}) \geq \left\{\frac{1}{2} - \frac{AN\ell}{4}\varepsilon^{\frac{4}{N\ell}}\right\}\|\nabla\vec{u}\|_2^2 - A^2c^2 - \frac{AA"^q}{q\varepsilon^q}mc^{(1-\sigma)(\ell+2)q}.$$

Taking ε such that $\dfrac{1}{2} - \dfrac{AN\ell}{4}\varepsilon^{\frac{4}{N\ell}} \geq 0$, we prove that $\mathcal{J}$ is bounded from below in $\vec{H}^1$. To show that all minimizing sequences of $(\mathcal{I})$ are bounded in $\vec{H}^1$, it suffices to take the latter inequality with a strict sign.

Remark 2.2. On the one hand, if we allow $\ell = \frac{4}{N}$ in $\mathcal{A}_0$, the minimization problem $(\mathcal{I})$ makes sense for sufficiently small values of c since in (2.1), we then have $\sigma = \frac{2}{\ell+2}$ and $(1-\sigma)(\ell+2) = \frac{4}{N}$. Therefore

$$\begin{aligned}\|u_i\|_{\ell+2}^{\ell+2} &\leq A"c^{(1-\sigma)(\ell+2)}\|\nabla u_i\|_2^2\\ &\leq A"c^{4/N}\|\nabla u_i\|_2^2.\end{aligned}$$

$$\mathcal{J}(\vec{u}) \geq \left\{\frac{1}{2} - AA"c^{4/N}\right\}\|\nabla\vec{u}\|_2^2 - Ac^2.$$

Thus, if $c < (\frac{1}{2AA"})^{N/4}$, the minimization problem $(\mathcal{I})$ is still well-posed. On the other hand, if $\ell > \frac{4}{N}$, we can prove that $\mathcal{I}_c = -\infty$.

Next, under a slight modifications of the argument we used above, we can easily obtain for $\mathcal{J}^\infty$ the following estimate

$$\begin{aligned}\mathcal{J}^\infty(\vec{u}) \quad \geq \quad & \left\{\frac{1}{2} - A^{(3)}\varepsilon^{\frac{4}{N\ell}}\right\}\|\nabla\vec{u}\|_2^2 - \frac{A^{(4)}m}{q_1\varepsilon^{q_1}}c^{(1-\sigma_1)(\beta+2)q_1}\\ - \quad & \frac{A^{(5)}mc^{(1-\sigma)(\ell+2)q}}{q\varepsilon^q},\end{aligned}$$

with $\sigma = \dfrac{N}{2}\dfrac{\beta}{\beta+2}$ and $\sigma_1 = \dfrac{N}{2}\dfrac{\ell}{\ell+2}$ and q_1 is also defined as in the previous proof. The assertion *iii)* is a straightforward consequence of the estimates of the second point. Therefore, we are kept with the proof of the last point. Consider $c > 0$ and a sequence $\{c^n\}_{n\in\mathbb{N}}$ such that $c^n \to c$. For any n, there exist $u_n \in \mathcal{S}_{c^n}$ such that

$$\mathcal{I}_{c^n} \leq \mathcal{J}(u_{n,1}, ..., u_{n,m}) \leq \mathcal{I}_{c^n} + \frac{1}{n}.$$

Thanks to the first estimate of the assertion ii), we can easily see that there exists a constant $K > 0$ such that $\|\vec{u}_n\|_{\vec{H}^1} \le K$ for all $n \in \mathbb{N}$. Now, we introduce $\vec{w}_n = (w_{n,1}, ..., w_{n,m})$ where $w_n = \frac{c}{c^n} u_n$. Then, we have obviously $\vec{w}_n \in \mathcal{S}_c$ and

$$\|\vec{u}_n - \vec{w}_n\|_{\vec{H}^1} \quad \le \quad \left| \frac{c}{c^n} - 1 \right| \|\vec{(u)}_n\|_{H^1}.$$

In particular, there exists n_1 such that

$$\|\vec{u}_n - \vec{w}_n\|_{\vec{H}^1} \le K + 1 \quad \text{ for all } n \ge n_1.$$

Now, it follows from the first assertion that

$$\|\mathcal{J}'(\vec{u})\|_{\vec{H}^{-1}} \le L(K) \quad \text{ for } \|\vec{u}\|_{\vec{H}^1} \le 2K + 1.$$

Therefore for all $n \ge n_1$, we have

$$\begin{aligned} |\mathcal{J}(\vec{w}_n) - \mathcal{J}(\vec{u}_n)| &= |\int_0^1 \frac{d}{dt} \mathcal{J}(t\vec{w}_n + (1-t)\vec{u}_n) dt|, \\ &\le \sup_{\|\vec{u}\|_{\vec{H}^1} \le 2K+1} |J'(\vec{u})|_{\vec{H}^{-1}} \|\vec{u}_n - \vec{w}_n\|_{\vec{H}^1}, \\ &\le L(K)K \left| 1 - \frac{c}{c^n} \right|. \end{aligned}$$

Eventually, we have

$$\mathcal{I}_c \ge \mathcal{J}(\vec{u}_n) - \frac{1}{n} \ge \mathcal{J}(\vec{w}_n) + KL(K) \left| 1 - \frac{c}{c^n} \right| - \frac{1}{n}.$$

