Orbital stability of standing waves of Two-component Bose-Einstein condensates with internal atomic Josephson Junction

H. Hajaiej

College of Sciences, King Saud University, 11451 Riyadh

Department of Mathematics

E.mail : hhajaiej@ksu.edu.sa

Abstract

In this paper, we prove existence, symmetry and uniqueness of standing waves for a coupled Gross-Pitaevskii equations modeling component Bose-Einstein condensates(BEC) with an internal atomic Josephson junction. We will then address the orbital stability of these standing waves and characterize their orbit.

1 Introduction

The dynamics of a model of a two-component (BEC) irradiated by an external electromagnetic field are given by the following two coupled nonlinear Schrödinger equations :

$$\left.\begin{array}{ll} i\partial_t\psi_j & = -\dfrac{1}{2}\Delta\psi_j + \dfrac{\gamma^2}{2}|x|^2\psi_j + \beta_{jj}|\psi_j|^2\psi_j + \beta_{ji}|\psi_i|^2\psi_j + \lambda\psi_i + \delta\psi_j \\ \psi_j(0,x) & = \psi_j^0 \end{array}\right\} \tag{1.1}$$

$i \neq j = (1,2), (t,x) \in \mathbb{R}\times\mathbb{R}^N, N = 1,2,3.$

$V(x) = \dfrac{\gamma^2}{2}|x|^2$ is the trapping potential, $\gamma > 0$.

$\beta_{12} = \beta_{21}$ is the inter-specific scattering length, while β_{11} and β_{22} are the intra ones. λ is the rabi frequency related to the external electric field. It is the effective frequency to realize the internal atomic Josephson junction by a Raman transition, δ is the detuning constant for the Raman transition. (1.1) arises in modelling BEC composed of atoms in two hyperfine states in the same harmonic map [1]. Recently, BEC with multiple species have been realized in experiments, ([2] and references therein) and many interesting phenomena, which do not appear in the single component BEC, have been observed in the multi-component BEC. The simplest multi-component BEC can be viewed as a binary mixture, which can be used as a model to produce atomic lazer. To our knowledge, the first experiment in this framework has been done quite recently, this has opened the way to many other groups of research who carried out the study of such problems for two-component BEC theoretically and experimentally.

In this paper, we consider a binary BEC model in which there is an irradiation with an electromagnetic field, this causes a Josephson-type oscillation between the two species. These condensates are extremely important in physics and nonlinear optics since it is possible to measure the relative phase of one component with respect to the other one [Lemma 2.1, 2]. Controlling the relative phase, it is also possible to produce vortices, [7] account is given in [1].

A standing wave for (1.1) is a function $(\psi_1, \psi_2) = (e^{-i\mu_1 t}\Phi_1, e^{-i\mu_2 t}\Phi_2)$ solving this NLS. Thus it satisfies the following $2 \times 2(\mathbb{C})$ elliptic system :

$$\begin{cases} \mu_1\Phi_1 &= \Big[-\frac{1}{2}\Delta + \frac{\gamma^2}{2}|x|^2 + \delta + \beta_{11}|\Phi_1|^2 + \beta_{12}|\Phi_2|^2 \Big]\Phi_1 + \lambda\Phi_2 \\ \mu_2\Phi_2 &= \Big[-\frac{1}{2}\Delta + \frac{\gamma^2}{2}|x|^2 + \delta + \beta_{22}|\Phi_2|^2 + \beta_{12}|\Phi_1|^2 \Big]\Phi_2 + \lambda\Phi_1 \end{cases} \tag{1.2}$$

Ground state solutions of (1.2) are the minimizes of the following constrained variational problem : For two prescribed real numbers c_1 and c_2

$$\hat{I}_{c_1,c_2} = \inf_{(\Psi_1,\Psi_2)\in\hat{S}_{c_1,c_2}} \hat{E}(\Psi_1, \Psi_2) \tag{1.3}$$

$$\hat{S}_{c_1,c_2} = \Big\{ (\Psi_1, \Psi_2) \in \Sigma_{\mathbb{C}}(\mathbb{R}^N) \times \Sigma_{\mathbb{C}}(\mathbb{R}^N) : \int |\Psi_1|^2 = c_1^2 \text{ and } \int |\Psi_2|^2 = c_2^2 \Big\}.$$

$$\sum(\mathbb{R}^N) = \Big\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^2 u^2(x)dx < \infty \Big\}$$

$$|u|^2_{\sum(\mathbb{R}^N)} = |u|_2^2 + |\nabla u|_2^2 + ||x|u|_2^2$$

$$\overline{\sum}_{\mathbb{C}}(\mathbb{R}^N) = \Big\{ z = (u, v) \simeq u + iv : (u, v) \in \sum(\mathbb{R}^N) \times \sum(\mathbb{R}^N) \Big\}$$

$$\|z\|^2_{\sum_{\mathbb{C}}(\mathbb{R}^N)} = \|z\|_2^2 + \|\nabla z\|_2^2 + \||x|z\|_2^2.$$

$$\hat{E}(\Psi) = \hat{E}_0(\Psi_1, \Psi_2) + 2\lambda \int Re(\Psi_1\overline{\Psi_2})dx,$$

with $\bar{f}$ denoting the conjugate part of f and Re(f) its real one.

