

Existence, Non-existence, Uniqueness of solutions for semilinear elliptic equations involving measures concentrated on boundary

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Abstract

The purpose of this paper is to study the weak solutions of the fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + \epsilon g(u) &= k \frac{\partial^\alpha \nu}{\partial \bar{n}^\alpha} && \text{in } \bar{\Omega}, \\ u &= 0 && \text{in } \bar{\Omega}^c, \end{aligned} \tag{0.1}$$

where $k > 0$, $\epsilon = 1$ or -1 , $(-\Delta)^\alpha$ with $\alpha \in (0, 1)$ is the fractional Laplacian defined in the principle value sense, Ω is a bounded C^2 open set in \mathbb{R}^N with $N \geq 2$, ν is a bounded Radon measure supported in $\partial\Omega$ and $\frac{\partial^\alpha \nu}{\partial \bar{n}^\alpha}$ is defined in the distribution sense, i.e.

$$\left\langle \frac{\partial^\alpha \nu}{\partial \bar{n}^\alpha}, \zeta \right\rangle = \int_{\partial\Omega} \frac{\partial^\alpha \zeta(x)}{\partial \bar{n}_x^\alpha} d\nu(x), \quad \forall \zeta \in C^\alpha(\mathbb{R}^N),$$

here \bar{n}_x denotes the unit inward normal vector at $x \in \partial\Omega$.

In this paper, we prove that (0.1) with $\epsilon = 1$ admits a unique weak solution when g is a continuous nondecreasing function satisfying

$$\int_1^\infty (g(s) - g(-s)) s^{-1 - \frac{N+\alpha}{N-\alpha}} ds < +\infty.$$

Our interest then is to analyse the properties of weak solution when $\nu = \delta_{x_0}$ with $x_0 \in \partial\Omega$, including the asymptotic behavior near x_0 and the limit of weak solutions as $k \rightarrow +\infty$. Furthermore, we show the optimality of the critical value $\frac{N+\alpha}{N-\alpha}$ in a certain sense, by proving the non-existence of weak solutions when $g(s) = s^{\frac{N+\alpha}{N-\alpha}}$.

The final part of this article is devoted to the study of existence for positive weak solutions to (0.1) when $\epsilon = -1$ and ν is a bounded nonnegative Radon measure supported in $\partial\Omega$. We employ the Schauder's fixed point theorem to obtain positive solution under the hypothesis that g is a continuous function satisfying

$$\int_1^\infty g(s) s^{-1 - \frac{N+\alpha}{N-\alpha}} ds < +\infty.$$

Key words: Fractional Laplacian; Radon measure; Dirac mass; Green kernel; Schauder's fixed point theorem.

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1 Introduction

1.1 Motivation

In 1991, a fundamental contribution of semilinear elliptic equations involving measures as boundary data is due to Gmira and Véron in [19], which studied the weak solutions for

$$\begin{aligned} -\Delta u + g(u) &= 0 & \text{in } \Omega, \\ u &= \mu & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded C^2 domain in \mathbb{R}^N and μ is a bounded Radon measure defined in $\partial\Omega$. A function u is said to be a weak solution of (1.1) if $u \in L^1(\Omega)$, $g(u) \in L^1(\Omega, \rho_{\partial\Omega} dx)$ and

$$\int_{\Omega} [u(-\Delta)\xi + g(u)\xi] dx = \int_{\partial\Omega} \frac{\partial\xi(x)}{\partial\vec{n}_x} d\mu(x), \quad \forall \xi \in C_0^{1,1}(\Omega), \tag{1.2}$$

where $\rho_{\partial\Omega}(x) = \text{dist}(x, \partial\Omega)$ and \vec{n}_x denotes the unit inward normal vector at point x . Gmira and Veron proved that problem (1.1) admits a unique weak solution when g is a continuous and nondecreasing function satisfying

$$\int_1^\infty [g(s) - g(-s)] s^{-1 - \frac{N+1}{N-1}} ds < +\infty. \tag{1.3}$$

Furthermore, the weak solution of (1.1) is approached by the classical solutions of (1.1) replacing μ by a sequence of regular functions $\{\mu_n\}$, which converge to μ in the distribution sense. Then this subject has been vastly expanded in recent works, see the papers of Marcus and Véron [22, 23, 24, 25], Bidaut-Véron and Vivier [5] and reference therein.

A very challenging question consists in studying the analogue elliptic problem involving fractional Laplacian defined by

$$(-\Delta)^\alpha u(x) = \lim_{\varepsilon \rightarrow 0^+} (-\Delta)_\varepsilon^\alpha u(x),$$

where

$$(-\Delta)_\varepsilon^\alpha u(x) = - \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} dz$$

for $\varepsilon > 0$. The main difficulty comes from how to define the boundary type data. Given a Radon measure μ defined in $\partial\Omega$, it is ill-posed that

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= 0 & \text{in } \Omega, \\ u &= \mu & \text{on } \partial\Omega, \\ u &= 0 & \text{in } \bar{\Omega}^c. \end{aligned}$$

Indeed, let $\{\mu_n\}$ be a sequence of regular functions defined in $\partial\Omega$ converging to the measure μ and a surprising result is that there is just zero solution for

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= 0 & \text{in } \Omega, \\ u &= \mu_n & \text{on } \partial\Omega, \\ u &= 0 & \text{in } \bar{\Omega}^c, \end{aligned}$$

which is in sharp contrast with Laplacian case, where (1.1) replacing μ by μ_n admits a unique nontrivial solution. On the other hand, it is also not proper to pose

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= 0 & \text{in } \Omega, \\ u &= \mu & \text{in } \Omega^c \end{aligned}$$

with μ being a Radon measure in Ω^c concentrated on $\partial\Omega$. In fact, letting functions $\{\mu_n\} \subset C_0^1(\Omega^c)$ converging to μ , the solution u_n of

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= 0 & \text{in } \Omega, \\ u &= \mu_n & \text{in } \Omega^c, \end{aligned}$$

is equivalent to the solution of

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= G_{\mu_n} & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned}$$

where

$$G_{\mu_n}(x) = \int_{\Omega^c} \frac{\mu_n(y)}{|x-y|^{N+2\alpha}} dy, \quad x \in \Omega,$$

see [10]. It could be seen that

$$\int_{\Omega} [u_n(-\Delta)^\alpha \xi + g(u_n)\xi] dx = \int_{\Omega} G_{\mu_n} \xi dx, \quad \forall \xi \in C_0^2(\Omega),$$

Then the limit of $\{u_n\}$ as $n \rightarrow \infty$ wouldn't be a weak solution as we desired, similar to (1.2).

Therefore, a totally different point of view has to be found to propose the fractional elliptic problem involving measure concentrated on boundary. Our idea is inspired by the study of elliptic equations with fractional Laplacian and Radon measure inside of Ω in [12], where the authors considered the equations

$$\begin{aligned} (-\Delta)^\alpha u + h(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \end{aligned} \tag{1.4}$$

for $\nu \in \mathfrak{M}(\Omega, \rho_{\partial\Omega}^\beta)$ with $\beta \in [0, \alpha]$ the space of Radon measure ν in Ω satisfying

$$\int_{\Omega} \rho_{\partial\Omega}^\beta(x) d|\nu(x)| < +\infty.$$

A function u is said to be a weak solution of (1.4), if $u \in L^1(\Omega)$, $h(u) \in L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)$ and

$$\int_{\Omega} [u(-\Delta)^\alpha \xi + h(u)\xi] dx = \int_{\Omega} \xi(x) d\nu(x), \quad \forall \xi \in \mathbb{X}_\alpha,$$

where $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ with $\alpha \in (0, 1)$ denotes the space of functions ξ satisfying:

- (i) $\text{supp}(\xi) \subset \bar{\Omega}$;
- (ii) $(-\Delta)^\alpha \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \leq C$ for some $C > 0$;
- (iii) there exist $\varphi \in L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)$ and $\varepsilon_0 > 0$ such that $|(-\Delta)_\varepsilon^\alpha \xi| \leq \varphi$ a.e. in Ω , for all $\varepsilon \in (0, \varepsilon_0]$.

A unique weak solution of (1.4) is obtained when the function h is continuous, nondecreasing and satisfies

$$\int_1^\infty (h(s) - h(-s))s^{-1-k_{\alpha,\beta}} ds < +\infty,$$

where

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if } \beta \in [0, \frac{N-2\alpha}{N}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if } \beta \in (\frac{N-2\alpha}{N}\alpha, \alpha]. \end{cases}$$

Motivated by the above results, we may approximate $\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}$ by a sequence measures defined in Ω and consider the limit of corresponding weak solutions. To this end, for a bounded Radon measure defined in $\bar{\Omega}$ with support in $\partial\Omega$, we observe that

$$\left\langle \frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}, \xi \right\rangle = \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\nu(x), \quad \xi \in \mathbb{X}_\alpha,$$

and

$$\frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} = \lim_{s \rightarrow 0^+} \frac{\xi(x + s\vec{n}_x) - \xi(x)}{s^\alpha} = \lim_{s \rightarrow 0^+} \xi(x + s\vec{n}_x) s^{-\alpha},$$

so $\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}$ could be approximated by measures $\{t^{-\alpha} \nu_t\}$ with support in $\{x \in \Omega : \rho_{\partial\Omega}(x) = t\}$ generated by ν , see Section 2 for details. Then we consider the limit of weak solutions as $t \rightarrow 0^+$ for the problem:

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= t^{-\alpha} \nu_t & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c. \end{aligned}$$

Here the limit of these weak solutions (if it exists) is called a weak solution of the following fractional elliptic problem with measure concentrated on boundary

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= \frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha} & \text{in } \bar{\Omega}, \\ u &= 0 & \text{in } \bar{\Omega}^c. \end{aligned}$$

This will be our main focus in this paper.

1.2 Statement of our problem and main results

Let $\alpha \in (0, 1)$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, Ω be a bounded smooth domain in \mathbb{R}^N with $N \geq 2$ and denote by $\mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$ the bounded Radon measure in $\bar{\Omega}$ with the support in $\partial\Omega$. Our purpose in this article is to investigate the existence, non-existence and uniqueness of weak solutions to semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + \epsilon g(u) &= k \frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha} & \text{in } \bar{\Omega}, \\ u &= 0 & \text{in } \bar{\Omega}^c, \end{aligned} \tag{1.5}$$

where $\epsilon = 1$ or -1 , $k > 0$, $(-\Delta)^\alpha$ is the fractional Laplacian and denote $\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}$ with $\nu \in \mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$ by

$$\left\langle \frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}, \xi \right\rangle = \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\nu(x), \quad \xi \in \mathbb{X}_\alpha,$$

with \vec{n}_x being the unit inward normal vector at x . We call g the absorption nonlinearity if $\epsilon = 1$, otherwise it is called as source nonlinearity.

Before starting our main theorems we make precise the notion of weak solution used in this article.

Definition 1.1 We say that u is a weak solution of (1.5), if $u \in L^1(\Omega)$, $g(u) \in L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)$ and

$$\int_{\Omega} [u(-\Delta)^\alpha \xi + \epsilon g(u)\xi] dx = k \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\nu(x), \quad \forall \xi \in \mathbb{X}_\alpha.$$

We notice that $\mathbb{X}_\alpha \supset C_0^2(\Omega)$ is the test functions space when we study semilinear fractional elliptic equations involving measures, which plays the same role as $C_0^{1,1}(\Omega)$ for dealing with second order elliptic equations with measures, see [12, 13, 14, 15]. Moreover, it follows from [27, Proposition 1.1] that ξ is C^α (α -Hölder continuous) in \mathbb{R}^N if $\xi \in \mathbb{X}_\alpha$.

Denote by G_α the Green kernel of $(-\Delta)^\alpha$ in $\Omega \times \Omega$ and by $\mathbb{G}_\alpha[\cdot]$ the Green operator defined as

$$\mathbb{G}_\alpha\left[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}\right](x) = \lim_{t \rightarrow 0^+} \int_{\partial\Omega} G_\alpha(x, y + t\vec{n}_y) t^{-\alpha} d\nu(y).$$

Now we are ready to state our first result for problem (1.5).