Thus $\liminf_{n \to +\infty} \mathcal{I}_{c^n} \ge \mathcal{I}_c$. On the other hand, there exists a sequence $\vec{u}_n \in \mathcal{S}_c$ such that $\mathcal{J}(\vec{u}_n) \xrightarrow[n \to +\infty]{} \mathcal{I}_c$ and, thanks to the first assertion, there exists $K > 0$ such that $\|\vec{u}_n\|_{\vec{H}^1} \le K$. Now, we set $w_n = \dfrac{c_n}{c} u_n$. following the argument above, we have $\vec{w}_n = (w_{n,1,...,}w_{n,m}) \in \mathcal{S}_{c_n}$, $c_n = (c_n^1, ..., c_n^m)$ and

$$\|\vec{u}_n - \vec{w}_n\|_{\vec{H}^1} \le K \left| 1 - \frac{c_n}{c} \right| \|\vec{(u)}_n\|_{H^1}.$$

Once again, as done previously, we get

$$|\mathcal{J}(\vec{w}_n) - \mathcal{J}(\vec{u}_n)| \le KL(K) \left| 1 - \frac{c_n}{c} \right|,$$

which implies that

$$\mathcal{I}_c \le \mathcal{J}(\vec{w}_n) \le \mathcal{J}(\vec{u}_n) + L(K)K \left| 1 - \frac{c_n}{c} \right|.$$

Thus $\limsup_{n \to +\infty} \mathcal{I}_{c^n} \le \mathcal{I}_c$ and we conclude. The equivalent assertion for $\mathcal{I}_\infty$ follows using the same argument. □

We shall need the following second technical Lemma

Lemma 2.3. *Let F such that $\mathcal{A}_0$ and $\mathcal{A}_1$ hold, then $\mathcal{I}_c < 0$.*

Proof. Let φ be a radial and radially decreasing function such that $\|\varphi\|_2 = 1$ and we set $\varphi_i = c_i\varphi$. Also, let $0 < \lambda \ll 1$ and $\vec{\Phi}_\lambda(x) = \lambda^{N/2}\vec{\Phi}(\lambda x) := \lambda^{N/2}(\varphi_1(\lambda x)), ..., \varphi_m(\lambda x))$. Then, we have

$$\begin{aligned}
\mathcal{J}(\vec{\Phi}_\lambda) &= \lambda^2\|\nabla\vec{\Phi}\|_2^2 - \int_{\mathbb{R}^N} F(x, \lambda^{N/2}\varphi_1(\lambda x), ..., \lambda^{N/2}\varphi_m(\lambda x))dx,\\
&\leq \lambda^2\|\nabla\vec{\Phi}\|_2^2 - \int_{|x|\geq R} F(x, \lambda^{N/2}\varphi_1(\lambda x), ..., \lambda^{N/2}\varphi_m(\lambda x))dx,\\
&\leq \lambda^2\|\nabla\vec{\Phi}\|_2^2 - \lambda^{\frac{N}{2}\alpha}\Delta\int_{|x|\geq R} |x|^{-t_i}\varphi_1^{\alpha_1}(\lambda x)...\varphi_m^{\alpha_m}(\lambda x)dx.
\end{aligned}$$

Applying the change of variable $y = \lambda x$ leads to that

$$\mathcal{J}(\vec{\Phi}_\lambda) \leq \lambda^2\|\nabla\vec{\Phi}\|_2^2 - \lambda^{\frac{N}{2}\alpha}\lambda^{-N}\Delta\lambda^{t_i}\int_{|y|\geq\lambda R} \varphi_1^{\alpha_1}(y), ..., \varphi_m^{\alpha_m}(y)dy.$$

Now, since $0 < \lambda \ll 1$, we get

$$\begin{aligned}
\mathcal{J}(\vec{\Phi}_\lambda) &\leq \lambda^2\|\nabla\vec{\Phi}\|_2^2 - \lambda^{\frac{N}{2}-N+t_i}\int_{|y|\geq R} |y|^{-t_i}\varphi_1^{\alpha_1}(y)...\varphi_m^{\alpha_m}(y)dy,\\
&\leq \lambda^2\{C_1 - \lambda^{\frac{N}{2}\alpha-N+t_i-2}C_2\}.
\end{aligned}$$

The result follows after observing that $\lambda \ll 1$ and $\frac{N}{2}\alpha - N + t_i - 2 > 0$. □

Remark 2.4. The strict negativity of the infimum is also discussed in Ref. [4] where the author provides other type of assumption ensuring this.

Now, we have the following Lemma and we refer to Ref. [7] for a proof.

Lemma 2.5. *We have the following facts*

i) If F satisfies $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$, then for all $c > 0$ and all $a \in (0, c)$ it holds $\mathcal{I}_c \leq \mathcal{I}_a + \mathcal{I}_{c-a}$.

ii) If F satisfies $\mathcal{A}_2, \mathcal{A}_4$ and $\mathcal{A}_1$ holds true for F^∞, then for all $c > 0$ and all $a \in (0, c)$ it holds $\mathcal{I}_c^\infty < \mathcal{I}_a^\infty + \mathcal{I}_{c-a}^\infty$.