$$\begin{aligned} \hat{E}_0(\psi_) &= \hat{E}_0(\Psi_1, \Psi_2) = \int_{\mathbb{R}^N} \frac{1}{2}\Big[|\nabla\Psi_1|_2^2 + |\nabla\Psi_2|_2^2 + \gamma^2|x|^2(|\Psi_1|^2 + |\Psi_2|^2) \Big] \\ &+\delta|\Psi_1|^2 + \frac{1}{2}\beta_{11}|\psi_1|^4 + \frac{1}{2}\beta_{22}|\Psi_2|^4 + \beta_{12}|\Psi_1|^2|\Psi_2|^2 \Big\} dx \end{aligned} \tag{1.5}$$

As proved in iii) and iv) of Lemma 2.1 of [2], solving the constrained minimization problem (1.3) is equivalent to study the auxiliary minimization problem :

$$\widetilde{\hat{I}}_{c_1,c_2} = \inf_{(\Psi_1,\Psi_2)\in\hat{S}_{c_1,c_2}} \widetilde{\hat{E}}(\Psi_1, \Psi_2) \tag{1.6}$$

where

$$\widetilde{\hat{E}}(\Psi) = \widetilde{\hat{E}}(\Psi_1, \Psi_2) = \hat{E}_0(\Psi) - 2|\lambda| \int_{\mathbb{R}^N} |\Psi_1||\Psi_2|dx \tag{1.7}$$

The main objective of the present work is to show the orbital stability of standing waves of (1.1). To reach this goal, we will first solve (1.6) for real-valued functions :

$$\widetilde{I}_{c_1,c_2} = \inf_{(u_1,u_2)\in S_{c_1,c_2}} \widetilde{E}(u_1, u_2) = \inf_{(u_1,u_2)\in S_{c_1,c_2}} \widetilde{E}(u). \tag{1.8}$$

$$\widetilde{E}(u) = \widetilde{E}(u_1, u_2) = \widetilde{E}_0(u_1, u_2) - 2|\lambda| \int |u_1||u_2|dx. \tag{1.9}$$

$$S_{c_1,c_2} = \left\{(u_1,u_2) \in \sum(\mathbb{R}^N) \times \sum(\mathbb{R}^N) : \int_{\mathbb{R}^N} u_1^2(x) = c_1^2 \text{ and } \int_{\mathbb{R}^N} u_2^2(x) = c_2^2\right\}.$$

We will first prove existence, symmetry uniqueness of minimizers of (1.8). Then we will use these qualitative properties to solve the constrained variational problem (1.6), which is in itself a key step to show the orbital stability of standing waves and to characterize their orbit.
It is extremely important to obtain stable solutions without using highly oscillating magnetic fields (which are very costful). These states are the most relevant in physics since they are the only ones that can be observed in the experiments.
As mentioned in [2], the 2-component (BEC) is also used as a model for producing coherent atomic lasers. Stable standing waves indicate that the propagation is perfect in these beams.

Nevertheless, this relevant issue has not been addressed in [2]. Note also that a crucial intermediate step to study the stability of ground state solutions is to establish the existence of the minimizers of the associated constrained variational problem (1.6). In [2], the authors only considered the very restrictive assumption on the constraint $\hat{S}_{c_1,c_2}$, i.e. $\int u_1^2 + \int u_2^2 = 1$. (Remark 1) b). This implies that one has to impose that $\mu_1 = \mu_2$ (1.6) for the standing waves and that the solutions must have small masses. This is of course inappropriate in view of the applications.

In this paper, we will concentrate our study on the critical case $N = 2$, which is, from the mathematical point of view, the most challenging case.
Our paper is organized as follows. In section 2, we will give some important definitions and preliminary results. Then we will derive some qualitative properties of the energy functional and the minimization problem, this will be the key ingredient to study the orbital stability of standing waves in the last section.
We will focus our study on the case $N = 2$ but we will give clear and complete indications about $N = 1$ and $N = 3$.

2 Notation, Definitions and Preliminary Results

$H^1(\mathbb{R}^N)$ is the usual Hilbert space

$$\sum(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^2|u|^2 dx < \infty\}$$

$$|u|^2_{\sum(\mathbb{R}^N)} = |u|_2^2 + |\nabla u|_2^2 + |xu|_2^2$$

$|u|_p$ is the standard norm of the $L^p(\mathbb{R}^N)$ space

$$H^1(\mathbb{R}^N, \mathbb{C}) = \{z = (u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\}.$$

We shall identify $z = (u,v)$ with $u + iv \in H^1(\mathbb{R}^N, \mathbb{C})$.
For $z \in H^1(\mathbb{R}^N, \mathbb{C})$, $\|z\|^2_{H^1(\mathbb{R}^N,\mathbb{C})} = \|z\|_2^2 + \|\nabla z\|_2^2$

$$\|z\|_2^2 = |u|_2^2 + |v|_2^2 \text{ and } \|\nabla z\|_2^2 = |\nabla u|_2^2 + |\nabla v|_2^2.$$