Theorem 1.1 Assume that $\epsilon = 1$, $k > 0$, $\nu \in \mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$ and g is a continuous nondecreasing function satisfying $g(0) \geq 0$ and

$$\int_1^\infty [g(s) - g(-s)] s^{-1 - \frac{N+\alpha}{N-\alpha}} ds < +\infty. \quad (1.6)$$

Then

- (i) problem (1.5) admits a unique weak solution u_ν ;
- (ii) the mapping $\nu \rightarrow u_\nu$ is increasing and

$$-k \mathbb{G}_\alpha\left[\frac{\partial^\alpha \nu_-}{\partial \vec{n}^\alpha}\right](x) \leq u_\nu(x) \leq k \mathbb{G}_\alpha\left[\frac{\partial^\alpha \nu_+}{\partial \vec{n}^\alpha}\right](x), \quad x \in \Omega, \quad (1.7)$$

where ν_+, ν_- are the positive and negative decomposition of ν such that $\nu = \nu_+ - \nu_-$;

(iii) if we assume additionally that g is C^β locally in \mathbb{R} with $\beta > 0$, then u_ν is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= 0 & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \setminus \text{supp}(\nu). \end{aligned} \quad (1.8)$$

We remark that

- (i) the second equality in (1.8) is understood in the sense that $u = 0$ in $\Omega^c \setminus \text{supp}(\nu)$ and u is continuous at every point in $\partial\Omega \setminus \text{supp}(\nu)$;
- (ii) the uniqueness requires the nondecreasing assumption on nonlinearity g , while the existence also holds without the nondecreasing assumption on g ;
- (iii) (1.6) is called as integral subcritical condition with critical value $\frac{N+\alpha}{N-\alpha}$, similar integral subcritical conditions see the references [5, 12, 13, 29].

Applied Theorem 1.1 when $\nu = \delta_{x_0}$ with $x_0 \in \partial\Omega$, problem (1.5) admits a unique non-negative weak solution when g satisfies the hypotheses in Theorem 1.1. Our second goal is to study the further properties of the weak solution.

Theorem 1.2 Assume that $\epsilon = 1$, $k > 0$, $\nu = \delta_{x_0}$ with $x_0 \in \partial\Omega$, g is a nondecreasing function in C^β locally in \mathbb{R} with $\beta > 0$ satisfying $g(0) \geq 0$ and (1.6). Let u_k be the weak solution of (1.5), then

(i)

$$\lim_{t \rightarrow 0^+} \frac{u_k(x_0 + t\vec{n}_{x_0})}{\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_{x_0}}{\partial \vec{n}^\alpha}\right](x_0 + t\vec{n}_{x_0})} = k. \quad (1.9)$$

(ii) if additionally $g(s) = s^p$ with $p \in (1 + \frac{2\alpha}{N}, \frac{N+\alpha}{N-\alpha})$, then the limit of $\{u_k\}$ as $k \rightarrow \infty$ exists in $\mathbb{R}^N \setminus \{x_0\}$, denoting u_∞ . Moreover, u_∞ is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \setminus \{x_0\} \end{aligned} \quad (1.10)$$

and satisfies

$$c_1 \leq u_\infty(x_0 + t\vec{n}_{x_0})t^{\frac{2\alpha}{p-1}} \leq c_2, \quad \forall t \in (0, \sigma_0), \quad (1.11)$$

where $c_2 > c_1 > 0$ and $\sigma_0 > 0$ small enough.

(iii) if we assume more that $g(s) = s^p$ with $p \in (0, 1 + \frac{2\alpha}{N}]$, then

$$\lim_{k \rightarrow \infty} u_k(x) = +\infty, \quad \forall x \in \Omega.$$

We notice that the limit of $\{u_k\}$ as $k \rightarrow \infty$ blows up every where in Ω when $g(u) = u^p$ with $1 < p \leq 1 + \frac{2\alpha}{N}$. This phenomena is different from the Laplacian case, which is caused by the nonlocal characteristic of the fractional Laplacian.

Theorem 1.1 and Theorem 1.2 show the existence and properties of weak solutions to (1.5) in the subcritical case. One natural question is what happens in the critical case, i.e., $g(s) = s^p$ with $p \geq \frac{N+\alpha}{N-\alpha}$. The results are given by:

Theorem 1.3 *Assume that $\epsilon = 1$, $k > 0$, $\Omega = B_1(e_N)$ with $e_N = (0, \dots, 0, 1)$, $\nu = \delta_0$ and $g(s) = s^p$ with $p = \frac{N+\alpha}{N-\alpha}$. Then problem (1.5) doesn't admit any weak solution.*

In general, the nonexistence of weak solution is obtained by capacity analysis for second order differential elliptic equations involving measures, see [29] and references therein. However, it is a very tough job to attain the nonexistence in the capacity framework by the nonlocal characteristic and the weak sense of $\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}$, which is weaker than Radon measure. In the proof of Theorem 1.3, we make use of the self-similar property in the half space.

The last goal of this paper is to consider the fractional elliptic problem (1.5) with source nonlinearity, that is, $\epsilon = -1$. In the last decades, semilinear elliptic problems with source nonlinearity and measure data

$$\begin{aligned} -\Delta u &= g(u) + k\nu & \text{in } \Omega, \\ u &= \mu & \text{on } \partial\Omega, \end{aligned} \quad (1.12)$$

have attracted numerous interests. There are three basic methods to obtain weak solutions. The first one is to iterate

$$u_{n+1} = \mathbb{G}_1[g(u_n)] + k\mathbb{G}_1[\nu], \quad \forall n \in \mathbb{N}$$

and look for a function v satisfying

$$v \geq \mathbb{G}_1[g(v)] + k\mathbb{G}_1[\nu].$$

When g is a pure power source, the existence results could be found in the references [3, 5, 6, 20, 29]. The second method is to apply duality argument to derive weak solution when the mapping $r \mapsto g(r)$ is nondecreasing, convex and continuous, see Baras-Pierre [4]. These two methods are very difficult to deal with for a general source nonlinearity. Recently, Chen-Felmer-Véron in [11] introduced a new method to solve problem (1.12) when g is a general nonlinearity, where the authors employed Schauder's fixed point theorem to obtain the uniform bound and then to approach the weak solution.

Here we develop the latter method to attain weak solution of (1.5) with $\epsilon = -1$ and the main results state as follows.

Theorem 1.4 *Let $\epsilon = -1$, $k > 0$ and $\nu \in \mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$ nonnegative with $\|\nu\|_{\mathfrak{M}^b(\bar{\Omega})} = 1$.*

(i) *Suppose that*

$$g(s) \leq c_3 s^{p_0} + \epsilon, \quad \forall s \geq 0, \quad (1.13)$$

for some $p_0 \in (0, 1]$, $c_3 > 0$ and $\epsilon > 0$. Assume more that c_3 is small enough when $p_0 = 1$.

Then problem (1.5) admits a nonnegative weak solution u_ν satisfying

$$u_\nu(x) \geq \mathbb{G}_\alpha \left[\frac{\partial^\alpha \nu}{\partial \bar{n}^\alpha} \right](x), \quad \forall x \in \Omega. \quad (1.14)$$

(ii) *Suppose that*

$$g(s) \leq c_4 s^{p_*} + \epsilon, \quad \forall s \in [0, 1] \quad (1.15)$$

and

$$g_\infty := \int_1^\infty g(s) s^{-1 - \frac{N+\alpha}{N-\alpha}} ds < +\infty, \quad (1.16)$$

where $c_4, \epsilon > 0$ and $p_* > 1$.

Then there exist $k_0, \epsilon_0 > 0$ depending on c_4, p_* and g_∞ such that for $k \in [0, k_0)$ and $\epsilon \in (0, \epsilon_0)$, problem (1.5) admits a nonnegative weak solution u_ν satisfying (1.14).

We remark that (i) it does not require any restrictions on parameters c_3, ϵ, k when $p_0 \in (0, 1)$ or on parameters ϵ, σ when $p_0 = 1$; (ii) the integral subcritical condition (1.16) has the same critical value with (1.6).

The rest of the paper is organized as follows. In Section 2 we study the properties of $\frac{\partial^\alpha \nu}{\partial \bar{n}^\alpha}$. Section 3 is devoted to prove Theorem 1.1. In Section 4 we analyse the properties of the weak solution for problem (1.5) when ν is Dirac mass. The nonexistence of weak solution in the critical case is addressed in Section 5. Finally we give the proof of Theorem 1.4 in Section 6.

2 General measure concentrated on boundary

In this section, we first build the one-to-one connection between the Radon measure space $\mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$ and the bounded Radon measure space $\mathfrak{M}^b(\partial\Omega)$.

On the one hand, for any $\mu \in \mathfrak{M}^b(\partial\Omega)$, we denote by $\tilde{\mu}$ the measure generated by μ extending inside Ω by zero, that is,

$$\tilde{\mu}(E) := \mu(E \cap \partial\Omega), \quad \forall E \subset \bar{\Omega} \text{ Borel set,}$$

then $\tilde{\mu} \in \mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$.

On the other hand, let $\tilde{\mu} \in \mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$, we see that

$$\tilde{\mu}(E) = \tilde{\mu}(E \cap \partial\Omega), \quad \forall E \subset \bar{\Omega} \text{ Borel set.}$$

Denote by μ a Radon measure such that $\mu(F) := \tilde{\mu}(F)$, $F \subset \partial\Omega$ Borel set. Then $\tilde{\mu}(E) = \mu(E \cap \partial\Omega)$ for any Borel set $E \subset \bar{\Omega}$ and

$$\|\tilde{\mu}\|_{\mathfrak{M}^b(\bar{\Omega})} = \|\mu\|_{\mathfrak{M}^b(\partial\Omega)}.$$

Now we make an approximation of $\frac{\partial^\alpha \nu}{\partial \bar{n}^\alpha}$ by a sequence Radon measure concentrated on one type of manifolds inside of Ω . Indeed, we observe that there exists $\sigma_0 > 0$ small such that

$$\Omega_t := \{x \in \Omega, \rho_{\partial\Omega}(x) > t\}$$

is a C^2 domain in \mathbb{R}^N for $t \in [0, \sigma_0]$ and for any $x \in \partial\Omega_t$, there exists a unique $x_\partial \in \partial\Omega$ such that $|x - x_\partial| = \rho_{\partial\Omega}(x)$. Conversely, for any $x \in \partial\Omega$, there exists a unique point $x_t \in \partial\Omega_t$ such that $|x - x_t| = \rho_{\partial\Omega_t}(x)$, where $t \in (0, \sigma_0)$ and $\rho_{\partial\Omega_t}(x) = \text{dist}(x, \partial\Omega_t)$. Then for any Borel set $E \subset \partial\Omega$, there exists unique $E_t \subset \partial\Omega_t$ such that $E_t = \{x_t : x \in E\}$.

In what follows, we always assume that $t \in [0, \sigma_0]$.

Denote by ν_t a Radon measure generated by ν as

$$\nu_t(E_t) = \nu(E),$$

and then ν_t is a bounded Radon measure with support in $\partial\Omega_t$ and

$$\nu_t(E) = \nu_t(E \cap \partial\Omega_t), \quad \forall E \subset \bar{\Omega} \text{ Borel set.}$$

In the distribution sense, we have that

$$\langle \nu_t, f \rangle = \int_{\partial\Omega_t} f(x) d\nu_t(x) = \int_{\partial\Omega} f(x + t\vec{n}_x) d\nu(x), \quad \forall f \in C_0(\Omega). \quad (2.1)$$

Then we observe that

$$\{x_t : x \in \text{supp}(\nu)\} = \text{supp}(\nu_t) \quad \text{and} \quad \|\nu_t\|_{\mathfrak{M}^b(\bar{\Omega})} = \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})}. \quad (2.2)$$

Now we are able to show an approximation of $\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}$.

Proposition 2.1 *The sequence of Radon measures $\{t^{-\alpha} \nu_t\}_t$ converges to $\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}$ as $t \rightarrow 0^+$ in the following distribution sense:*

$$\lim_{t \rightarrow 0^+} \int_{\partial\Omega_t} \xi(x) t^{-\alpha} d\nu_t(x) = \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\nu(x), \quad \forall \xi \in \mathbb{X}_\alpha.$$

Proof. It follows from [27, Proposition 1.1], that $\xi \in C^\alpha(\mathbb{R}^N)$ if $\xi \in \mathbb{X}_\alpha$. This together with the fact that $\text{supp}(\xi) \subset \bar{\Omega}$, $\frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha}$ is well-defined for any $x \in \partial\Omega$ and for $x_t \in \partial\Omega_t$, implies that there exists a unique $x \in \partial\Omega$ such that

$$x_t = x + t\vec{n}_x \quad \text{and} \quad |x - x_t| = \rho_{\partial\Omega}(x_t),$$

then

$$\xi(x + t\vec{n}_x) t^{-\alpha} = \frac{\xi(x + t\vec{n}_x) - \xi(x)}{t^\alpha},$$

which implies that

$$\xi(\cdot + t\vec{n}) t^{-\alpha} \rightarrow \frac{\partial^\alpha \xi(\cdot)}{\partial \vec{n}^\alpha} \quad \text{as } t \rightarrow 0^+ \quad \text{in } C(\bar{\Omega}).$$

Along with (2.1), we have that

$$\begin{aligned} & \left| \int_{\partial\Omega_t} \xi(x) t^{-\alpha} d\nu_t(x) - \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\nu(x) \right| \\ &= \left| \int_{\partial\Omega} \xi(x + t\vec{n}_x) t^{-\alpha} d\nu(x) - \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\nu(x) \right| \\ &\leq \int_{\partial\Omega} \left| \xi(x + t\vec{n}_x) t^{-\alpha} - \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} \right| d|\nu(x)| \\ &\rightarrow 0 \quad \text{as } t \rightarrow 0^+, \end{aligned}$$

which ends the proof. \square

We note that Proposition 2.1 shows that $\frac{\partial^\alpha \nu}{\partial \bar{n}^\alpha}$ is approximated by a sequence Radon measure with support in Ω in the distribution sense and this provides a new method to derive weak solution of (1.5) by considering the limit of the weak solutions to

$$\begin{aligned} (-\Delta)^\alpha u + \epsilon g(u) &= kt^{-\alpha} \nu_t & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c. \end{aligned}$$

To end this section, we give an upper bound for $\mathbb{G}_\alpha[\frac{\partial^\alpha |\nu|}{\partial \bar{n}^\alpha}]$.