As a consequence, we have the following

Lemma 2.6. *Let F satisfies $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_5, \mathcal{A}_1$ hold true for F^∞, then for all $c > 0$ and all $a \in (0, c)$ we have*

$$I_c < I_a + I_{c-a}^\infty.$$

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. For that purpose, let $\{\vec{u}_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of the problem $(\mathcal{I}_\infty)$. We proceed by concentration-compactness scenario's elimination. First of all, we prove that vanishing does not occur. We proceed by contradiction and assume that vanishing holds true. Therefore, using Lemma I.1 of Ref. [7] that $\||\vec{u}_n|\|_p \xrightarrow[n\to+\infty]{} 0$ as for all $p \in (2, 2^\star)$. Thanks to assumption $\mathcal{A}_4$, we have

$$\int_{\mathbb{R}^N} F^\infty(x, \vec{u}_n(x))\, dx \leq \left\{ \||\vec{u}_n|\|_{\beta+2}^{\beta+2} + \||\vec{u}_n|\|_{\ell+2}^{\ell+2} \right\}.$$

Therefore, $\int_{\mathbb{R}^N} F^\infty(x, \vec{u}_n(x)\, dx \xrightarrow[n\to+\infty]{} 0$, hence $\liminf_{n\to+\infty} \mathcal{J}^\infty(\vec{u}_n) \geq 0$, contradicting the fact that $I_c^\infty < 0$. Thus vanishing does not occur. Now, we use the notation introduced in the appendix and eliminate the dichotomy scenario. For all $n \geq n_0$, we have

$$\begin{aligned}
\mathcal{J}^\infty(\vec{u}_n) - \mathcal{J}^\infty(\vec{v}_n) - \mathcal{J}^\infty(\vec{w}_n) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla \vec{u}_n|^2 - |\nabla \vec{v}_n|^2\right) dx \\
&- \int_{\mathbb{R}^N} \left(F^\infty(x, \vec{u}_n) - F^\infty(x, \vec{v}_n) - F^\infty(x, \vec{w}_n)\right) dx, \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla \vec{u}_n|^2 - |\nabla \vec{v}_n|^2\right) dx - \int_{\mathbb{R}^N} \left(F^\infty(x, \vec{u}_n) - F^\infty(x, \vec{v}_n + \vec{w}_n)\right) dx, \\
&\geq -\varepsilon - \int_{\mathbb{R}^N} \left(F^\infty(x, \vec{u}_n) - F^\infty(x, \vec{v}_n + \vec{w}_n)\right) dx.
\end{aligned}$$

In the estimate above, we used the fact that $\mathrm{Supp}\, \vec{v}_n \cap \mathrm{Supp}\, \vec{w}_n = \emptyset$. Now since the sequences $\{\vec{w}_n\}_{n\in\mathbb{N}}, \{\vec{v}_n\}_{n\in\mathbb{N}}$ and $\{\vec{w}_n\}_{n\in\mathbb{N}}$ are bounded in $\vec{H}^1$, it follows from the proof of Lemma 2.1 that there exist $C, K > 0$ such that

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} (F^\infty(x, \vec{u}_n) \right. &- \left. F^\infty(x, \vec{v}_n + \vec{w}_n))\, dx\right| \\
&\leq \sup_{\|\vec{u}\|_{\vec{H}^1} \leq K} \sum_{i=1}^m \|\partial_i F^\infty(x, \vec{u})\|_{\vec{H}^{-1}} \|\vec{u}_n - (\vec{v}_n + \vec{w}_n)\|_{\vec{H}^1}, \\
&\leq \sup_{\|\vec{u}\|_{\vec{H}^1} \leq K} \sum_{i=1}^m \|\partial_i^1 F^\infty(x, \vec{u})\|_{\vec{L}^2} \|\vec{u}_n - (\vec{v}_n + \vec{w}_n)\|_{\vec{L}^2}, \\
&+ \sup_{\|\vec{u}\|_{\vec{H}^1} \leq K} \sum_{i=1}^m \|\partial_i^2 F^\infty(x, \vec{u})\|_{\vec{L}^p} \|\vec{u}_n - (\vec{v}_n + \vec{w}_n)\|_{\vec{L}^{p'}}, \\
&\leq C \sup_{\|\vec{u}\|_{\vec{H}^1} \leq K} \|\vec{u}\|_{\vec{L}^2} \|\vec{u}_n - (\vec{v}_n + \vec{w}_n)\|_{\vec{L}^2} \\
&+ C \sup_{\|\vec{u}\|_{\vec{H}^1} \leq K} \|\vec{u}\|_{L^q}^{1+\frac{4}{N}} \|\vec{u}_n - (\vec{v}_n + \vec{w}_n)\|_{\vec{L}^{p'}}, \\
&\leq C_1 K \|\vec{u}_n - (\vec{v}_n + \vec{w}_n)\|_{\vec{L}^2} \\
&+ C_2 K^{1+\frac{4}{N}} \|\vec{u}_n - (\vec{v}_n + \vec{w}_n)\|_{\vec{L}^{p'}}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\mathcal{J}^\infty(\vec{v}_n) - \mathcal{J}^\infty(\vec{v}_n) - \mathcal{J}^\infty(\vec{w}_n) \geq -\varepsilon &- C_1 K \|\vec{u}_n - (\vec{v}_n + \vec{w}_n)\|_{\vec{L}^2} \\
&- C_2 K^{1+\frac{4}{N}} \|\vec{u}_n - (\vec{v}_n + \vec{w}_n)\|_{\vec{L}^{p'}}.
\end{aligned}$$