Here and elsewhere $|\ |_q$ denotes the usual norm in $L^q(\mathbb{R}^N)$ and $\|\ \|_q$ is the standard norm in $L^q(\mathbb{R}^N, \mathbb{C})$.

$$\sum_{\mathbb{C}}(\mathbb{R}^N) = \left\{z \in H^1(\mathbb{R}^N, \mathbb{C}) \int |x|^2|z|^2 dx < \infty\right\}.$$

$\sum(\mathbb{R}^N) \times \sum(\mathbb{R}^N)$ and $\sum_{\mathbb{C}}(\mathbb{R}^N) \times \sum_{\mathbb{C}}(\mathbb{R}^N)$ are equipped with the standard cartesian norms.
For fixed real numbers c_1 and c_2, we define

$$Z_{c_1,c_2} = \{(z_1, z_2) \in \hat{S}_{c_1,c_2} : \widetilde{\hat{E}}(z_1, z_2) = \widetilde{\hat{I}}_{c_1,c_2}\} \tag{2.1}$$

$$W_{c_1,c_2} = \{(u_1, u_2) \in S_{c_1,c_2}, u_1 \text{ and } u_2 > 0 \text{ and } \widetilde{E}(u_1, u_2) = \widetilde{I}_{c_1,c_2}\}.$$

We say that Z_{c_1,c_2} is stable if :

$$Z_{c_1,c_2} \neq \emptyset$$

and $\forall\ w = (w_1, w_2) \in Z_{c_1,c_2}$, $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that for any $\psi_0 = (\Psi_0^1, \Psi_0^2) \in \sum_{\mathbb{C}}(\mathbb{R}^N) \times \sum_{\mathbb{C}}(\mathbb{R}^N)$ satisfying :

$$\begin{cases} \|\Psi_0 - w\|_{\sum_{\mathbb{C}(\mathbb{R}^N)}} < \delta \\ \inf\limits_{z \in Z_{c_1,c_2}} \|\Psi(t,.) - z\|_{\Sigma_{\mathbb{C}}(\mathbb{R}^n)} < \varepsilon \end{cases} \tag{2.2}$$

for all $t \in \mathbb{R}$, where $\Psi(t,.)$ is the unique solution of (1.1) corresponding to the initial condition Ψ_0. (Note that in [1], the authors have solved the Cauchy problem (1.1) under the assumptions of Lemma 2.3 below).
Lemma 2.1. [Lemma 4.4,8]
$\sum(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$ for any q such that $2 \leq q < \frac{2N}{N-2}$.
Lemma 2.2 Let $N = 2$,

$$(A_1) \begin{cases} \beta_{ij} < 0, 1 \leq i, j \leq 2 \text{ and} \\ \beta_{11}c_1^2 + \beta_{12}c_1c_2 > -c_b \\ \beta_{22}c_2^2 + \beta_{12}c_1c_2 > -c_b \end{cases}$$

c_b is defined as the best constant in the Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^2} u^4 \leq \frac{1}{c_b}|\nabla u|_2^2|u|_2^2 \tag{2.3}$$

Then :

1. The minimization problem (1.8) is well-posed and any minimizing sequence of (1.8) is bounded in $\sum(\mathbb{R}^2) \times \sum(\mathbb{R}^2)$.

2. Any minimizing sequence of (1.8) is relatively compact in $\sum(\mathbb{R}^2) \times \sum(\mathbb{R}^2)$, i.e, $\forall\ u_n = (u_{n,1}, u_{n,2}) \subset S_{c_1,c_2}$ such that $E(u_{n,1}, u_{n,2}) \to I_{c_1,c_2}$, then there exists $u = (u_1, u_2) \in \sum(\mathbb{R}^2) \times \sum(\mathbb{R}^2)$ such that $u_n \to u$ in $\sum(\mathbb{R}^2) \times \sum(\mathbb{R}^2)$ (up to a subsequence).

3. The functionals $\widetilde{E}$ and $\widetilde{\hat{E}}$ are C^1 in $\sum(\mathbb{R}^2) \times \sum(\mathbb{R}^2)$ (resp. $\sum_{\mathbb{C}}(\mathbb{R}^2) \times \sum_{\mathbb{C}}(\mathbb{R}^2)$)

4. $(c_1, c_2) \to \widetilde{I}_{c_1,c_2}$ is continuous .

Proof

1. Let $(u_1, u_2) \in S_{c_1,c_2}$.
First using Gagliardo-Nirenberg inequality, we know that

$$\int_{\mathbb{R}^2} |u_1|^4 \leq \frac{1}{c_b}|\nabla u_1|_2^2|u_1|_2^2 = \frac{1}{c_b}|\nabla u_1|_2^2 c_1^2 \tag{2.4}$$

and

$$\int_{\mathbb{R}^2} |u_2|^4 \leq \frac{1}{c_b}|\nabla u_2|_2^2|u_2|_2^2 = \frac{1}{c_b}|\nabla u_2|_2^2 c_2^2.$$

On the other hand, by Hardy inequality, we have that :

$$\int_{\mathbb{R}^2} |u_1|^2|u_2|^2 \leq \left(\int_{\mathbb{R}^2} |u_1|^4\right)^{1/2} \left(\int_{\mathbb{R}^2} |u_2|^4\right)^{1/2} \leq \frac{c_1 c_2}{c_b}|\nabla u_1|_2|\nabla u_2|_2 \tag{2.5}$$