Lemma 2.1 *Let $\nu \in \mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$, then there exists $c_5 > 0$ such that*

$$\mathbb{G}_\alpha[\frac{\partial^\alpha |\nu|}{\partial \bar{n}^\alpha}](x) \leq \int_{\partial\Omega} \frac{c_5}{|x-y|^{N-\alpha}} d|\nu|(y), \quad x \in \Omega.$$

Proof. From [5, Theorem 1.1], there exists $c_5 > 0$ independent of t such that for any $(x, y) \in \Omega \times \partial\Omega_t$, $x \neq y$,

$$G_\alpha(x, y) \leq c_5 \frac{\rho_{\partial\Omega}^\alpha(y)}{|x-y|^{N-\alpha}} = \frac{c_5 t^\alpha}{|x-y|^{N-\alpha}}. \quad (2.3)$$

Then for $x \in \Omega$,

$$\begin{aligned} \mathbb{G}_\alpha[\frac{\partial^\alpha |\nu|}{\partial \bar{n}^\alpha}](x) &= \lim_{t \rightarrow 0^+} \int_{\partial\Omega_t} G_\alpha(x, y) t^{-\alpha} d|\nu_t|(y) \\ &\leq \lim_{t \rightarrow 0^+} \int_{\partial\Omega_t} \frac{c_5}{|x-y|^{N-\alpha}} d|\nu_t|(y) \\ &= \int_{\partial\Omega} \frac{c_5}{|x-y|^{N-\alpha}} d|\nu|(y). \end{aligned}$$

We complete the proof. \square

3 Absorption Nonlinearity

In this section, our goal is to prove the existence and uniqueness of weak solution for fractional elliptic problem (1.5) with $\epsilon = 1$. To this end, we first consider the properties of weak solution of

$$\begin{aligned} (-\Delta)^\alpha u + g_n(u) &= kt^{-\alpha} \nu_t & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \quad (3.1)$$

where $t \in (0, \sigma_0)$, ν_t is given in (2.1) and $\{g_n\}$ are a sequence of C^1 nondecreasing functions defined on \mathbb{R} such that $g_n(0) = g(0) \geq 0$,

$$|g_n| \leq g, \quad \sup_{s \in \mathbb{R}} |g_n(s)| = n \quad \text{and} \quad \lim_{n \rightarrow \infty} \|g_n - g\|_{L_{loc}^\infty(\mathbb{R})} = 0. \quad (3.2)$$

The existence and uniqueness of weak solution to (3.1) is stated as follows.

Proposition 3.1 *Assume that $k > 0$, $\alpha \in (0, 1)$, g_n is a C^1 nondecreasing function satisfying $g_n(0) \geq 0$ and (3.2). Then for $t \in (0, \sigma_0)$, problem (3.1) admits a unique weak solution $u_{n, k\nu_t}$ such that*

$$|u_{n, k\nu_t}| \leq k \mathbb{G}_\alpha[t^{-\alpha} |\nu_t|] \quad \text{a.e. in } \Omega$$

and

$$\|g_n(u_{n,k\nu_t})\|_{L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)} \leq ck \|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t]\|_{L^1(\Omega)}, \quad (3.3)$$

where $c > 0$ independent of t , k and n .

Furthermore, for any fixed $n \in \mathbb{N}$, $t \in (0, \sigma_0)$ and $k > 0$, the mapping $\nu \mapsto u_{n,k\nu_t}$ is increasing.

Proof. For any $t > 0$, we observe that $kt^{-\alpha}\nu_t$ is a bounded Radon measure in Ω and g_n is bounded, then it follows from [12, Theorem 1.1] that problem (3.1) admits a unique weak solution $u_{n,k\nu_t}$. Moreover, $kt^{-\alpha}\nu_t$ is increasing with respect to ν_t and ν_t is increasing with respect to ν by the definition of ν_t , then applying [12, Theorem 1.1], we have that for any fixed $t \in (0, \sigma_0)$ and $k > 0$, the mapping $\nu \mapsto u_{n,k\nu_t}$ is increasing. \square

To simplify the notation, we always write $u_{n,k\nu_t}$ by $u_{n,t}$ in this section. In order to consider the limit of $\{u_{n,t}\}$ as $t \rightarrow 0^+$, we introduce some auxiliary lemmas which are the key steps to obtain $\{g_n(u_{n,t})\}$ uniformly integrable with respect to t . For $\lambda > 0$, let us set

$$S_\lambda = \{x \in \Omega : \mathbb{G}_\alpha[t^{-\alpha}|\nu_t|](x) > \lambda\} \quad \text{and} \quad m(\lambda) = \int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx. \quad (3.4)$$

Lemma 3.1 For $\nu \in \mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$ and any $t \in (0, \sigma_0)$, there exists $c_6 > 0$ independent of t such that

$$m(\lambda) \leq c_6 \lambda^{-\frac{N}{N-\alpha}}. \quad (3.5)$$

Proof. For $\Lambda > 0$ and $y \in \partial\Omega_t$ with $t \in (0, \sigma_0/2)$, we denote

$$A_\Lambda(y) = \{x \in \Omega \setminus \{y\} : G_\alpha(x, y) > \Lambda\} \quad \text{and} \quad m_\Lambda(y) = \int_{A_\Lambda(y)} \rho_{\partial\Omega}^\alpha(x) dx.$$

For any $(x, y) \in \Omega \times \partial\Omega_t$, $x \neq y$, it infers by (2.3) that

$$A_\Lambda(y) \subset \left\{ x \in \Omega \setminus \{y\} : \frac{c_5 t^\alpha}{|x-y|^{N-\alpha}} > \Lambda \right\} \subset B_r(y),$$

where $r = (\frac{c_5 t^\alpha}{\Lambda})^{\frac{1}{N-\alpha}}$. Thus, $\rho_{\partial\Omega}(x) \leq R_0$ for some $R_0 > 0$ such that $\Omega \subset B_{R_0}(0)$ and

$$m_\Lambda(y) \leq R_0^\alpha \int_{B_r(y)} dx \leq c_7 t^{\frac{N\alpha}{N-\alpha}} \Lambda^{-\frac{N}{N-\alpha}}, \quad (3.6)$$

where $c_7 > 0$ independent of t .

For $y \in \partial\Omega_t$, we have that

$$\int_{S_\lambda} G_\alpha(x, y) \rho_{\partial\Omega}^\alpha(x) dx \leq \int_{A_\Lambda(y)} G_\alpha(x, y) \rho_{\partial\Omega}^\alpha(x) dx + \Lambda \int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx.$$

By integration by parts, we obtain

$$\int_{A_\Lambda(y)} G_\alpha(x, y) \rho_{\partial\Omega}^\alpha(x) dx = \Lambda m_\Lambda(y) + \int_\Lambda^\infty m_s(y) ds \leq c_8 t^{\frac{N\alpha}{N-\alpha}} \Lambda^{1-\frac{N}{N-\alpha}},$$

where $c_8 > 0$ independent of t . Thus,

$$\int_{S_\lambda} G_\alpha(x, y) \rho_{\partial\Omega}^\alpha(x) dx \leq c_8 t^{\frac{N\alpha}{N-\alpha}} \Lambda^{1-\frac{N}{N-\alpha}} + \Lambda \int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx.$$

Choose $\Lambda = t^\alpha (\int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx)^{-\frac{N-\alpha}{N}}$ and then

$$\int_{S_\lambda} G_\alpha(x, y) \rho_{\partial\Omega}^\alpha(x) dx \leq c_9 t^\alpha (\int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx)^{\frac{\alpha}{N}},$$

where $c_9 = c_8 + 1$. Therefore,

$$\begin{aligned} \int_{S_\lambda} \mathbb{G}_\alpha[t^{-\alpha}|\nu_t|](x) \rho_{\partial\Omega}^\alpha(x) dx &= \int_{\Omega} \int_{S_\lambda} G_\alpha(x, y) \rho_{\partial\Omega}^\alpha(x) dx t^{-\alpha} d|\nu_t(y)| \\ &\leq c_9 \int_{\Omega} d|\nu_t(y)| (\int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx)^{\frac{\alpha}{N}} \\ &\leq c_9 \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})} (\int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx)^{\frac{\alpha}{N}}. \end{aligned}$$

As a consequence,

$$\lambda m(\lambda) \leq c_9 \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})} m(\lambda)^{\frac{\alpha}{N}},$$

which implies (3.5). This ends the proof. \square

From Lemma 3.1, it implies that

$$\|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|\|_{M^{\frac{N}{N-\alpha}}(\Omega, \rho_{\partial\Omega}^\alpha dx)} \leq c_9 \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})}, \quad (3.7)$$

where $M^{\frac{N}{N-\alpha}}(\Omega, \rho_{\partial\Omega}^\alpha dx)$ is Marcinkiewicz space with exponent $\frac{N}{N-\alpha}$. The definition and properties of Marcinkiewicz space see the references [2, 9, 12, 29].

In next lemma, the uniformly regularity plays an important role in our approximation of weak solution.

Lemma 3.2 *Assume that u_t is a weak solution of (3.1) replacing g_n by g , a continuous nondecreasing function satisfying $g(0) \geq 0$. Then for any compact subsets $\mathcal{K} \subset \Omega$, there exist $t_0 > 0$, $\beta > 0$ small and $c_{10} > 0$ independent of t such that for $t \in (0, t_0]$,*

$$\|u_t\|_{C^\beta(\mathcal{K})} \leq c_{10} \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})}. \quad (3.8)$$

Moreover, if g is C^β locally in \mathbb{R} , then there exists $c_{11} > 0$ independent of t such that

$$\|u_t\|_{C^{2\alpha+\beta}(\mathcal{K})} \leq c_{11} \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})}. \quad (3.9)$$

Proof. We observe from Proposition 3.1 that

$$|u_t| \leq \mathbb{G}_\alpha[t^{-\alpha}|\nu_t|] \quad \text{a.e. in } \Omega. \quad (3.10)$$

For compact set \mathcal{K} in Ω , there exists $t_0 > 0$ such that

$$\mathcal{K}_{5t_0} \subset \Omega,$$

where $\mathcal{K}_r := \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{K}) < r\}$ with $r > 0$. Then $\mathcal{K}_{4t_0} \cap \partial\Omega_t = \emptyset$ for any $t \in (0, t_0]$ and

$$\|g(u_t)\|_{L^\infty(\mathcal{K}_{3t_0})} \leq \|g(\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|])\|_{L^\infty(\mathcal{K}_{3t_0})}.$$

Since $t^{-\alpha}\nu_t$ is a bounded Radon measure in Ω , there exists a sequence $\{f_n\} \subset C_0^2(\Omega)$ such that f_n converges to $t^{-\alpha}\nu_t$ in the distribution sense and for some $N_{t_0} > 0$ such that for $n \geq N_{t_0}$, $\text{supp}(f_n) \cap \mathcal{K}_{3t_0} = \emptyset$.