Given any $\delta > 0$, using the properties of the sequences $\{\vec{v}_n\}_{n\in\mathbb{N}}$ and $\{\vec{w}_n\}_{n\in\mathbb{N}}$, we can find $\varepsilon_\delta \in (0,\delta)$ such that

$$\mathcal{J}^\infty(\vec{u}_n) - \mathcal{J}^\infty(\vec{v}_n) - \mathcal{J}^\infty(\vec{w}_n) \geq -\delta.$$

Now let $a_n^2(\delta) = \int_{\mathbb{R}^N} v_n^2\, dx$ and $b_n^2(\delta) = \int_{\mathbb{R}^N} w_n^2\, dx$, passing to a subsequences if necessary, we may suppose that $a_n^2(\delta) \xrightarrow[n\to+\infty]{} a^2(\delta)$ and $b_n^2(\delta) \xrightarrow[n\to+\infty]{} b^2(\delta)$ where $|a^2(\delta) - a^2| \leq \varepsilon_\delta < \delta$ and $|b^2(\delta) - (c^2 - a^2)| \leq \varepsilon_\delta < \delta$. Recalling that $c \mapsto \mathcal{I}_c^\infty$ is continuous, we find that

$$\begin{aligned} \mathcal{I}_c^\infty \geq \lim_{n\to+\infty} \mathcal{J}^\infty(\vec{u}_n) &\geq \liminf\{\mathcal{J}^\infty(\vec{v}_n) + \mathcal{J}^\infty(\vec{w}_n)\} - \delta, \\ &\geq \liminf\{\mathcal{I}^\infty_{a,(\delta)} + \mathcal{I}^\infty_{b,(\delta)}\} - \delta, \\ &\geq \mathcal{I}^\infty_{a,(\delta)} + \mathcal{I}^\infty_{b,(\delta)}\} - \delta. \end{aligned}$$

Eventually, letting δ goes to zero and using again the continuity of $\mathcal{I}_c^\infty$, we get

$$\mathcal{I}_c^\infty \geq \mathcal{I}_a^\infty + \mathcal{I}^\infty_{\sqrt{c^2-a^2}}.$$

This contradicts Lemma 2.5. Thus, dichotomy does not occur and we conclude that the compactness scenario holds true. Hence, there exists $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that for all $\varepsilon > 0$ we have

$$\int_{B(y_n,R(\varepsilon))} u_n^2\, dx \geq c^2 - \varepsilon.$$

For all $n \in \mathbb{N}$, we can choose $z_n \in \mathbb{Z}^N$ such that $y_n - z_n \in [0,1]^N$. Now we set $\vec{v}_n(x) = \vec{u}_n(x + z_n)$, we certainly have that $\|\vec{v}_n\|_{\vec{H}^1} = \|\vec{u}_n\|_{\vec{H}^1}$ is bounded. Therefore, passing to a subsequence if necessary, we may assume that $\vec{v}_n \rightharpoonup \vec{v}$ in $\vec{H}^1$. In particular $\vec{v}_n \rightharpoonup \vec{v}$ weakly in $\vec{L}^2$ and $\|v_n\|_2^2 = c^2$. However, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\vec{v}|^2\, dx \geq \int_{B(0,R(\varepsilon)+\sqrt{N})} |\vec{v}|^2\, dx &= \lim_{n\to+\infty} \int_{B(0,R(\varepsilon)+\sqrt{N})} |\vec{v}_n|^2\, dx \\ &= \lim_{n\to+\infty} \int_{B(z_n,R(\varepsilon)+\sqrt{N})} |\vec{v}_n|^2\, dx. \end{aligned}$$

Since $|y_n - z_n| \leq \sqrt{N}$, we have

$$\int_{B(z_n,R(\varepsilon)+\sqrt{N})} |\vec{u}_n|^2\, dx \geq \int_{B(y_n,R(\varepsilon))} |\vec{u}_n|^2\, dx \geq c^2 - \varepsilon.$$

Hence, for all $\varepsilon > 0$ we have

$$\|\vec{v}\|^2_{\vec{L}^2} \geq c^2 - \varepsilon \Rightarrow \|\vec{v}\|^2_{\vec{L}^2} \geq c^2. \tag{3.1}$$