It follows by Young inequality that :

$$\int_{\mathbb{R}^2} |u_1|^2|u_2|^2 \leq \frac{1}{2c_b} c_1 c_2[|\nabla u_1|_2^2 + |\nabla u_2|_2^2] \tag{2.6}$$

On the other hand, we can easily prove that

$$-2|\lambda| \int_{\mathbb{R}^2} |u_1||u_2| \geq -2|\lambda| c_1 c_2 \tag{2.7}$$

Combining (2.4) to (2.7), we get :

$$\begin{aligned} \widetilde{E}(u) = \widetilde{E}(u_1, u_2) \quad & \geq |\nabla u_1|_2^2\{\frac{1}{2} + \frac{1}{2}\beta_{11}\frac{c_1^2}{c_b} + \frac{1}{2}\beta_{12}\frac{c_1 c_2}{c_b}\} \\ & + |\nabla u_2|_2^2\{1/2 + \tfrac{1}{2}\beta_{22}\tfrac{c_2^2}{c_b} + \tfrac{1}{2}\beta_{12}\tfrac{c_1 c_2}{c_b}\} - |\delta||u_1|_2^2 \\ & -2|\lambda| c_1 c_2 + \frac{\gamma^2}{2}\int_{\mathbb{R}^2} |x|^2(|u_1|^2 + |u_2|^2). \end{aligned} \tag{2.8}$$

(A_1) enables us to conclude that the energy functional $\widetilde{E}$ is bounded from below in $\sum(\mathbb{R}^N) \times \sum(\mathbb{R}^N)$.

Remark 1

a) If there exists $\beta_{ij} \geq 0$, then 1) still holds true by replacing $\beta_{jj} \geq 0$ by 0 in the assumption (A_1).

b) In [2], the boundedness from below of the energy functional $\widetilde{E}$ has been proved differently (page 56, line 9).
More precisely : Combining Cauchy and Gagliardo Nirenberg inequalities, the authors have proved that

$$\begin{aligned} \int_{\mathbb{R}^2} \beta_{11}|u_1|^4 \quad & + \beta_{22}|u_2|^4 + 2\beta_{12}|u_1|^2|u_2|^2 dx \geq \\ & -c_b \int_{\mathbb{R}^2} (\sqrt{|u_1|^2 + |u_2|^2})^4 dx \geq -\int_{\mathbb{R}^2} (\sqrt{|u_1|^2 + |u_2|^2})^2 dx \int_{\mathbb{R}^2} (\nabla\sqrt{|u_1|^2 + |u_2|^2})^2 \\ & \geq -\int_{\mathbb{R}^2} |\nabla u_1|^2 + |\nabla u_2|^2. \end{aligned} \tag{2.8$'$}$$

provided that

$$\left.\begin{array}{ll} \beta_{11} & > -c_b \\ \beta_{22} & > -c_b \\ \beta_{12} & \geq -c_b - \sqrt{\beta_{11} + c_b}\sqrt{\beta_{22} + c_b} \\ \text{and} & \\ c_1^2 & + c_2^2 < 1. \end{array}\right\}. \tag{A_1'}$$

It seems that if one uses their approach, it is necessary to impose the very restrictive condition : $c_1^2 + c_2^2 < 1$.

Nevertheless the two approaches are equivalent if one considers the same one-constrained minimization problem (1.8) with "their" constraint $\int u_1^2 + u_2^2 = 1$.

c) The case $N = 1$ is immediate since the Gagliardo Nirenberg inequality is not critical there.

However for $N = 3$, our approach only applies when all the constants β_{ij} are positive. Additionally if $c_1^2 + c_2^2 < 1$ then using the same approach developed in [2] ((2.8)'), we can easily prove that 1) and 2) still hold true if we have the following assumption :

$$\left.\begin{array}{l}\beta_{11} > 0\\ \beta_{22} > 0\\ \beta_{11}\beta_{22} - \beta_{12}^2 > 0.\end{array}\right\} \qquad (A_1)_{N=3}$$

$$\beta_{11} > 0, \beta_{12} > 0 \text{ and } \beta_{22} > 0. \qquad (A_1')_{N=3}$$

d) Let us finally emphasize that all the results of this section hold true provided that the constrained minimization problem is well-posed.

Proof of 2)