We may assume that g is C^β locally in \mathbb{R} . (In fact, we can choose a sequence of nondecreasing functions $\{g_n\} \subset C^\beta(\mathbb{R})$ such that $g_n(0) \geq 0$, $|g_n(s)| \leq |g(s)|$ for $s \in \mathbb{R}$ and $g_n \rightarrow g$ locally in \mathbb{R} as $n \rightarrow \infty$.) Let w_n be the classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + g_n(u) &= f_n & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c. \end{aligned} \quad (3.11)$$

By the uniqueness of weak solution to (3.1), we obtain that, up to some subsequence,

$$u_t = \lim_{n \rightarrow \infty} w_n \quad \text{a.e. in } \Omega. \quad (3.12)$$

We observe that $0 \leq w_n = \mathbb{G}_\alpha[f_n] - \mathbb{G}_\alpha[g(w_n)] \leq \mathbb{G}_\alpha[f_n]$ and $\mathbb{G}_\alpha[f_n]$ converges to $\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]$ uniformly in any compact set of $\Omega \setminus \partial\Omega_t$ and in $L^1(\Omega)$, then there exists $c_{11} > 0$ independent of n and t such that

$$\|w_n\|_{L^\infty(\mathcal{K}_{3t_0})} \leq c_{11} \|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^\infty(\mathcal{K}_{3t_0})}, \quad \|w_n\|_{L^1(\Omega)} \leq c_{11} \|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^1(\Omega)}.$$

By [14, Lemma 3.1], for $\beta \in (0, 2\alpha)$, there exists $c_{12} > 0$ independent of n and t , such that

$$\begin{aligned} \|w_n\|_{C^\beta(\mathcal{K}_{2t_0})} &\leq c_8 [\|w_n\|_{L^1(\Omega)} + \|g(w_n)\|_{L^\infty(\mathcal{K}_{3t_0})} + \|w_n\|_{L^\infty(\mathcal{K}_{3t_0})}] \\ &\leq c_{12} [\|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^1(\Omega)} + \|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^\infty(\mathcal{K}_{3t_0})} + \|g(\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|])\|_{L^\infty(\mathcal{K}_{3t_0})}]. \end{aligned}$$

It follows by [27, Corollary 2.4] that there exist $c_{13}, c_{14} > 0$ such that

$$\begin{aligned} \|w_n\|_{C^{2\alpha+\beta}(\mathcal{K})} &\leq c_{13} [\|w_n\|_{L^1(\Omega)} + \|g(w_n)\|_{C^\beta(\mathcal{K}_{2t_0})} + \|w_n\|_{C^\beta(\mathcal{K}_{2t_0})}] \\ &\leq c_{14} [\|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^1(\Omega)} + \|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^\infty(\mathcal{K}_{3t_0})} \\ &\quad + \|g\|_{C^\beta([0, \|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^\infty(\mathcal{K}_{3t_0})})} \|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{C^\beta(\mathcal{K}_{3t_0})}]. \end{aligned} \quad (3.13)$$

Therefore, together with (3.12) and the Arzela-Ascoli Theorem, it follows that $u_t \in C^{2\alpha+\epsilon}(\mathcal{K})$ for $\epsilon \in (0, \beta)$. Then $w_n \rightarrow u_t$ and $f_n \rightarrow 0$ uniformly in any compact subset of $\Omega \setminus \partial\Omega_t$ as $n \rightarrow \infty$. It infers by [14, Lemma 3.1] that

$$\begin{aligned} \|u_t\|_{C^\beta(\mathcal{K})} &\leq c_8 [\|u_t\|_{L^1(\Omega)} + \|g(u_t)\|_{L^\infty(\mathcal{K}_{3t_0})} + \|u_t\|_{L^\infty(\mathcal{K}_{3t_0})}] \\ &\leq c_{12} [\|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^1(\Omega)} + \|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^\infty(\mathcal{K}_{3t_0})} + \|g(\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|])\|_{L^\infty(\mathcal{K}_{3t_0})}]. \end{aligned}$$

We next claim that $\|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^1(\Omega)}$, $\|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^\infty(\mathcal{K}_{3t_0})}$ are uniformly bounded. In fact, for $x \in \mathcal{K}$ and $y \in \partial\Omega_t$ with $t \in (0, t_0)$, we have that $|x - y| \geq 3t_0$. By (2.3), it implies that

$$\begin{aligned} \mathbb{G}_\alpha[t^{-\alpha}|\nu_t|](x) &\leq \int_{\partial\Omega_t} \frac{c_5}{|x-y|^{N-\alpha}} d|\nu_t(y)| \\ &\leq c_5 t_0^{\alpha-N} \|\nu_t\|_{\mathfrak{M}^b(\bar{\Omega})} = c_5 t_0^{\alpha-N} \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})} \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} \|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^1(\Omega)} &\leq \int_\Omega \int_{\partial\Omega_t} \frac{c_5}{|x-y|^{N-\alpha}} d|\nu_t(y)| dx \\ &= \int_{\partial\Omega_t} \int_\Omega \frac{c_5}{|x-y|^{N-\alpha}} dx d|\nu_t(y)| \leq c_{15} \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})}, \end{aligned} \quad (3.15)$$

which implies that

$$\|u_t\|_{C^\beta(\mathcal{K})} \leq c_{15} \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})},$$

where $c_{15} > 0$ independent of t .

Moreover, if g is C^β locally in \mathbb{R} , similar to (3.13) it implies by (3.14) and (3.15) that

$$\|u_t\|_{C^{2\alpha+\beta}(\mathcal{K})} \leq c_{16}\|\nu\|_{\mathfrak{M}^b(\bar{\Omega})},$$

where $c_{16} > 0$ independent of t . We conclude by Theorem 2.2 in [10] that u_t is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= 0 & \text{in } \Omega \setminus \partial\Omega_t, \\ u &= 0 & \text{in } \Omega^c. \end{aligned} \quad (3.16)$$

This ends the proof. \square

Proposition 3.2 *Assume that $k > 0$ and $\{g_n\}$ are a sequence of C^1 nondecreasing functions defined on \mathbb{R} such that $g_n(0) = g(0)$ and (3.2). Then problem*

$$\begin{aligned} (-\Delta)^\alpha u + g_n(u) &= k \frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha} & \text{in } \bar{\Omega}, \\ u &= 0 & \text{in } \bar{\Omega}^c \end{aligned} \quad (3.17)$$

admits a unique weak solution u_n satisfying

$$-k\mathbb{G}_\alpha\left[\frac{\partial^\alpha \nu_-}{\partial \vec{n}^\alpha}\right](x) \leq u_n(x) \leq k\mathbb{G}_\alpha\left[\frac{\partial^\alpha \nu_+}{\partial \vec{n}^\alpha}\right](x), \quad x \in \Omega, \quad (3.18)$$

where ν_+, ν_- are the positive and negative decomposition of ν such that $\nu = \nu_+ - \nu_-$. Furthermore,

$$\|g_n(u_n)\|_{L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)} \leq k\|\mathbb{G}_\alpha\left[\frac{\partial^\alpha |\nu|}{\partial \vec{n}^\alpha}\right]\|_{L^1(\Omega)} \quad (3.19)$$

and u_n is a classical solution of (1.8) replacing g by g_n .

Proof. *To prove the existence of weak solution.* Since ν_t is a bounded Radon measure with $\text{supp}(\nu_t) \subset \partial\Omega_t$ for $t \in (0, \sigma_0)$, then by Proposition 3.1, we have that problem (3.1) admits a unique weak solution $u_{n,t}$ such that

$$|u_{n,t}| \leq \mathbb{G}_\alpha[t^{-\alpha}|\nu_t|] \quad \text{a.e. in } \Omega, \quad \int_\Omega |g_n(u_{n,t})|\rho_{\partial\Omega}^\alpha dx \leq k\|\mathbb{G}_\alpha[t^{-\alpha}|\nu_t|]\|_{L^1(\Omega)} \quad (3.20)$$

and

$$\int_\Omega [u_{n,t}(-\Delta)^\alpha \xi + g_n(u_{n,t})\xi] dx = \int_{\partial\Omega_t} t^{-\alpha} \xi(x) d\nu_t(x), \quad \forall \xi \in \mathbb{X}_\alpha. \quad (3.21)$$

For any compact set $\mathcal{K} \subset \Omega$, there exists $t_0 \in (0, \sigma_0)$ such that

$$\mathcal{K} \subset \Omega_t \quad \text{and} \quad \text{dist}(\mathcal{K}, \partial\Omega_t) \geq t_0, \quad \forall t \in (0, t_0].$$

By Lemma 3.2, we observe that for some $\beta \in (0, \alpha)$

$$\|u_{n,t}\|_{C^\beta(\mathcal{K})} \leq c_5 t_0^{-N+2\alpha} \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})}.$$

Therefore, up to some subsequence, there exists u_n such that

$$\lim_{t \rightarrow 0^+} u_{n,t} = u_n \quad \text{a.e. in } \Omega.$$

Then $g_n(u_{n,t})$ converges to $g_n(u_n)$ almost every in Ω as $t \rightarrow 0^+$. By (3.20) and (3.7), we have that $\{u_{n,t}\}_t$ is relatively compact in $L^1(\Omega)$, up to subsequence,

$$u_{n,t} \rightarrow u_n \quad \text{in } L^1(\Omega) \quad \text{as } t \rightarrow 0^+$$

and then

$$g_n(u_{n,t}) \rightarrow g_n(u_n) \quad \text{in } L^1(\Omega, \rho_{\partial\Omega}^\alpha dx) \quad \text{as } t \rightarrow 0^+.$$

By Proposition 2.1,

$$\int_{\partial\Omega_t} t^{-\alpha} \xi(x) d\nu_t(x) \rightarrow \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\nu(x) \quad \text{as } t \rightarrow 0^+,$$

Passing to the limit as $t \rightarrow 0^+$ in the identity (3.21), it implies that

$$\int_{\Omega} [u_n(-\Delta)^\alpha \xi + g_n(u_n)\xi] dx = k \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\nu(x), \quad \forall \xi \in \mathbb{X}_\alpha.$$

This implies that u_n is a weak solution of (3.17). We see that (3.19) follows by (3.3) and Lemma 3.1. Moreover, by the facts that $u_n = \lim_{t \rightarrow 0^+} u_{n,t}$ and

$$-k\mathbb{G}_\alpha[t^{-\alpha}\nu_-] \leq u_{n,t} \leq k\mathbb{G}_\alpha[t^{-\alpha}\nu_+] \quad \text{in } \Omega,$$

we have that (3.18) holds.

To prove that $u_n = 0$ in $\Omega^c \setminus \text{supp}(\nu)$. Let $x_0 \in \partial\Omega \setminus \text{supp}(\nu)$ and $x_s = x_0 + s\vec{n}_{x_0}$ with $s \in (0, \sigma_0)$. We only have to prove that $\lim_{s \rightarrow 0^+} u(x_s) = 0$. From [5, Theorem 1.1], for any $(x, y) \in \Omega \times \partial\Omega_t$, $x \neq y$,

$$G_\alpha(x, y) \leq c_5 \frac{\rho_{\partial\Omega}^\alpha(y)\rho_{\partial\Omega}^\alpha(x)}{|x-y|^N} = c_5 \frac{\rho_{\partial\Omega}^\alpha(x)t^\alpha}{|x-y|^N}. \quad (3.22)$$

For some $s_0 > 0$ and any $s \in (0, s_0)$, we observe that $\text{dist}(x_s, \text{supp}(\nu)) \geq \frac{1}{2}\text{dist}(x_0, \text{supp}(\nu))$ and

$$\begin{aligned} \mathbb{G}_\alpha[t^{-\alpha}|\nu_t|](x_s) &\leq c_5 \int_{\partial\Omega} \frac{\rho_{\partial\Omega}^\alpha(x_s)}{|x_s-y|^N} d|\nu|(y) \\ &= c_5 s^\alpha \int_{\partial\Omega \setminus \text{supp}(\nu)} \frac{1}{|x_s-y|^N} d|\nu|(y) \\ &\leq c_5 2^N s^\alpha \text{dist}(x_0, \text{supp}(\nu))^{-N} \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})} \\ &\rightarrow 0 \quad \text{as } s \rightarrow 0^+. \end{aligned}$$

Together with the facts that

$$u_n = \lim_{t \rightarrow 0^+} u_{n,t} \quad \text{and} \quad |u_{n,t}| \leq \mathbb{G}_\alpha[t^{-\alpha}|\nu_t|], \quad (3.23)$$

we derive that $u_n = 0$ in $\Omega^c \setminus \text{supp}(\nu)$.