On the other hand $\|\vec{v}\|_{\vec{L}^2} \leq \liminf_{n\to+\infty} \|\vec{v}_n\|_{\vec{L}^2}$, thus

$$\|\vec{v}\|_{\vec{L}^2} \leq c. \tag{3.2}$$

Thus combining (3.1) and (3.2), we get $\|\vec{v}\|^2_{\vec{L}^2} = c^2$, thus $\|\vec{v} - \vec{v}_n\|_{\vec{L}^2} \to 0$ as $n \to +\infty$. Furthermore, by the periodicity of F^∞, we see that $\mathcal{J}^\infty(\vec{u}_n) = \mathcal{J}^\infty(\vec{v}_n) \xrightarrow[n\to+\infty]{} \mathcal{I}_c^\infty$ and $\vec{v}_n \xrightarrow[n\to+\infty]{} \vec{v}$ in $\vec{L}^p$ for all $p \in [2, 2^\star)$. It follows that $\vec{v}_n \xrightarrow[n\to+\infty]{} \vec{v}$ in $\vec{H}^1$ and consequently $\displaystyle\int_{\mathbb{R}^N} F^\infty(x, \vec{v}_n)\, dx \xrightarrow[n\to+\infty]{}$

$\displaystyle\int_{\mathbb{R}^N} F^\infty(x,\vec{v})\,dx$ which implies that $\mathcal{J}^\infty(\vec{v}) = \mathcal{I}_c^\infty$.

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let $(\vec{u}_n)_{n\in\mathbb{N}}$ denotes a minimizing sequence of $(\mathcal{I})$ and we will again make use of the notation introduced in the appendix. As before, we start by showing that vanishing does not occur by proceeding by contradiction. Indeed, if it occurs, it follows from Lemma I.1 of Ref. [7] that $\||\vec{u}_n|\|_p \xrightarrow[n\to+\infty]{} 0$ for $p \in (2, 2^\star)$. Combining $\mathcal{A}_0$ and $\mathcal{A}_3$, we get that for all $\delta > 0$ there exists $R_\delta > 0$ such that for all $|x| \geq R_\delta$ we have

$$F(x,\vec{s}) \leq \delta(|\vec{s}|^2 + |\vec{s}|^{\alpha+2}) + A'(|\vec{s}|^{\beta+2} + |\vec{s}|^{\ell+2}).$$

Thus,

$$\int_{|x|\geq R_\delta} F(x,\vec{u}_n)\,dx \leq \delta(\|\vec{u}_n\|_2^2 + \|\vec{u}_n\|_{\alpha+2}^{\alpha+2}) + A'(\|\vec{u}_n\|_{\beta+2}^{\beta+2} + \|\vec{u}_n\|_{\ell+2}^{\ell+2}).$$

Therefore,

$$\limsup_{n\to+\infty} \int_{|x|\geq R_\delta} F(x,\vec{u}_n)\,dx \leq \delta c^2.$$

On the other hand

$$\begin{aligned}\int_{|x|\leq R_\delta} F(x,\vec{u}_n)dx &\leq A\int_{|x|\leq R_\delta} \left(|\vec{u}_n|^2 + |\vec{u}_n|^{\ell+2}\right)\,dx,\\ &\leq A\left(\|\vec{u}_n\|_{\ell+2}^{\ell+2}|R_\delta|^{\frac{\ell}{\ell+2}} + \|\vec{u}_n\|_{\ell+2}^{\ell+2}\right) \xrightarrow[n\to+\infty]{} 0.\end{aligned}$$

Hence, for any $\delta > 0$ we see that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} F(x,\vec{u}_n)\,dx < \delta c^2,$$

and so

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} F(x,\vec{u}_n)\,dx = 0.$$

The contradiction follows since we know that $\mathcal{J}(\vec{u}_n) \xrightarrow[n\to+\infty]{} \mathcal{I}_c < 0$. Now, we show that dichotomy does not occur. We argue again by contradiction and suppose first that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded. We write

$$\begin{aligned}
J(\vec{u}_n) - J(\vec{v}_n) - J^\infty(\vec{w}_n) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla\vec{w}_n|^2 - |\nabla\vec{v}_n|^2 - \nabla\vec{w}_n|^2\right) dx \\
&- \int_{\mathbb{R}^N} \left(F(x,\vec{u}_n) - F(x,\vec{v}_n) - F(x,\vec{w}_n)\right) dx \\
&+ \int_{\mathbb{R}^N} \left(F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n)\right) dx, \\
&\geq -\varepsilon - \int_{\mathbb{R}^N} \left(F(x,\vec{u}_n) - F(x,\vec{v}_n+\vec{w}_n)\right) dx \\
&+ \int_{\mathbb{R}^N} \left(F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n)\right) dx, \\
&\geq -\varepsilon - \int_{\mathbb{R}^N} \left(F(x,\vec{u}_1) - F(x,\vec{v}_n+\vec{w}_n)\right) dx \\
&+ \int_{|x-y_n|\geq R_n} \left(F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n)\right) dx.
\end{aligned}$$

We used the fact that $\operatorname{Supp}\vec{v}_n \cap \operatorname{Supp}\vec{w}_n = \emptyset$. Now using the same argument as before, it follows that given $\delta > 0$, we can choose $\varepsilon = \varepsilon_\delta \in (0,\delta)$ such that

$$-\varepsilon - \int_{\mathbb{R}^N} \left(F(x,\vec{u}_n) - F(x,\vec{v}_n+\vec{w}_n)\right) \geq -\delta.$$