By 1), we can conclude that any minimizing sequence $u_n = (u_{n,1}, u_{n,2})$ of (1.8) is bounded in $\sum(\mathbb{R}^2) \times \sum(\mathbb{R}^2)$. Therefore up to a subsequence (that we will also denote by (u_n)), there exists $(u_1, u_2) \in \sum(\mathbb{R}^2) \times \sum(\mathbb{R}^2)$ such that $u_{n,1} \rightharpoonup u_1$ and $u_{n,2} \rightharpoonup u_2$ in $\sum(\mathbb{R}^2)$.
By Lemma 2.1, $u_{n,1} \to u_1$ and $u_{n,2} \to u_2$ in $L^p(\mathbb{R}^2)$, $\forall\, 2 \leq p < \infty$. (2.9)
Now note that by the lower semi-continuity of the norm $|\;|_{\sum(\mathbb{R}^N)}$, we certainly have :

$$\begin{aligned}\tfrac{1}{2}|\nabla u_1|_2^2 \; &+ \tfrac{1}{2}|\nabla u_2|_2^2 + \tfrac{\gamma^2}{2}\int_{\mathbb{R}^2} |x|^2(u_1|^2 + |u_2|^2)dx \\ &\leq \liminf \Big(\tfrac{1}{2}|\nabla u_{n,1}|_2^2 + \tfrac{1}{2}|\nabla u_{n,2}|_2^2 + \tfrac{\gamma^2}{2}\int_{\mathbb{R}^2} |x|^2(|u_{n,1}|^2 + |u_{n,2}|^2)dx\Big)\end{aligned} \qquad (2.10)$$

On the other hand, by (2.9), we have that :

$$\begin{aligned}\int_{\mathbb{R}^2} |u_{n,1}|^4 &\to \int |u_1|^4 \\ \int_{\mathbb{R}^2} |u_{n,2}|^4 &\to \int |u_2|^4\end{aligned} \qquad (2.11)$$

Thus using the dominated convergence theorem, we can deduce that.

$$\int_{\mathbb{R}^2} |u_{n,1}|^2|u_{n,2}|^2 \to \int_{\mathbb{R}^2} |u_1|^2|u_2|^2 \qquad (2.12)$$

Indeed since $u_n \to u$ in $L^4(\mathbb{R}^2) \times L^4(\mathbb{R}^2)$, there exist a subsequence $(u_{nj,1}) \subset L^4(\mathbb{R}^2)$ and a function $h \in L^4(\mathbb{R}^2)$ such that $u_{nj,1} \to u_1$ almost every where with $|u_{nj,1}| \leq h$.
Similarly, we can find $(u_{nj,2}$ and $k \subset L^4(\mathbb{R}^2)$ such that $u_{n,2} \to u_2$ a.e with $|u_{nj,2}| \leq k)$

$$\int_{\mathbb{R}^2} |u_{nj,1}|^2|u_{nj,2}|^2 \leq \int_{\mathbb{R}^2} h^2k^2dx \leq (\int h^4)^{1/2}(\int k^4)^{1/2} < \infty.$$

Therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} |u_{nj,1}|^2|u_{nj,2}|^2 \leq \int_{\mathbb{R}^2} h^2k^2dx \leq (\int h^4)^{1/2}(\int k^4)^{1/2} < \infty$$

In the same manner, we can prove that $\lim_{n\to\infty}\int |u_{n,1}||u_{n,2}| = \int |u_1||u_2|$. (2.13)
Combining (2.10) to (2.13), we obtain :

$$\widetilde{E}(u) = \widetilde{E}(u_1, u_2) \leq \liminf \widetilde{E}(u_{n,1}, u_{n,2}) = \widetilde{I}_{c_1,c_2}. \qquad (2.14)$$

But

$$\int u_1^2 = \lim_{n\to\infty} \int_{\mathbb{R}^2} u_{n,1}^2 = c_1^2 \quad \text{and} \quad \int u_2^2 = \lim_{n\to\infty} \int_{\mathbb{R}^2} u_{n,2}^2 = c_2^2.$$

Thus $u = (u_1, u_2) \in S_{c_1,c_2}$ with $\widetilde{E}(u) = \widetilde{E}(u_1, u_2) = \widetilde{I}_{c_1,c_2}$.

Remark 2 :

- In part 2) of the Lemma, we have also proved that any minimizing sequence of (1.8) is relatively compact in $\sum(\mathbb{R}^N) \times \sum(\mathbb{R}^N)$
- The proofs of 3) and 4) goes exactly in the same way as in [Proposition 3.2, 4].

Lemma 2.3 Under (A_1), all the minimizers of (1.3) are non-negative radial and radially decreasing.

Proof. First note that $\widetilde{E}(|u_1|, |u_2|) \leq \widetilde{E}(u_1, u_2)$ for any $(u_1, u_2) \in \sum(\mathbb{R}^N) \times \sum(\mathbb{R}^N)$. Therefore, we can suppose without class of generality that u_1 and u_2 are non-negative.

On the other hand, using rearrangement inequalities, [5], we know that for any f, g non-negative $\in \sum(\mathbb{R}^N)$, we have :

$$\int f^2 = \int (f^*)^2$$

$$\int f^4 = \int (f^*)^4$$

$$\int fg \leq \int f^* g^*$$

$$\beta_{12} \int (f^*)^2 (g^*)^2 \leq \beta_{12} \int f^2 g^2$$

$$\int |x|^2 (f^*)^2 < \int |x|^2 f^2,$$

and

$$|\nabla f^*|_2 \leq |\nabla f|_2.$$

Lemma 2.4. If

$$(A_2) \begin{cases} \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix} \text{ is positive semi-definite} \\ \text{and at least } : \begin{array}{ll} \beta_{11} - \beta_{22} & \neq 0 \text{ or} \\ \beta_{11} - \beta_{12} & \neq 0 \text{ or} \\ \delta & \neq 0 \text{ or} \\ \lambda & \neq 0 \end{array} \end{cases}$$

Then (1.8) has a unique minimizer.