To prove the uniqueness of weak solution. Let u_1, u_2 be two weak solutions of (3.17) and $w = u_1 - u_2$. Then $(-\Delta)^\alpha w = g_n(u_2) - g_n(u_1)$ and $g_n(u_2) - g_n(u_1) \in L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)$. By Kato's inequality, see Proposition 2.4 in [12], for $\xi \in \mathbb{X}_\alpha$, $\xi \geq 0$, we have that

$$\int_{\Omega} |w|(-\Delta)^\alpha \xi dx + \int_{\Omega} [g_n(u_1) - g_n(u_2)] \text{sign}(w) \xi dx \leq 0.$$

Combining with $\int_{\Omega} [g_n(u_1) - g_n(u_2)] \text{sign}(w) \xi dx \geq 0$, then we have

$$w = 0 \quad \text{a.e. in } \Omega.$$

Regularity of u_n . Since g_n is C^1 in \mathbb{R} , then by (3.9), we have

$$\|u_n\|_{C^{2\alpha+\beta}(\mathcal{K})} \leq c_{17}\|\nu\|_{\mathfrak{M}^b(\bar{\Omega})}, \quad (3.24)$$

for any compact set \mathcal{K} and some $\beta \in (0, \alpha)$. Then u_n is $C^{2\alpha+\beta}$ locally in Ω . Together with the fact that $u_{n,t}$ is classical solution of (3.16), we derive by Theorem 2.2 in [10] that u_n is a classical solution of (1.8). \square

For $\lambda > 0$, let us define

$$\tilde{S}_\lambda = \{x \in \Omega : \mathbb{G}_\alpha[\frac{\partial^\alpha|\nu|}{\partial\vec{n}^\alpha}](x) > \lambda\} \quad \text{and} \quad \tilde{m}(\lambda) = \int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x)dx. \quad (3.25)$$

Lemma 3.3 For $\nu \in \mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$, then there exist $\lambda_0 > 1$ and $c_{18} > 0$ such that for any $\lambda \geq \lambda_0$,

$$\tilde{m}(\lambda) \leq c_{18}\lambda^{-\frac{N+\alpha}{N-\alpha}}. \quad (3.26)$$

Proof. From Lemma 2.1, we see that

$$\mathbb{G}_\alpha[\frac{\partial^\alpha|\nu|}{\partial\vec{n}^\alpha}](x) \leq \int_{\partial\Omega} \frac{c_5}{|x-y|^{N-\alpha}} d|\nu(y)|, \quad x \in \Omega.$$

For $\Lambda > 0$ and $y \in \partial\Omega$, we denote

$$\tilde{A}_\Lambda(y) = \{x \in \Omega : \frac{c_5}{|x-y|^{N-\alpha}} > \Lambda\} \quad \text{and} \quad \tilde{m}_\Lambda(y) = \int_{\tilde{A}_\Lambda(y)} \rho_{\partial\Omega}^\alpha(x)dx.$$

For any $(x, y) \in \Omega \times \partial\Omega$, it infers by(2.3) that

$$\tilde{A}_\Lambda(y) \subset B_{r_0}(y),$$

where $r_0 = (\frac{c_5}{\Lambda})^{\frac{1}{N-\alpha}}$.

Since Ω is C^2 , there exists $\Lambda_0 > 1$ such that for $\Lambda > \Lambda_0$ such that

$$\rho_{\partial\Omega}(x) \leq |x-y|, \quad \forall x \in \tilde{A}_\Lambda(y)$$

and

$$\tilde{m}_\Lambda(y) \leq \int_{B_{r_0}(y)} |x-y|^\alpha dx \leq c_{19}\Lambda^{-\frac{N+\alpha}{N-\alpha}}. \quad (3.27)$$

For $y \in \partial\Omega$, we have that

$$\int_{\tilde{S}_\lambda} \frac{c_5}{|x-y|^{N-\alpha}} \rho_{\partial\Omega}^\alpha(x)dx \leq \int_{\tilde{A}_\Lambda(y)} \frac{c_5}{|x-y|^{N-\alpha}} \rho_{\partial\Omega}^\alpha(x)dx + \Lambda \int_{\tilde{S}_\lambda} \rho_{\partial\Omega}^\alpha(x)dx.$$

By integration by parts, we obtain

$$\begin{aligned} \int_{\tilde{A}_\Lambda(y)} \frac{c_5}{|x-y|^{N-\alpha}} \rho_{\partial\Omega}^\alpha(x)dx &= \Lambda \tilde{m}_\Lambda(y) + \int_\Lambda^\infty \tilde{m}_s(y)ds \\ &\leq c_{20}\Lambda^{1-\frac{N+\alpha}{N-\alpha}}, \end{aligned}$$

where $c_{20} > 0$. Thus,

$$\int_{\tilde{S}_\lambda} \frac{c_5}{|x-y|^{N-\alpha}} \rho_{\partial\Omega}^\alpha(x)dx \leq c_{20}\Lambda^{1-\frac{N+\alpha}{N-\alpha}} + \Lambda \int_{\tilde{S}_\lambda} \rho_{\partial\Omega}^\alpha(x)dx.$$

Since $S_{\tilde{\lambda}_1} \subset S_{\tilde{\lambda}_2}$ if $\lambda_1 \geq \lambda_2$ and

$$\lim_{\lambda \rightarrow 0^+} \int_{\tilde{S}_\lambda} \rho_{\partial\Omega}^\alpha(x) dx = 0,$$

then there exists $\lambda_0 > 0$ such that

$$\left(\int_{\tilde{S}_{\lambda_0}} \rho_{\partial\Omega}^\alpha(x) dx \right)^{-\frac{N-\alpha}{N+\alpha}} \geq \Lambda_0$$

and for $\lambda \geq \lambda_0$, we may choose $\Lambda = (\int_{\tilde{S}_\lambda} \rho_{\partial\Omega}^\alpha(x) dx)^{-\frac{N-\alpha}{N+\alpha}} \geq \Lambda_0$ and then

$$\int_{\tilde{S}_\lambda} \frac{c_5}{|x-y|^{N-\alpha}} \rho_{\partial\Omega}^\alpha(x) dx \leq c_{21} \left(\int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx \right)^{\frac{2\alpha}{N+\alpha}},$$

where $c_{21} = c_{20} + 1$. Therefore,

$$\begin{aligned} \int_{\tilde{S}_\lambda} \mathbb{G}_\alpha \left[\frac{\partial^\alpha |\nu|}{\partial \vec{n}^\alpha} \right] (x) \rho_{\partial\Omega}^\alpha(x) dx &\leq \int_{\partial\Omega} \int_{\tilde{S}_\lambda} \frac{c_5}{|x-y|^{N-\alpha}} \rho_{\partial\Omega}^\alpha(x) dx d|\nu(y)| \\ &\leq c_{21} \int_{\partial\Omega} d|\nu(y)| \left(\int_{\tilde{S}_\lambda} \rho_{\partial\Omega}^\alpha(x) dx \right)^{\frac{2\alpha}{N+\alpha}} \\ &\leq c_{21} \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})} \left(\int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx \right)^{\frac{2\alpha}{N+\alpha}}. \end{aligned}$$

As a consequence,

$$\lambda \tilde{m}(\lambda) \leq c_{21} \|\nu\|_{\mathfrak{M}^b(\bar{\Omega})} \tilde{m}(\lambda)^{\frac{2\alpha}{N+\alpha}},$$

which implies (3.26). This ends the proof. \square

To estimate the nonlinearity in $L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)$, we have to introduce an auxiliary lemma as follows.

Lemma 3.4 *Assume that $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function satisfying*

$$\int_1^\infty g(s) s^{-1-p} ds < +\infty \tag{3.28}$$

for some $p > 0$. Then there is a sequence real positive numbers $\{T_n\}$ such that

$$\lim_{n \rightarrow \infty} T_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} g(T_n) T_n^{-p} = 0.$$

Assume additionally that g is nondecreasing, then

$$\lim_{T \rightarrow \infty} g(T) T^{-p} = 0.$$

Proof. The first argument see [13, Lemma 3.1] and second see [12, Lemma 3.1]. \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. *To prove the existence of weak solution. Take $\{g_n\}$ a sequence of C^1 nondecreasing functions defined on \mathbb{R} satisfying $g_n(0) = g(0)$ and (3.2). By Proposition 3.2, problem (3.17) admits a unique weak solution u_n such that*

$$|u_n| \leq \mathbb{G}_\alpha \left[\frac{\partial^\alpha |\nu|}{\partial \vec{n}^\alpha} \right] \quad \text{a.e. in } \Omega$$

and

$$\int_{\Omega} [u_n(-\Delta)^{\alpha}\xi + g_n(u_n)\xi]dx = k \int_{\partial\Omega} \frac{\partial^{\alpha}\xi(x)}{\partial\vec{n}_x^{\alpha}}d\nu(x), \quad \forall \xi \in \mathbb{X}_{\alpha}. \quad (3.29)$$

For any compact set $\mathcal{K} \subset \Omega$, we observe from Lemma 3.2 that for some $\beta \in (0, \alpha)$,

$$\|u_n\|_{C^{\beta}(\mathcal{K})} \leq c_{22}\|\nu\|_{\mathfrak{M}^b(\bar{\Omega})}.$$

Therefore, up to some subsequence, there exists u_{ν} such that

$$\lim_{n \rightarrow \infty} u_n = u_{\nu} \quad \text{a.e. in } \Omega.$$

Then $g_n(u_n)$ converge to $g(u_{\nu})$ a.e. in Ω as $n \rightarrow \infty$. By Lemma 3.3 and (3.19), we have that

$$u_n \rightarrow u_{\nu} \text{ in } L^1(\Omega), \quad \|g_n(u_n)\|_{L^1(\Omega, \rho_{\partial\Omega}^{\alpha}dx)} \leq c_{23}\|\mathbb{G}_{\alpha}[\frac{\partial^{\alpha}|\nu|}{\partial\vec{n}^{\alpha}}]\|_{L^1(\Omega)}$$

and

$$\tilde{m}(\lambda) \leq c_{18}\lambda^{-\frac{N+\alpha}{N-\alpha}} \quad \text{for } \lambda > \lambda_0,$$

where

$$\tilde{m}(\lambda) = \int_{\tilde{S}_{\lambda}} \rho_{\partial\Omega}^{\alpha}(x)dx \quad \text{with } \tilde{S}_{\lambda} = \{x \in \Omega : \mathbb{G}_{\alpha}[\frac{\partial^{\alpha}|\nu|}{\partial\vec{n}^{\alpha}}] > \lambda\}.$$

For any Borel set $E \subset \Omega$, we have that

$$\begin{aligned} \int_E |g_n(u_n)|\rho_{\partial\Omega}^{\alpha}(x)dx &\leq \int_{E \cap \tilde{S}_{\frac{\lambda}{k}}^c} g\left(k\mathbb{G}_{\alpha}[\frac{\partial^{\alpha}|\nu|}{\partial\vec{n}^{\alpha}}]\right)\rho_{\partial\Omega}^{\alpha}(x)dx + \int_{E \cap \tilde{S}_{\frac{\lambda}{k}}} g\left(k\mathbb{G}_{\alpha}[\frac{\partial^{\alpha}|\nu|}{\partial\vec{n}^{\alpha}}]\right)\rho_{\partial\Omega}^{\alpha}(x)dx \\ &\leq \tilde{g}\left(\frac{\lambda}{k}\right) \int_E \rho_{\partial\Omega}^{\alpha}(x)dx + \int_{\tilde{S}_{\frac{\lambda}{k}}} \tilde{g}\left(k\mathbb{G}_{\alpha}[\frac{\partial^{\alpha}|\nu|}{\partial\vec{n}^{\alpha}}]\right)\rho_{\partial\Omega}^{\alpha}(x)dx \\ &\leq \tilde{g}\left(\frac{\lambda}{k}\right) \int_E \rho_{\partial\Omega}^{\alpha}(x)dx + \tilde{m}\left(\frac{\lambda}{k}\right) \tilde{g}\left(\frac{\lambda}{k}\right) + \int_{\frac{\lambda}{k}}^{\infty} \tilde{m}(s)d\tilde{g}(s), \end{aligned}$$

where $\tilde{g}(r) = g(|r|) - g(-|r|)$.