Therefore, we get

$$\mathcal{J}(\vec{u}_n) - \mathcal{J}(\vec{v}_n) - \mathcal{J}^\infty(\vec{w}_n) \geq -\delta + \int_{|x-y_n|\geq R_n} \left(F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n)\right) dx.$$

Given any $\eta > 0$, we can find $R > 0$ such that for all $\vec{s}$ and $|x| \geq R$

$$|F^\infty(x,\vec{s}) - F(x,\vec{s})| \leq \eta(|\vec{s}|^2 + |\vec{s}|^{\alpha+2}).$$

Now, since $R_n \xrightarrow[n\to+\infty]{} +\infty$ and we are supposing that $\{y_n\}_{n\in\mathbb{N}}$ is bounded, we have for n large enough

$$\{x : |x-y_n| \geq R_n\} \subset \{x : |x| \geq R\}.$$

From this and the boundedness of $\{\vec{w}_n\}_{n\in\mathbb{N}}$ in $\vec{H}^1$, it follows that

$$\lim_{n\to+\infty} \int_{|x-y_n|\geq R_n} \left(F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n)\right) dx = 0.$$

Now, let $a_n^2 = \sum_{i=1}^m \int_{\mathbb{R}^N} v_{n,i}^2\,dx$ and $b_n^2 = \sum_{i=1}^m \int_{\mathbb{R}^N} w_{n,i}^2\,dx$. Passing to a subsequences if necessary, we may suppose that $a_n^2(\delta) \xrightarrow[n\to+\infty]{} a^2(\delta)$ and $b_n^2(\delta) \xrightarrow[n\to+\infty]{} b^2(\delta)$ where $|a^2(\delta)-a^2| \leq \varepsilon_\delta < \delta$ and $|b^2(\delta)-(c^2-a^2)| \leq \varepsilon_\delta < \delta$. Recalling that the mappings $c \mapsto \mathcal{I}_c$ and $c \mapsto \mathcal{I}_c^\infty$ are continuous we find that

$$\begin{aligned}
\mathcal{I}_c &= \lim_{n\to+\infty} \mathcal{J}(\vec{u}_n) \geq \liminf_{n\to+\infty}\{\mathcal{J}(\vec{v}_n) + \mathcal{J}^\infty(\vec{w}_n)\} - \delta, \\
&\geq \liminf_{n\to+\infty}\{\mathcal{I}_{a_1(\delta),\ldots,a_n(\delta)} + \mathcal{I}_{b_1(\delta),\ldots,b_n(\delta)}\} - \delta.
\end{aligned}$$

Thus, $\mathcal{I}_c \geq \mathcal{I}_{a(\delta)} + \mathcal{I}_{b(\delta)} - \delta$. Letting $\delta \to 0$ we get

$$\mathcal{I}_c \geq \mathcal{I}_a + \mathcal{I}_{\sqrt{c^2-a^2}}.$$

Therfore, the sequence $\{y_n\}_{n\in\mathbb{N}}$ cannot be bounded and passing to a subsequence if necessary, we may suppose that $|y_n| \to +\infty$. Now we obtain a contradiction with Lemma 2.5 by using similar arguments applied to $\mathcal{J}(\vec{u}_n) - \mathcal{J}^\infty(\vec{v}_n) - \mathcal{J}(\vec{w}_n)$ to show that $\mathcal{I}_c \geq \mathcal{I}_a + \mathcal{I}_{\sqrt{c^2-a^2}}$ and therefore prove that dichotomy cannot occur. Eventually, the compactness occurs. According to the appendix, there exists $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that for all $\varepsilon > 0$

$$\int_{B(y_n,R(\varepsilon))} \left(u_{n,1}^2 + ... + u_{n,m}^2\right) dx \geq c^2 - \varepsilon.$$

Let us first prove that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded . By contradiction, if it is not the case, we may assume that $|y_n| \xrightarrow[n\to+\infty]{} +\infty$ by passing to a subsquence. Now we can choose $z_n \in \mathbb{Z}^N$ such that $y_n - z_n \in [0,1]^N$. Setting $\vec{v}_n(x) = \vec{u}_n(x+z_n)$, we can suppose that $\vec{v}_n \rightharpoonup \vec{v}$ weakly in $\vec{H}^1$ and

$$\|\vec{v}_n - \vec{v}\|_{\vec{L}^2} \xrightarrow[n\to+\infty]{} 0 \quad \text{for} \quad 2 \leq p \leq 2^\star,$$

$$\mathcal{J}^\infty(\vec{v}_n) = \mathcal{J}^\infty(\vec{u}_n).$$

On the other hand, we have

$$\begin{aligned}
\mathcal{J}(\vec{u}_n) - \mathcal{J}^\infty(\vec{u}_n) &= \int_{\mathbb{R}^N} \left(F^\infty(x,\vec{u}_n) - F(x,\vec{u}_n)\right) dx, \\
&= \int_{\mathbb{R}^N} \left(F^\infty(x,\vec{v}_n) - F(x-z_n,\vec{v}_n)\right) dx.
\end{aligned}$$