Proof [Lemma 2.2, 2]

3 Orbital stability of standing waves of (1.1) (when $\lambda = 0$)

In this section we will restrict ours study to the case $\lambda = 0$ since we only have conservation of each mass in this case as noticed in (4.1) and (4.2) of [7]. We assume that (A_1) and (A_2) hold true.

Theorem 3.1

1. For any $c_1, c_2; \widetilde{I}_{c_1,c_2} = \widetilde{\widehat{I}}_{c_1,c_2}, Z_{c_1,c_2} \neq \emptyset$ and Z_{c_1,c_2} is stable

2. For any $z = (z_1, z_2) \in Z_{c_1,c_2}, |z| = (|z_1|, |z_2|) \in W_{c_1,c_2}$ and

$$Z_{c_1,c_2} = \left\{(e^{i\theta_1}w_1, e^{i\theta_2}w_2), (\theta_1, \theta_2) \in \mathbb{R}^2\right\},$$

where (w_1, w_2) is the unique solution of (1.8).

Proof

1. As suggested in [3], to show the orbital stability of the standing waves of (1.1), it suffices to prove that : $Z_{c_1,c_2} \neq \emptyset$ and any minimizing sequence

$$\left\{ \begin{array}{l} z_n = (z_{n,1}, z_{n,2}) \in \Sigma_{\mathbb{C}}(\mathbb{R}^2) \times \Sigma_{\mathbb{C}}(\mathbb{R}^2) \text{ such that } \|z_{n,1}\|_2 \to c_1 \text{ and} \\ \|z_{n,2}\|_2 \to c_2 \text{ and } \widetilde{\hat{E}}(z_n) \to \widetilde{\hat{I}}_{c_1,c_2} \end{array} \right. \tag{3.1}$$

is relatively compact in $\sum_{\mathbb{C}}(\mathbb{R}^2) \times \sum_{\mathbb{C}}(\mathbb{R}^2)$.
Let $z_n = (z_{n,1}, z_{n,2})$ (with $z_{n,1} = (u_{n,1}, v_{n,1}), z_{n,2} = (u_{n,2}, v_{n,2})) \subset \sum_{\mathbb{C}}(\mathbb{R}^2) \times \sum_{\mathbb{C}}(\mathbb{R}^2)$ be a sequence such that $\|z_{n,1}\|_2 \to c_1$ $\|z_{n,2}\|_2 \to c_2$ and $\widetilde{\hat{E}}(z_{n,1}, z_{n,2}) \to \widetilde{\hat{I}}_{c_1,c_2}$.

Our first goal is to prove that $\{z_n\}$ has a subsequence which is convergent in $\sum_{\mathbb{C}}(\mathbb{R}^2) \times \sum_{\mathbb{C}}(\mathbb{R}^2)$.
By Lemma 2.2, it can be immediately deduced that $\{z_n\}$ is bounded in $\sum_{\mathbb{C}}(\mathbb{R}^2) \times \sum_{\mathbb{C}}(\mathbb{R}^2)$, therefore passing to a subsequence, one can suppose that :

$$u_{n,i} \rightharpoonup u_i \text{ and } v_{n,i} \rightharpoonup v_i \text{ in } \sum(\mathbb{R}^2), 1 \leq i \leq 2. \tag{3.2}$$

Now set $\rho_{n,i} = |z_{n,i}| = (u_{n,i}^2 + v_{n,i}^2)^{1/2}$.
It follows that $\{\rho_{n,i}\} \subset \sum(\mathbb{R}^2)$ and that for all $n \in \mathbb{N}$ and $1 \leq i, j \leq 2$:

$$\partial_j \rho_{n,i} = \left\{ \begin{array}{ll} \dfrac{u_{n,i}(x)\partial_j u_{n,i}(x) + v_{n,i}(x)\partial_j v_{n,i}(x)}{(u_{n,i}^2 + v_{n,i}^2)^{1/2}} & \text{if } u_{n,i}^2 + v_{n,i}^2 > 0 \\ 0 & \text{elsewhere} \end{array} \right.$$

Thus

$$\begin{aligned} \widetilde{\hat{E}}(z_n) - \widetilde{E}(\rho_n) &= \frac{1}{2}\{\|\nabla z_n\|_2^2 - \|\nabla |z_n|\|_2^2\} \\ &= \frac{1}{2}\{\|\nabla z_{n,1}\|_2^2 + \|\nabla z_{n,2}\|_2^2 - |\nabla \rho_{n,1}|_2^2 - |\nabla \rho_{n,2}|_2^2\} \\ &= \frac{1}{2}\{|\nabla v_{n,1}|_2^2 + |\nabla v_{n,1}|_2^2 + |\nabla u_{n,2}|_2^2 + |\nabla u_{n,2}|_2^2 - |\nabla \rho_{n,1}|_2^2 - |\nabla \rho_{n,2}|_2^2\} \\ &= \frac{1}{2}\sum_{i=1}^{2} \int_{\{u_{n,i}^2 + v_{n,i}^2 > 0\}} \sum_{j=1}^{2} (\frac{u_{n,i}\partial_j v_{n,i} - v_{n,i}\partial v_{n,i})^2}{u_{n,i}^2 + v_{n,i}^2} \geq 0 \end{aligned} \tag{3.3}$$