On the other hand,

$$\int_{\frac{\lambda}{k}}^{\infty} \tilde{g}(s)d\tilde{m}(s) = \lim_{T \rightarrow \infty} \int_{\frac{\lambda}{k}}^T \tilde{g}(s)d\tilde{m}(s).$$

Thus,

$$\begin{aligned} \tilde{m}\left(\frac{\lambda}{k}\right) \tilde{g}\left(\frac{\lambda}{k}\right) + \int_{\frac{\lambda}{k}}^T \tilde{m}(s)d\tilde{g}(s) &\leq c_{24}\tilde{g}\left(\frac{\lambda}{k}\right) \left(\frac{\lambda}{k}\right)^{-\frac{N+\alpha}{N-\alpha}} + c_{24} \int_{\frac{\lambda}{k}}^T s^{-\frac{N+\alpha}{N-\alpha}} d\tilde{g}(s) \\ &\leq c_{25}T^{-\frac{N+\alpha}{N-\alpha}} \tilde{g}(T) + \frac{c_{24}}{\frac{N+\alpha}{N-\alpha} + 1} \int_{\frac{\lambda}{k}}^T s^{-1-\frac{N+\alpha}{N-\alpha}} \tilde{g}(s)ds. \end{aligned}$$

By assumption (1.6) and Lemma 3.4 with $p = \frac{N+\alpha}{N-\alpha}$, $T^{-\frac{N+\alpha}{N-\alpha}} \tilde{g}(T) \rightarrow 0$ when $T \rightarrow \infty$, therefore,

$$\tilde{m}\left(\frac{\lambda}{k}\right) \tilde{g}\left(\frac{\lambda}{k}\right) + \int_{\frac{\lambda}{k}}^{\infty} \tilde{m}(s) d\tilde{g}(s) \leq \frac{c_{24}}{\frac{N+\alpha}{N-\alpha} + 1} \int_{\frac{\lambda}{k}}^{\infty} s^{-1-\frac{N+\alpha}{N-\alpha}} \tilde{g}(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when $\lambda \rightarrow \infty$. The conclusion follows: for any $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\frac{c_{24}}{\frac{N+\alpha}{N-\alpha} + 1} \int_{\frac{\lambda}{k}}^{\infty} s^{-1-\frac{N+\alpha}{N-\alpha}} \tilde{g}(s) ds \leq \frac{\epsilon}{2}.$$

For λ fixed, there exists $\delta > 0$ such that

$$\int_E \rho_{\partial\Omega}^{\alpha}(x) dx \leq \delta \implies \tilde{g}\left(\frac{\lambda}{k}\right) \int_E \rho_{\partial\Omega}^{\alpha}(x) dx \leq \frac{\epsilon}{2},$$

which implies that $\{g_n \circ u_n\}$ is uniformly integrable in $L^1(\Omega, \rho_{\partial\Omega}^{\alpha} dx)$. Then $g_n \circ u_n \rightarrow g \circ u_{\nu}$ in $L^1(\Omega, \rho_{\partial\Omega}^{\alpha} dx)$ by Vitali convergence theorem.

Passing to the limit as $n \rightarrow +\infty$ in the identity (3.29), it implies that

$$\int_{\Omega} [u_{\nu}(-\Delta)^{\alpha} \xi + g(u_{\nu}) \xi] dx = \int_{\partial\Omega} \frac{\partial^{\alpha} \xi(x)}{\partial \vec{n}_x^{\alpha}} d\nu(x), \quad \forall \xi \in \mathbb{X}_{\alpha}.$$

Then u_{ν} is a weak solution of (1.5). Moreover, it follows by the fact

$$-k \mathbb{G}_{\alpha} \left[\frac{\partial^{\alpha} \nu_{-}}{\partial \vec{n}^{\alpha}} \right] \leq u_n \leq k \mathbb{G}_{\alpha} \left[\frac{\partial^{\alpha} \nu_{+}}{\partial \vec{n}^{\alpha}} \right] \quad \text{in } \Omega.$$

which, together with $u_{\nu} = \lim_{n \rightarrow +\infty} u_n$, implies (1.7).

The arguments including $u_n = 0$ in $\Omega^c \setminus \text{supp}(\nu)$, uniqueness and regularity follow the proof of Proposition 3.2. \square

The proof of the existence of weak solution is divided into two steps: the first step is to get weak solution u_n to (1.5) with truncated nonlinearity g_n and then to prove the limit of $\{u_n\}$ as $n \rightarrow \infty$ is our desired weak solution. This is due to the estimate in Lemma 3.1 where we only could get exponent $\frac{N}{N-\alpha}$ and in the second step, we make use of Lemma 3.3, the critical exponent of the nonlinearity g could be up to $\frac{N+\alpha}{N-\alpha}$.

4 Isolated singularity on boundary

For simplicity, we assume that $x_0 = 0$ and \vec{n}_0 is the unit inward normal vector at the origin in what follows and u_k is the weak solution of (1.5).

4.1 Weak singularity

In this subsection, we prove Theorem 1.2 part (i). The regularity refers to Theorem 1.1 in the case that $\nu = \delta_0$ with $0 \in \partial\Omega$ and our main work is to prove (1.9). We start our analysis with an auxiliary lemma.

Lemma 4.1 *Under the hypotheses of Theorem 1.2 part (i), we assume more that $x_s = s\vec{n}_0 \in \Omega$ for $s > 0$ small, then there exists $c_{26} > 1$ such that*

$$\frac{1}{c_{26}} s^{-N+\alpha} \leq \mathbb{G}_{\alpha} \left[\frac{\partial^{\alpha} \delta_0}{\partial \vec{n}^{\alpha}} \right] (x_s) \leq c_{26} s^{-N+\alpha}$$

and

$$\lim_{s \rightarrow 0^+} \mathbb{G}_{\alpha} \left[g \left(\mathbb{G}_{\alpha} \left[k \frac{\partial^{\alpha} \delta_0}{\partial \vec{n}^{\alpha}} \right] (x_s) \right) \right] s^{N-\alpha} = 0.$$

Proof. It follows by Lemma 2.1 with $\nu = \delta_0$ that

$$\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right](x) \leq \frac{c_5}{|x|^{N-\alpha}}, \quad \forall x \in \Omega, \quad (4.1)$$

in particular,

$$\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right](x_s) \leq \frac{c_5}{s^{N-\alpha}}.$$

Let $y_t = t\vec{n}_0$ with $t \in (0, s/2)$, then

$$|y_t - x_s| = s - t > \frac{s}{2} = \frac{1}{2} \max\{\rho_{\partial\Omega}(y_t), \rho_{\partial\Omega}(x_s)\}$$

and apply [5, Theorem 1.2] to derive that there exists $c_{27} > 0$ such that

$$G_\alpha(x_s, y_t) \geq c_{27} \frac{\rho_{\partial\Omega}^\alpha(y_t) \rho_{\partial\Omega}^\alpha(x_s)}{|x_s - y_t|^N}. \quad (4.2)$$

Thus,

$$\mathbb{G}_\alpha[t^{-\alpha} \delta_{y_t}](x_s) \geq \frac{c_{27} s^\alpha}{|s - t|^N},$$

which implies that

$$\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right](x_s) \geq \frac{c_{27}}{s^{N-\alpha}}.$$

(ii) By (4.1) and monotonicity of g , we have that

$$\begin{aligned} \mathbb{G}_\alpha\left[g\left(\mathbb{G}_\alpha\left[k \frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right]\right)\right](x_s) s^{N-\alpha} &\leq \int_\Omega G_\alpha(x_s, y) g\left(\frac{c_5 k}{|y|^{N-\alpha}}\right) dy s^{N-\alpha} \\ &\leq \int_\Omega \frac{c_5}{|x_s - y|^{N-\alpha}} g\left(\frac{c_5 k}{|y|^{N-\alpha}}\right) dy s^{N-\alpha} \\ &= c_5 s^{N-\alpha} \left[\int_{B_{\frac{s}{2}}(x_s)} \frac{|y|^\alpha}{|x_s - y|^{N-\alpha}} g\left(\frac{c_5 k}{|y|^{N-\alpha}}\right) dy \right. \\ &\quad \left. + \int_{\Omega \setminus B_{\frac{s}{2}}(x_s)} \frac{|y|^\alpha}{|x_s - y|^{N-\alpha}} g\left(\frac{c_5 k}{|y|^{N-\alpha}}\right) dy \right] \\ &:= A_1(s) + A_2(s). \end{aligned}$$

For $y \in B_{\frac{s}{2}}(x_s)$, we have $\frac{s}{2} \leq |y| \leq \frac{3s}{2}$ and by applying Lemma 3.4, we derive that

$$\begin{aligned} A_1(s) &\leq c_5 s^{N+\alpha} g\left(\frac{2^{N-\alpha} c_5 k}{s^{N-\alpha}}\right) \int_{B_{1/2}(\vec{n}_0)} \frac{|z|^\alpha}{|\vec{n}_0 - z|^{N-\alpha}} dz \\ &= c_5 r^{-\frac{N+\alpha}{2}} g(2^{N-\alpha} c_5 r k) \int_{B_{1/2}(\vec{n}_0)} \frac{|z|^\alpha}{|\vec{n}_0 - z|^{N-\alpha}} dz \\ &\rightarrow 0 \quad \text{as } r \rightarrow +\infty, \end{aligned}$$

where $r = s^{\alpha-N}$. We next claim that $A_2(s) \rightarrow 0$ as $s \rightarrow 0^+$. In fact, for $y \in B_{\frac{s}{2}}(0)$, we see that $|x_s - y| > s/2$ and

$$s^{N-\alpha} \int_{B_{\frac{s}{2}}(0)} \frac{|y|^\alpha}{|x_s - y|^{N-\alpha}} g\left(\frac{c_5 k}{|y|^{N-\alpha}}\right) dy \leq 2^{N-\alpha} \int_{B_{\frac{s}{2}}(0)} |y|^\alpha g\left(\frac{c_5 k}{|y|^{N-\alpha}}\right) dy$$

$$\begin{aligned}
&= c_{28} \int_0^{\frac{s}{2}} r^\alpha g\left(\frac{c_5 k}{r^{N-\alpha}}\right) r^{N-1} dr \\
&= \frac{c_{28}}{N-\alpha} \int_{s^{-\frac{1}{N-\alpha}}}^\infty \tau^{-1-\frac{N+\alpha}{N-\alpha}} g(c_5 k \tau) d\tau \\
&\rightarrow 0 \quad \text{as } s \rightarrow 0^+,
\end{aligned}$$

where the converging used (1.6). For $y \in \Omega \setminus (B_{\frac{s}{2}}(0) \cup B_{\frac{s}{2}}(x_s))$, we have that $|y - x_s| > \frac{1}{4}|y|$ and

$$\begin{aligned}
&s^{N-\alpha} \int_{\Omega \setminus (B_{\frac{s}{2}}(0) \cup B_{\frac{s}{2}}(x_s))} \frac{|y|^\alpha}{|x_s - y|^{N-\alpha}} g\left(\frac{c_5 k}{|y|^{N-\alpha}}\right) dy \\
&\leq s^{N-\alpha} \int_{B_R(0) \setminus B_s(0)} |y|^{2\alpha-N} g\left(\frac{c_5 k}{|y|^{N-\alpha}}\right) dy \\
&= c_{29} s^{N-\alpha} \int_s^R \tau^{2\alpha-1} g(c_5 k \tau^{\alpha-N}) d\tau \\
&= \frac{s^{2\alpha-1} g(c_5 k s^{\alpha-N})}{(N-\alpha) s^{\alpha-N-1}} \quad (\text{L'Hospital's Rule}) \\
&= \frac{c_{29}}{N-\alpha} s^{N+\alpha} g(c_5 k s^{\alpha-N}) \\
&\rightarrow 0 \quad \text{as } s \rightarrow 0^+,
\end{aligned}$$

for some $R > 0$ such that $\Omega \subset B_R(0)$ and $c_{29} > 0$. Then

$$A_2(s) \rightarrow 0 \quad \text{as } s \rightarrow 0^+.$$

Therefore,

$$\lim_{s \rightarrow 0^+} \mathbb{G}_\alpha[g(\mathbb{G}_\alpha[k \frac{\partial^\alpha \delta_0}{\partial \bar{n}^\alpha})]](x_s) s^{N-\alpha} = 0. \quad (4.3)$$

The proof ends. \square

Proof of Theorem 1.2 (i). The existence, uniqueness and regularity follow by Theorem 1.1. We only need to prove (1.9) to complete the proof. We observe that

$$\begin{aligned}
k \mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \bar{n}^\alpha}\right](x_s) \geq u_k(x_s) &\geq k \mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \bar{n}^\alpha}\right](x_s) - \mathbb{G}_\alpha[g(u_k)](x_s) \\
&\geq k \mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \bar{n}^\alpha}\right](x_s) - \mathbb{G}_\alpha[g(k \mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \bar{n}^\alpha}\right])](x_s),
\end{aligned}$$

where $s > 0$ small. Together with Lemma 4.1, (1.9) holds. \square

4.2 Strong singularity for $p \in (1 + \frac{2\alpha}{N}, \frac{N+\alpha}{N-\alpha})$

In this subsection, we consider the limit of $\{u_k\}$ as $k \rightarrow \infty$, where u_k is the weak solution of

$$\begin{aligned}
(-\Delta)^\alpha u + u^p &= k \frac{\partial^\alpha \delta_0}{\partial \bar{n}^\alpha} \quad \text{in } \bar{\Omega}, \\
u &= 0 \quad \text{in } \bar{\Omega}^c,
\end{aligned}$$

here $0 \in \partial\Omega$ and $p \in (1 + \frac{2\alpha}{N}, \frac{N+\alpha}{N-\alpha})$. From Theorem 1.1 (iii), we know that u_k is a classical solution of

$$\begin{aligned}
(-\Delta)^\alpha u + u^p &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega^c \setminus \{0\}.
\end{aligned} \quad (4.4)$$

In order to study the limit of $\{u_k\}$ as $k \rightarrow \infty$, we have to obtain a super solution of (4.4). To this end, we consider the function

$$w_p(x) = |x|^{-\frac{2\alpha}{p-1}}, \quad x \in \mathbb{R}^N \setminus \{0\}. \quad (4.5)$$

Lemma 4.2 *Assume that $p \in (1 + \frac{2\alpha}{N}, \frac{N+\alpha}{N-\alpha})$ and w_p is defined in (4.5). Then there exists $\lambda_0 > 0$ such that $\lambda_0 w_p$ is a super solution of (4.4).*

Proof. For $p \in (1 + \frac{2\alpha}{N}, \frac{N+\alpha}{N-\alpha})$, we have that $-\frac{2\alpha}{p-1} \in (-N, -N + 2\alpha)$ and from [17], it shows that there exists $c(p) < 0$ such that

$$(-\Delta)^\alpha w_p(x) = c(p)|x|^{-\frac{2\alpha}{p-1}-2\alpha}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

thus, taking $\lambda_0 = |c(p)|^{\frac{1}{p-1}}$, we derive that

$$(-\Delta)^\alpha(\lambda_0 w_p) + (\lambda_0 w_p)^p = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Together with $\lambda_0 w_p > 0$ in Ω^c , $\lambda_0 w_p$ is a super solution of (4.4). The proof ends. \square

We observe that the super solution $\lambda_0 w_p$ constructed in Lemma 4.2 could control the asymptotic behavior of u_∞ near the origin, but for $\partial\Omega \setminus \{0\}$, $\lambda_0 w_p$ does not provide enough information for us. To control the behavior of u_∞ on $\partial\Omega \setminus \{0\}$, we have to construct new super solutions. For any given $y_0 \in \partial\Omega \setminus \{0\}$, we denote $\eta_0 : \mathbb{R}^N \rightarrow [0, 1]$ a C^2 functions such that

$$\eta_0(x) = \begin{cases} 0, & x \in B_r(y_0), \\ 1, & x \in \mathbb{R}^N \setminus B_{2r}(y_0), \end{cases} \quad (4.6)$$

where $r = \frac{|y_0|}{8}$.