Now, given $\varepsilon > 0$ it follows from $\mathcal{A}_3$ that there exists $R > 0$ such that

$$\begin{aligned}
|\int_{|x-z_n|\geq R} (F^\infty(x,\vec{v}_n) &- F(x-z_n,\vec{v}_n))\, dx| \\
&= \left|\int_{|x-z_n|\geq R} (F^\infty(x-z_n,\vec{v}_n) - F(x-z_n,\vec{v}_n))\, dx\right|, \\
&\leq \varepsilon \int_{|x-z_n|\geq R} \left(|\vec{v}_n|^2 + |\vec{v}_n|^{\alpha+2}\right) dx, \\
&\leq \varepsilon C \left(\|\vec{v}_n\|_{\vec{H}^1}^2 + \|\vec{v}_n\|_{\vec{H}^1}^{\alpha+2}\right) \leq \varepsilon D,
\end{aligned}$$

since $\vec{v}_n$ is bounded in $\vec{H}^1$. Next, since $|z_n| \xrightarrow[n\to+\infty]{} +\infty$, there exists $n_R > 0$ such that for all $n \geq n_R$ we have

$$\begin{aligned}
\Big|\int_{|x-z_n|\leq R} (F^\infty(x,\vec{v}_n) &- F(x-z_n,\vec{v}_n))\,dx\Big| \\
&\leq |\int_{|x|\geq\frac{1}{2}|z_n|} (F^\infty(x,\vec{v}_n) - F(x-z_n,\vec{v}_n))\,dx, \\
&\leq K\int_{|x|\geq\frac{1}{2}|z_n|} \left(|\vec{v}_n|^2 + |\vec{v}_n|^{\ell+2}\right)dx, \\
&\leq K\int_{|x|\geq\frac{1}{2}|z_n|} \left(|\vec{v}|^2 + |\vec{v}|^{\ell+2}\right)dx \\
&+ K\int_{|x|\geq\frac{1}{2}|z_n|} \left(|\vec{v}-\vec{v}_n|^2 + |\vec{v}-\vec{v}_n|^{\ell+2}\right)dx, \\
&\leq K\int_{|x|\geq\frac{1}{2}|z_n|} \left(|\vec{v}|^2 + |\vec{v}|^{\ell+2}\right)dx \\
&+ K\int_{\mathbb{R}^N} \left(|\vec{v}-\vec{v}_n|^2 + |\vec{v}-\vec{v}_n|^{\ell+2}\right)dx.
\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty} |\int_{|x-z_n|\geq R_n} (F^\infty(x,\vec{v}_n) - F(x-z_n,\vec{v}_n))\,dx| = 0.$$

Thus, for all $\varepsilon > 0$, we get $\liminf_{n\to+\infty}\{\mathcal{J}(\vec{u}_n) - \mathcal{J}^\infty(\vec{u}_n)\} \geq -\varepsilon D$ and so

$$\mathcal{I}_c = \lim_{n\to+\infty} \mathcal{J}(\vec{u}_n) \geq \liminf_{n\to+\infty} \mathcal{J}^\infty(\vec{u}_n) \geq \mathcal{I}_c^\infty.$$

We reach a contradiction with the fact that $\mathcal{I}_c < \mathcal{I}_c^\infty$. Hence $\{y_n\}_{n\in\mathbb{N}}$ is bounded. Setting $\rho = \sup_{n\in\mathbb{N}}|y_n|$, it follows that for all $\varepsilon > 0$

$$\begin{aligned}
\int_{B(0,R(\varepsilon)+\rho)} \left(u_{n,1}^2 + ... + u_{n,m}^2\right)dx &\geq \int_{B(y_n,R(\varepsilon))} \left(u_{n,1}^2 + ... + u_{n,m}^2\right)dx, \\
&\geq c^2 - \varepsilon.
\end{aligned}$$

Thus, for all $\varepsilon > 0$

$$\begin{aligned}
\int_{\mathbb{R}^N} |\vec{u}|^2\,dx &\geq \int_{B(0,R(\varepsilon)+\rho)} |\vec{u}|^2\,dx, \\
&= \lim_{n\to+\infty}\int_{B(0,R(\varepsilon)+\rho)} |\vec{u}_n|^2\,dx \geq c^2 - \varepsilon.
\end{aligned}$$

Hence $\displaystyle\int (u_1^2+..+u_m^2)\,dx \geq c^2$. On the other hand $\displaystyle\int (u_1^2+u_2^2+...+u_m^2)\,dx \leq c^2$. Thus $\vec{u} \in \mathcal{S}_c$ and $\|\vec{u}_n - \vec{u}\|_{\vec{L}^2} \xrightarrow[n\to+\infty]{} 0$. By the boundedness of $\{\vec{u}_n\}_{n\in\mathbb{N}}$ in $\vec{H}^1$, it follows that $\vec{u}_n \xrightarrow[n\to+\infty]{} \vec{u}$ in $\vec{L}^p$ for all $p \in [2, 2^\star]$. Therefore $\lim_{n\to+\infty}\int F(x,\vec{u}_n)\,dx = \int F(x,\vec{u})\,dx$ implying that $\mathcal{J}(\vec{u}) = \mathcal{I}_c$.