Hence $\widetilde{\hat{I}}_{c_1,c_2} = \lim\limits_{n\to\infty} \widetilde{\hat{E}}(z_n) \geq \limsup \widetilde{E}(\rho_n)$.
Taking into account that, we obtain :

$$\|z_{n,i}\|_2^2 = |\rho_{n,i}|_2^2 = c_{n,i}^2 \to c_i^2 \quad \forall\, 1 \leq i \leq 2, \tag{3.4}$$

Thus using Lemma 2.2 4), we obtain that :

$$\liminf \widetilde{E}(\rho_n) \geq \liminf \widetilde{I}_{c_{n,1},c_{n,2}} \geq \widetilde{I}_{c_1,c_2} \geq \widetilde{\hat{I}}_{c_1,c_2},$$

and hence

$$\lim_{n\to\infty} \widetilde{E}(\rho_n) = \lim_{n\to\infty} \widetilde{\hat{E}}(z_n) = \widetilde{\hat{I}}_{c_1,c_2} = \widetilde{I}_{c_1,c_2}. \tag{3.5}$$

On the other hand (3.3) implies that for any $1 \le i \le 2$, we have :

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |\nabla u_{n,i}|^2 + |\nabla v_{n,i}|^2 - |\nabla (u_{n,i}^2 + v_{n,i}^2)^{1/2}|^2 dx = 0. \tag{3.6}$$

(3.2) together with (3.6) imply that $\forall\, 1 \le i \le 2$.

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |\nabla u_{n,i}|^2 + |\nabla v_{n,i}|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^2} |\nabla (u_{n,i}^2 + v_{n,i}^2)^{1/2}|^2 \tag{3.7}$$

which is equivalent to say that

$$\lim_{n\to\infty} \|\nabla z_n\|_2^2 = \lim_{n\to\infty} \Big|\nabla |z_n|\Big|_2^2. \tag{3.8}$$

Now using (3.4), (3.5) and Remark 1, $\rho_n = (\rho_{n,1}, \rho_{n,2})$ is relatively compact in $\sum(\mathbb{R}^2) \times \sum(\mathbb{R}^2)$. Thus, there exist $\rho_1, \rho_2 \in \sum(\mathbb{R}^2)$ such that :

$$\begin{cases} (u_{n,i}^2 + v_{n,i}^2)^{1/2} \text{ converges to } \rho_j \text{ in } \sum(\mathbb{R}^2) : \ \forall\, 1 \le i \le 2 \\ |\rho_j|_2 = c_j \text{ and} \\ \widetilde{E}(\rho_1, \rho_2) = \widetilde{I}_{c_1,c_2} \end{cases} \tag{3.9}$$

Let us first prove that $\rho_i = |z_i| = (u_i^2 + v_i^2)^{1/2}$; (u_i and v_i are given in (3.2)).
By (3.2), we know that $u_{n,i} \to u_i$ and $v_{n,i} \to v_i$ in $L^2(B(0,R))$, and we can easily see that :

$$[(u_{n,i}^2 + v_{n,i}^2)^{1/2} - (u_i^2 + v_i^2]^2 \le |u_{n,i} - u_i|^2 + |v_{i,n} - v_i|^2,$$

therefore

$$(u_{n,i}^2 + v_{n,i}^2)^{1/2} \to (u_i^2 + v_i^2)^{1/2} \text{ in } L^2(B(0,r)) \quad \forall\, R > 0.$$

But $(u_{n,i}^2 + v_{n,i}^2)^{1/2} \to \rho_i$ in $L^2(B(0,R)$, this certainly implies that $|z_i| = \rho_i \ \ \forall\, 1 \le i \le 2$.
On the other hand $\|z_{n,i}\|_2 = \|z_{n,i}\|_2 \to c_i = \|z_i\| = ||z_i||_2$.
Therefore the proof of the first part of Theorem 3.1 is complete if we show that

$$\lim_{n\to\infty} \|\nabla z_{n,i}\|_2^2 \to \|\nabla z_i\|_2^2 \quad \forall\, 1 \le i \le 2.$$

From (3.6), we have that

$$\lim_{n\to\infty} \|\nabla z_{n,i}\|_2 = \lim_{n\to\infty} |\nabla |z_{n,i}||_2 \quad \text{and} \quad \lim_{n\to+\infty} |\nabla |z_{n,i}||_2 = |\nabla |z_i\|_2^2.$$

Hence by the lower semi-continuity of $|\ |_2$, we have :

$$\|\nabla z_i\|_2^2 \le \lim |\nabla z_{n,i}\|_2^2 = \lim |\nabla |z_{n,i}\|_2^2 = |\nabla |z_i\|_2^2 \tag{3.10}$$

Finally, replacing $z_{n,i}$ by z_i in (3.3), we see that :

$$\|\nabla z_i\|_2^2 \ge |\nabla |z_i\|_2^2 \quad \forall\, 1 \le i \le 2.$$

Now using (3.2), we know that $z_{n,i} \to z_i$ in $\sum_{\mathbb{C}}(\mathbb{R}^2)$. Thus $z_{n,i} \to z_i$ in $\sum_{\mathbb{C}}(\mathbb{R}^2)$ $\forall\, 1 \le i \le 2$.
Proof of 2) Let $z = (z_1, z_2) \in Z_{c_1,c_2}$ with $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$.