Lemma 4.3 *Assume that $p \in (1 + \frac{2\alpha}{N}, \frac{N+\alpha}{N-\alpha})$ and $w_{\lambda,j} = \lambda \tilde{w}_p + j \eta_1$, where $\lambda, j > 0$, $\tilde{w}_p = w_p \eta_0$ in \mathbb{R}^N and $\eta_1 = \mathbb{G}_\alpha[1]$.*

Then there exist $\lambda_1 > 0$ and $j_1 > 0$ depending on $|y_0|$ such that w_{λ_1, j_1} is a super solution of (4.4).

Proof. For $x \in \Omega \setminus B_{4r}(y_0)$, we have that $\tilde{w}_p(x) = w_p(x)$ and

$$\begin{aligned} (-\Delta)^\alpha \tilde{w}_p(x) &= - \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{\tilde{w}_p(z) - w_p(x)}{|z-x|^{N+2\alpha}} dz \\ &= (-\Delta)^\alpha w_p(x) - \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{\tilde{w}_p(z) - w_p(z)}{|z-x|^{N+2\alpha}} dz \\ &\geq (-\Delta)^\alpha w_p(x) - \int_{B_{2r}(y_0)} \frac{w_p(z)}{|z-x|^{N+2\alpha}} dz \\ &\geq c(p)|x|^{-\frac{2\alpha}{p-1}-2\alpha} - c_{30} r^{-\frac{2\alpha}{p-1}-2\alpha}, \end{aligned}$$

where $c_{30} > 0$ and the last inequality used the facts $|z-x| \geq 2r$ and $w_p(z) \leq r^{-\frac{2\alpha}{p-1}}$. For $x \in B_{2r}(0) \setminus \{0\}$, take $\lambda = \lambda_0$ from Lemma 4.2 and $j \geq c_{30} \lambda_0 r^{-\frac{2\alpha}{p-1}-2\alpha}$, then we have

$$(-\Delta)^\alpha w_{\lambda,j}(x) + w_{\lambda,j}^p(x) \geq -c(p)\lambda_0 |x|^{-\frac{2\alpha}{p-1}-2\alpha} + w_p^p(x) \geq 0.$$

We observe that there exists $c_{31} > 0$ dependent of r such that

$$|(-\Delta)^\alpha \tilde{w}_p| \leq c_{31} \quad \text{in } \Omega \setminus B_{2r}(0),$$

then take $j \geq c_{31}\lambda_0$, we have that

$$(-\Delta)^\alpha w_{\lambda_0, j} \geq 0, \quad \forall x \in \Omega \setminus B_{2r}(0).$$

Therefore, letting $\lambda_1 = \lambda_0$ and $j_1 = \max\{c_{31}\lambda_0, c_{30}\lambda_0 r^{-\frac{2\alpha}{p-1}-2\alpha}\}$, we have that

$$(-\Delta)^\alpha w_{\lambda_1, j_1} + w_{\lambda_1, j_1}^p \geq 0 \quad \text{in } \Omega.$$

The proof ends. \square

Let $x_s = s\vec{n}_0 \in \Omega$ and a set

$$A_r = \bigcup_{s \in (0, r)} B_{\frac{s}{8}}(x_s).$$

It is obvious that A_r is a cone with the vertex at the origin.

Lemma 4.4 *Assume that $p \in (0, \frac{N+\alpha}{N-\alpha})$, then there exists $c_{32} > 0$ such that for any $x \in A_{r_0}$,*

$$\mathbb{G}_\alpha[(\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}])^p](x) \leq \begin{cases} c_{32}|x|^{-(N-\alpha)p+2\alpha} & \text{if } p \in (\frac{2\alpha}{N-\alpha}, \frac{N+\alpha}{N-\alpha}), \\ -c_{32} \ln|x| & \text{if } p = \frac{2\alpha}{N-\alpha}, \\ c_{32} & \text{if } p \in (0, \frac{2\alpha}{N-\alpha}). \end{cases} \quad (4.7)$$

Proof. Since $\partial\Omega$ is C^2 , then for $r_0 \in (0, 1/2)$ small enough, we observe that for any $x \in B_{\frac{s}{8}}(x_s)$ with $s \in (0, r_0)$,

$$\frac{3s}{4} \leq \rho_{\partial\Omega}(x) \leq \frac{5s}{4}$$

and for any $t \in (0, \frac{s}{8})$,

$$|x - x_t| \geq \frac{5s}{8} \geq \frac{1}{2} \max\{\rho_{\partial\Omega}(x), \rho_{\partial\Omega}(x_t)\}.$$

Then it follows by [5, Theorem 1.1, Theorem 1.2] that there exists $c_{33} > 1$ such that

$$\frac{1}{c_{33}} s^{\alpha-N} t^\alpha \leq G_\alpha(x, x_t) \leq c_{33} s^{\alpha-N} t^\alpha, \quad \forall x \in B_{\frac{s}{8}}(x_s). \quad (4.8)$$

Thus, there exists $c_{34} > 0$ independent of s, t such that

$$\frac{1}{c_{34}} s^{-N+\alpha} \leq \mathbb{G}_\alpha[t^{-\alpha} \delta_{x_t}](x) \leq c_{34} s^{-N+\alpha}, \quad \forall x \in B_{\frac{s}{8}}(x_s),$$

which implies that

$$\frac{1}{c_{34}} s^{-N+\alpha} \leq \mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}](x) \leq c_{34} s^{-N+\alpha}, \quad \forall x \in B_{\frac{s}{8}}(x_s). \quad (4.9)$$

From Lemma 2.1, it shows that for any $x \in \Omega$,

$$\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}](x) \leq c_5 |x|^{-N+\alpha}, \quad \forall x \in \Omega. \quad (4.10)$$

It follows by (2.3) and (4.10) that

$$\begin{aligned} \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}])^p](x_s) &\leq c_5^p \int_\Omega G_\alpha(x_s, y) \frac{1}{|y|^{(N-\alpha)p}} dy \\ &\leq c_5^{p+1} \int_\Omega \frac{|y|^\alpha}{|x_s - y|^{N-\alpha}} \frac{1}{|y|^{(N-\alpha)p}} dy \\ &= c_5^{p+1} s^{2\alpha - (N-\alpha)p} \int_{\tilde{\Omega}_s} \frac{1}{|\vec{n}_0 - z|^{N-\alpha}} \frac{1}{|z|^{(N-\alpha)p-\alpha}} dz \\ &= c_5^{p+1} s^{2\alpha - (N-\alpha)p} \left[\int_{\tilde{\Omega}_s \cap B_{1/2}(\vec{n}_0)} \frac{1}{|\vec{n}_0 - z|^{N-\alpha}} \frac{1}{|z|^{(N-\alpha)p-\alpha}} dz \right. \\ &\quad \left. + \int_{\tilde{\Omega}_s \cap B_{\frac{1}{2}}(\vec{n}_0)} \frac{1}{|\vec{n}_0 - z|^{N-\alpha}} \frac{1}{|z|^{(N-\alpha)p-\alpha}} dz \right] \\ &:= c_5^{p+1} s^{2\alpha - (N-\alpha)p} [I_1(s) + I_2(s)], \end{aligned}$$

where $\Omega_s = \{sz : z \in \Omega\}$.

We observe that

$$I_1(s) \leq c_{35} \int_{B_{1/2}(\vec{n}_0)} \frac{1}{|\vec{n}_0 - z|^{N-\alpha}} dz \leq c_{36}$$

and since $(N - \alpha)p - \alpha < N$ by $p \in (0, \frac{N+\alpha}{N-\alpha})$, then

$$\begin{aligned} I_2(s) &\leq c_{37} \int_{\tilde{\Omega}_s} \frac{1}{|z|^{(N-\alpha)p-\alpha}(1+|z|)^{N-\alpha}} dz \\ &\leq c_{37} \int_{B_{\frac{R}{s}}(0) \setminus B_{\frac{1}{2}}(0)} \frac{1}{|z|^{(N-\alpha)p-2\alpha+N}} dz \\ &\leq \begin{cases} c_{38} s^{(N-\alpha)p-2\alpha} & \text{if } p \in (\frac{2\alpha}{N-\alpha}, \frac{N+\alpha}{N-\alpha}), \\ -c_{38} \ln s & \text{if } p = \frac{2\alpha}{N-\alpha}, \\ c_{38} & \text{if } p \in (0, \frac{2\alpha}{N-\alpha}), \end{cases} \end{aligned}$$

where $c_{35}, c_{36}, c_{37}, c_{38} > 0$ and $R > 0$ such that $\Omega \subset B_R(0)$. Then (4.7) holds. \square

Proof of Theorem 1.2 (ii). For $p \in (1 + \frac{2\alpha}{N}, \frac{N+\alpha}{N-\alpha})$, we have that

$$-\frac{2\alpha}{p-1} \in (-N, -N + \alpha)$$

and it follows by Lemma 2.1 that

$$u_k(x) \leq k \mathbb{G}_\alpha \left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha} \right](x) \leq \frac{c_5 k}{|x|^{N-\alpha}}, \quad x \in \Omega.$$

Then $\lim_{x \in \Omega, |x| \rightarrow 0} \frac{u_k(x)}{w_p(x)} = 0$ and we claim that

$$u_k \leq \lambda_0 w_p \quad \text{in } \Omega.$$

In fact, if it fails, then there exists $z_0 \in \Omega$ such that

$$(u_k - \lambda_0 w_p)(z_0) = \inf_{\Omega} (u_k - \lambda_0 w_p) = \operatorname{ess\,inf}_{\mathbb{R}^N} (u_k - \lambda_0 w_p) < 0.$$

Then we have $(-\Delta)^\alpha (u_k - \lambda_0 w_p)(z_0) < 0$, which contradicts the fact that

$$(-\Delta)^\alpha (u_k - \lambda_0 w_p)(z_0) = \lambda_0 w_p^p(z_0) - u_k^p(z_0) > 0.$$

By monotonicity of the mapping $k \rightarrow u_k$, there holds

$$u_\infty(x) := \lim_{k \rightarrow \infty} u_k(x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

which is a classical solution of (4.5) and

$$u_\infty(x) \leq \lambda_0 w_p(x) = \lambda_0 |x|^{-\frac{2\alpha}{p-1}}, \quad \forall x \in \Omega.$$

By applying Lemma 4.3, we obtain that u_∞ is continuous up to the boundary except the origin.