Appendix

In this appendix, we present the concentration-compactness Lemma in the multi-constrained setting for the reader convenience. Let $\{\vec{u}_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of the problem $(\mathcal{I})$, we introduce its associate concentration function

$$Q_n(R) = \sup_{y\in\mathbb{R}^N} \int_{B_R+y} \rho_n^2(\xi)d\xi,$$

where $\rho_n^2(\xi) = |\vec{u}_n|^2 = \sum_{i=1}^{m} u_{n,i}^2(\xi)$. Applying the concentration compactness method (see page 136-137 of Ref. [6] and page 272-273 of Ref. [7]), one of the following alternatives occur :

Vanishing

That is, $\limsup_{y\in\mathbb{R}^N} \int_{y+B_R} |\vec{u}_n|^2 = 0$.

Dichotomy

That is, for all $1 \leq i \leq m$, there exists $a_i \in (0, c_i)$ such that for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and two bounded sequences in $\vec{H}^1$ denoted by $\{\vec{v}_n\}_{n\in\mathbb{N}}$ and $\{\vec{w}_n\}_{n\in\mathbb{N}}$ (all depending on ε) such that for all $n \geq n_0$, we have for all $1 \leq i \leq m$ and $p \in (2, 2^\star]$

$$\left| \int_{\mathbb{R}^N} v_{n,i}^2\, dx - a_i^2 \right| < \varepsilon, \quad \left| \int_{\mathbb{R}^N} w_{n,i}^2\, dx - (c_i^2 - a_i^2) \right| < \varepsilon,$$

$$\int_{\mathbb{R}^N} \left(|\nabla \vec{u}_n|^2 - |\nabla \vec{v}_n|^2 - |\nabla \vec{w}_n|^2 \right) dx \geq -2\varepsilon,$$

$$\|u_{n,i} - (v_{n,i} + w_{n,i})\|_p \leq 4\varepsilon.$$

Furthermore, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $\{R_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ such that $\lim_{n\to+\infty} R_n = +\infty$, $\mathrm{dist}(\mathrm{Supp}|v_{n,i}|, \mathrm{Supp}|w_{n,i}|) \xrightarrow[n\to+\infty]{} +\infty$ and

$$\begin{cases} v_{n,i} = u_{n,i} & \text{if} \quad |x - y_n| \leq R_0, \\ |v_{n,i}| \leq |u_{n,i}| & \text{if} \quad R_0 \leq |x - y_n| \leq 2R_0, \\ v_{n,i} = 0 & \text{if} \quad |x - y_n| \geq 2R_0, \\ w_{n,i} = 0 & \text{if} \quad |x - y_n| \leq R_n, \\ |w_{n,i}| \leq |u_{n,i}| & \text{if} \quad R_n \leq |x - y_n| \leq 2R_n, \\ w_{n,i} = u_{n,i} & \text{if} \quad |x - y_n| \geq 2R_n. \end{cases}$$

Compactness

That is, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that for all $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$\int_{B(y_n, R(\varepsilon))} |\vec{u}_n|^2\, dx \geq \sum_{i=1}^{m} c_i^2 - \varepsilon.$$

As suggested and stated by Lions in Ref. [7], page 137-138, to get the above properties, it suffices to apply his method to ρ_n. Decomposing ρ_n in the classical setting and thus simultaneously $u_{n,i}$, leads to the properties of the splitting sequences $\vec{v}_n$ and $\vec{w}_n$, mentioned above.

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References

[1] A. Ambrosetti and E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations.* J. London Math. Soc., **75**, 67-82 (2007)

[2] T. Bartsch and Z.Q. Wand, *Note on ground state of nonlinear Schrdïngier systems* J. Partial Differential Equations, **19**, 200-207 (2006)

[3] R. Cipolatti and W Zumpichiatti, *On the existence and regularity of ground states for nonlinear system of coupled Schrödinger equations.* Comput. Appl. Math., **18**, 19-36 (1999)

[4] H. Hajaiej, *Symmetric ground states solutions of m-coupled nonlinear Schrödinger equation.* Nonlinear Analysis: Methods, Theory and Applications, vol. **71**, no. 2, (15 November 2009)

[5] T.C. Lin and J. Wei, *Ground states of N coupled nonlinear Schrödinger equations in* $\mathbb{R}^N, N \geq 3$. Comm. Math. Phy., **255**, 629-653 (2005)

[6] P.L. Lions, *The concentration - compactness principle in the calculus of variations. The locally compact case: Part 1*, p 109 -145, Ann. Inst. H. Poincaré, Vol **1**, n 4, (1984)

[7] P.L. Lions, *The concentration - compactness principle in the calculus of variations. The locally compact case: Part 2*, p 223 - 283, Ann. Inst. H. Poincaré, Vol **1**, n 4, (1984)

[8] L.A. Maia, E. Montefusco and B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system.* J. Diff. Eqs., **229**, 743-767 (2006)

[9] L.A. Maia, E. Montefusco and B. Pellacci, *Orbital stability of ground state solutions of coupled nonlinear Schrödinger equations* Preprint

[10] S.V. Mankov, *On the theory of two-dimensional stationary self-focusing of electromagnetic waves.* Sov. Phgs.,JETP **38**, 248-253 (1974)

[11] C.R. Menyuk, *Nonlinear pulse propagation in birefringent optical bers.* IEEE J. Quantum Electron., **23**,174-176 (1987)

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