Let $\rho_1 = (u_1^2 + v_2^2)^{1/2}$ and $\rho_2 = (u_2^2 + v_2^2)^{1/2}$. By the latter, we certainly have that

$$\forall\, 1 \le i \le 2 \ \forall\, 1 \le j \le 2 \ \int_{\mathbb{R}^2} \left(\frac{u_i \partial_j v_i - v_i \partial_j u_i}{u_i^2 + v_i^2} \right)^2 dx = 0 \tag{3.11}$$

On the other hand $\widetilde{\hat{E}}(z_1, z_2) = \widetilde{\hat{I}}_{c_1,c_2}$, which implies that there exists a Lagrange multiplier $\alpha \in \mathbb{C}$ such that

$$\widetilde{\hat{E}}(z)\xi = \frac{\alpha}{2} \sum_{i=1}^{2} z_i \bar{\xi}_i + \overline{\xi z_i} \text{ for all } \xi \in \mathbb{C} \times \mathbb{C}.$$

By elementary regularity theory and maximum principle, we can prove that u_i and $v_i \in C^0(\mathbb{R}^2) \cap \sum(\mathbb{R}^2)$ and $\rho_i > 0$.

Set $\Omega = \{x \in \mathbb{R}^2 : u_i(x) = 0\}$ then Ω is closed since u_i is continuous. Let us prove that it is also open.

Let $x \in \Omega$. Using the fact that $v_i(x) > 0$, we can find a ball B centered in x_0 such that $v_i(x) \neq 0$ for any $x \in B$.

Thus for $x \in B$

$$(\frac{v_i \partial_j v_i - v_i \partial_j v_i)^2}{u_i^2 + v_i^2} = [\partial_j (\frac{u_i}{v_i}]^2 \frac{v_i^4}{u_i^2 + v_i^2} \text{ for } 1 \le i, j \le 2.$$

This implies that

$$\int_B |\nabla(\frac{u_i}{v_i})|^2 \, \frac{v_i^4}{u_i^2 + v_i^2} = 0. \tag{3.8}$$

Hence $\nabla(\frac{u_i}{v_i}) = 0$ on $B \Rightarrow \ \exists\, C$ such that $\frac{u_i}{v_i} = C$ on B. Since $x_0 \in B \Rightarrow C \equiv 0$.

Therefore Ω is also an open set of $\mathbb{R}^N$. Hence we have proved that for $1 \le i \le 2$, these are two alternatives :

1. $u_i \equiv 0$ or $u_i > 0$ or $\mathbb{R}^2$
2. $v_i \equiv 0$ or $v_i > 0$ or $\mathbb{R}^2$.

Now let $z_i = e^{i\sigma_i} w_i$, $\sigma_i \in \mathbb{R}$, $w_i \in W_{c_1,c_2}$. Thus $|z_i|_2 = c_i$ and $\widetilde{E}(z_1, z_2) = \hat{I}_{c_1,c_2}$. Then $\{e^{i\sigma_i} w_i : \sigma_i \in \mathbb{R}, w_i \in W_{c_1,c_2}\} \subset Z_{c_1,c_2}$.

Conversely for $z_i = (u_i, v_i)$ such that $(z_1, z_2) \in Z_{c_1,c_2}$, set $w_i = |z_i|$. Then $\widetilde{\hat{E}}(z_1, z_2) = \widetilde{E}(w_1, w_2) = \widetilde{\hat{I}}_{c_1,c_2} = \hat{I}_{c_1,c_2}$ and $(w_1, w_2) \in W_{c_1,c_2}$.

We now have four possible alternatives. We will discuss one in details, the three others can be shown following exactly the same ideas.

Suppose that v_1 and $v_2 \neq 0$ for all $x \in \mathbb{R}^2$.

In this case, it follows that $\nabla(\frac{u_i}{v_i}) = 0$ on $\mathbb{R}^2$.

Thus we can find 2 constants $K_1, K_2 \in \mathbb{R}$ such that

$$u_1 \equiv K_1 v_1 \quad \text{ and } \quad u_2 \equiv K_2 v_2.$$

Therefore $w_1 = (K_1 + i)v_1$ and $w_1 = |K_1 + i||v_1|$.

Let $\theta_1 \in \mathbb{R}$ such that $K_1 + i = |K_1 + i|e^{\theta_1}$ and let $\varphi_1 = 0$ if $v_i > 0$ and $\varphi_1 = \pi$ if $v_1 < 0$ on $\mathbb{R}^2$. Setting $\sigma_1 = \theta_1 + \varphi_1$, $z_1 = (K_1 + i)v_1 = |K_1 + i|e^{\theta_1}|v_1|e^{i\varphi_1} = w_1 e^{i\sigma_1}$.

Similarly $z_2 = w_2 e^{i\sigma_2}$ with $w = (w_1, w_2)$.

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