Finally, we claim that there exists $c_{39} > 0$ and $t_0 < \sigma_0$ such that

$$u_\infty(x_t) \geq c_{39} t^{-\frac{2\alpha}{p-1}}, \quad \forall t \in (0, t_0), \quad (4.11)$$

where $x_t = t\vec{n}_0 \in \Omega$. Indeed, let $r_k = (\sigma^{-1}k)^{\frac{p-1}{(N-\alpha)p-N-\alpha}}$, where $\sigma > 0$ will be chosen later, then $k = \sigma r_k^{\frac{(N-\alpha)p-N-\alpha}{p-1}}$ and for $x \in A_{r_0} \cap [B_{r_k}(0) \setminus B_{\frac{r_k}{2}}(0)]$, we apply Lemma 4.4 with $p \in (1 + \frac{2\alpha}{N}, \frac{N+\alpha}{N-\alpha})$ that

$$\begin{aligned} u_k(x) &\geq k\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right](x) - k^p\mathbb{G}_\alpha\left[\left(\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right]\right)^p\right](x) \\ &\geq c_5 k|x|^{\alpha-N}[1 - c_{40}k^{p-1}|x|^{(\alpha-N)p+\alpha+N}] \\ &\geq c_5 \sigma r_k^{-\frac{2\alpha}{p-1}}[1 - c_{40}\sigma^{p-1}r_k^{p-1}(r_k/2)^{(\alpha-N)p+\alpha+N}] \\ &\geq c_5 \sigma r_k^{-\frac{2\alpha}{p-1}}[1 - c_{40}\sigma^{p-1}2^{(N-\alpha)p-\alpha-N}] \\ &\geq \frac{c_5 \sigma}{2}|x|^{-\frac{2\alpha}{p-1}}, \end{aligned}$$

where we choose σ such that $c_{40}\sigma^{p-1}2^{(N-\alpha)p-\alpha-N} = \frac{1}{2}$. Then for any $x \in A_{r_0} \cap B_{r_k}^c(0)$, there exists $k > 0$ such that $x \in A_{r_0} \cap [B_{r_k}(0) \setminus B_{\frac{r_k}{2}}(0)]$ and then

$$u_\infty(x) \geq u_k(x) \geq \frac{c_5 \sigma}{2}|x|^{-\frac{2\alpha}{p-1}}, \quad \forall x \in A_{r_0} \cap B_{r_k}^c(0).$$

This ends the proof. \square

4.3 The limit of $\{u_k\}$ blows up when $p \in (0, 1 + \frac{2\alpha}{N}]$

In this subsection, we derive the blow-up behavior of the limit of $\{u_k\}$ when $p \in (0, 1 + \frac{2\alpha}{N}]$. To this end, we first do precise estimate for u_k .

Lemma 4.5 *Assume that $g(s) = s^p$ with $p \in (1, \frac{N}{N-\alpha}]$ and u_k is the solution of (1.5) obtained by Theorem 1.1. Then there exist $c_{41} > 0$, $r_0 \in (0, \frac{1}{4})$ and $\{r_k\}_k \subset (0, r_0)$ satisfying $r_k \rightarrow 0$ as $k \rightarrow \infty$ such that*

$$u_k(x) \geq \frac{c_{41}|x|^{-N}}{-\ln(|x|)}, \quad \forall x \in A_{r_0} \cap B_{r_k}^c(0). \quad (4.12)$$

Proof. *To prove (4.12) in the case of $p \in (\frac{2\alpha}{N-\alpha}, 1 + \frac{2\alpha}{N})$. Let $r_j = j^{-\frac{1}{\alpha}}$ with $j \in (k_0, k)$, then $j = r_j^{-\alpha}$. Applying Lemma 4.4 with $p \in (\frac{2\alpha}{N-\alpha}, 1 + \frac{2\alpha}{N})$ and (4.9), we have that for $x \in A_{r_0} \cap [B_{r_j}(0) \setminus B_{\frac{r_j}{2}}(0)]$,*

$$\begin{aligned} u_j(x) &\geq j\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right](x) - j^p\mathbb{G}_\alpha\left[\left(\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right]\right)^p\right](x) \\ &\geq c_{34}^{-1}j r_j^{\alpha-N} - c_{32}j^p|x|^{(\alpha-N)p+2\alpha} \\ &\geq c_{34}^{-1}r_j^{-N} - c_{32}r_j^{-\alpha p - (N-\alpha)p+2\alpha} \\ &\geq \frac{1}{2c_{34}}|x|^{-N}, \end{aligned}$$

where the last inequality holds since $-\alpha p - (N-\alpha)p + 2\alpha > -N$ and $r_j \rightarrow 0$ as $j \rightarrow \infty$. Then for any $x \in A_{r_0} \cap B_{r_k}^c(0)$, there exists $j \in (k_0, k)$ such that $x \in A_{r_0} \cap [B_{r_j}(0) \setminus B_{\frac{r_j}{2}}(0)]$ and then

$$u_k(x) \geq u_j(x) \geq \frac{1}{2c_{34}}|x|^{-N}, \quad \forall x \in A_{r_0} \cap B_{r_k}^c(0).$$

To prove (4.12) in the case of $p \in (0, \frac{2\alpha}{N-\alpha}]$. Let $r_j = j^{-\frac{1}{\alpha}}$ with $j \in (k_0, k)$, then $j = r_j^{-\alpha}$ and for $x \in A_{r_0} \cap [B_{r_j}(0) \setminus B_{\frac{r_j}{2}}(0)]$, we have that

$$u_j(x) \geq j\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right](x) - j^p\mathbb{G}_\alpha\left[\left(\mathbb{G}_\alpha\left[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}\right]\right)^p\right](x)$$

$$\begin{aligned}
&\geq c_{34}^{-1} j |x|^{\alpha-N} - c_{32} j^p \\
&\geq c_{34}^{-1} r_j^{-N} - c_{32} r_j^{-\alpha p} \\
&\geq \frac{1}{2c_{34}} |x|^{-N},
\end{aligned}$$

where the last inequality holds since $-\alpha p > -N$ and $r_j \rightarrow 0$ as $j \rightarrow \infty$. For any $x \in A_{r_0} \cap B_{r_k}^c(0)$, there exists $j \in (k_0, k)$ such that $x \in A_{r_0} \cap [B_{r_j}(0) \setminus B_{\frac{r_j}{2}}(0)]$ and then

$$u_k(x) \geq u_j(x) \geq \frac{1}{2c_{34}} |x|^{-N}, \quad \forall x \in A_{r_0} \cap B_{r_k}^c(0).$$

To prove (4.12) in the case of $p = 1 + \frac{2\alpha}{N}$. Let $\rho_j = j^{-\frac{1}{\alpha}}$ and $r_j = \frac{\rho_j}{[-\log(\rho_j)]^{\frac{1}{\alpha}}}$, then $j = \rho_j^{-\alpha}$ and applied Lemma 4.4 for $x \in A_{r_0} \cap [B_{r_j}(0) \setminus B_{\frac{r_j}{2}}(0)]$,

$$\begin{aligned}
u_j(x) &\geq j \mathbb{G}_\alpha \left[\frac{\partial^\alpha \delta_0}{\partial \bar{n}^\alpha} \right](x) - j^p \mathbb{G}_\alpha \left[\left(\mathbb{G}_\alpha \left[\frac{\partial^\alpha \delta_0}{\partial \bar{n}^\alpha} \right] \right)^p \right](x) \\
&\geq c_{34}^{-1} j |x|^{\alpha-N} - c_{32} j^p |x|^{(\alpha-N)p+2\alpha} \\
&\geq c_{34}^{-1} \rho_j^{-N} (-\log \rho_j)^{\frac{N-\alpha}{\alpha}} - c_{42} \rho_j^{-N} (-\log \rho_j)^{\frac{(N-\alpha)p-2\alpha}{\alpha}} \\
&= c_{34}^{-1} \rho_j^{-N} (-\log \rho_j)^{\frac{N-\alpha}{\alpha}} \left[1 - c_{42} (-\log \rho_j)^{\frac{(N-\alpha)p-2\alpha}{\alpha} - \frac{N-\alpha}{\alpha}} \right] \\
&\geq c_{34}^{-1} \frac{r_j^{-N}}{-\log \rho_j} \left[1 - c_{42} (-\log \rho_j)^{\frac{(N-\alpha)p-N-\alpha}{\alpha}} \right] \\
&\geq \frac{c_{34} |x|^{-N}}{-2 \log |x|},
\end{aligned}$$

where $c_{42} > 0$ and we used the facts that $\log(\rho_j) \leq c \log r_j \leq c \log |x|$ and $\frac{(N-\alpha)p-N-\alpha}{\alpha} < 0$. Then for any $x \in A_{r_0} \cap B_{r_k}^c(0)$, there exists $j \in (k_0, k)$ such that $x \in A_{r_0} \cap [B_{r_j}(0) \setminus B_{\frac{r_j}{2}}(0)]$ and then

$$u_k(x) \geq \frac{c_{34} |x|^{-N}}{-2 \log |x|}, \quad x \in A_{r_0} \cap B_{r_k}^c(0).$$

The proof ends. \square

Proof of Theorem 1.2 (iii). It derives by Lemma 4.5 that

$$\pi_k := \int_{B_{r_0}(0)} u_k(x) \geq c_{41} \int_{A_{r_0} \cap B_{r_k}^c(0)} \frac{|x|^{-N}}{-\log |x|} dx \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (4.13)$$

Fix $y_0 \in \Omega \setminus \bar{B}_{r_0}(0)$, it follows by Lemma 2.4 in [15] that problem

$$\begin{aligned}
(-\Delta)^\alpha u + u^p &= 0 && \text{in } B_{\varrho_0}(y_0), \\
u &= 0 && \text{in } \mathbb{R}^N \setminus (B_{\varrho_0}(y_0) \cup B_{r_0}(0)), \\
u &= u_k && \text{in } B_{r_0}(0)
\end{aligned} \quad (4.14)$$

admits a unique solution w_k , where $\varrho_0 = \min\{\rho_{\partial\Omega}(y_0), |y_0| - r_0\}$. By Lemma 2.2 in [15],

$$u_k \geq w_k \quad \text{in } B_{\varrho_0}(y_0). \quad (4.15)$$

Let $\tilde{w}_k = w_k - u_k \chi_{B_{r_0}(0)}$, then $\tilde{w}_k = w_k$ in $B_{\varrho_0}(y_0)$ and for $x \in B_{\varrho_0}(y_0)$,

$$\begin{aligned}
(-\Delta)^\alpha \tilde{w}_k(x) &= -\lim_{\epsilon \rightarrow 0^+} \int_{B_{\varrho_0}(y_0) \setminus B_\epsilon(x)} \frac{w_k(z) - w_k(x)}{|z-x|^{N+2\alpha}} dz \\
&\quad + \lim_{\epsilon \rightarrow 0^+} \int_{B_{\varrho_0}^c(y_0) \setminus B_\epsilon(x)} \frac{w_k(x)}{|z-x|^{N+2\alpha}} dz \\
&= -\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{w_k(z) - w_k(x)}{|z-x|^{N+2\alpha}} dz + \int_{B_{r_0}(0)} \frac{u_k(z)}{|z-x|^{N+2\alpha}} dz \\
&\geq (-\Delta)^\alpha w_k(x) + c_{42} \pi_k,
\end{aligned}$$

where $c_{42} = (|y_0| + r_0)^{-N-2\alpha}$ and the last inequality follows by the fact of

$$|z - x| \leq |x| + |z| \leq |y_0| + r_0 \quad \text{for } z \in B_{\frac{1}{4}}(0), x \in B_{\frac{1}{4}}(y_0).$$

Therefore,

$$\begin{aligned} (-\Delta)^\alpha \tilde{w}_k(x) + \tilde{w}_k^p(x) &\geq (-\Delta)^\alpha w_k(x) + w_k^p(x) + c_{42}\pi_k \\ &= c_{42}\pi_k, \quad x \in B_{\varrho_0}(y_0), \end{aligned}$$

that is, \tilde{w}_k is a super solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= c_{42}\pi_k \quad \text{in } B_{\varrho_0}(y_0), \\ u &= 0 \quad \text{in } B_{\varrho_0}^c(y_0). \end{aligned} \tag{4.16}$$

Let η_1 be the solution of

$$\begin{aligned} (-\Delta)^\alpha u &= 1 \quad \text{in } B_{\varrho_0}(y_0), \\ u &= 0 \quad \text{in } B_{\varrho_0}^c(y_0). \end{aligned}$$

Then $(c_{42}\pi_k)^{\frac{1}{p}} \frac{\eta_1}{2 \max_{\mathbb{R}^N} \eta_1}$ is sub solution of (4.16) for k large enough. By Lemma 2.2 in [15], we have that

$$\tilde{w}_k(x) \geq (c_{42}\pi_k)^{\frac{1}{p}} \frac{\eta_1(x)}{2 \max_{\mathbb{R}^N} \eta_1}, \quad \forall x \in B_{\varrho_0}(y_0),$$

which implies that

$$w_k(y) \geq c_{43}(c_{42}\pi_k)^{\frac{1}{p}}, \quad \forall y \in B_{\frac{\varrho_0}{2}}(y_0),$$

where $c_{43} = \min_{x \in B_{\varrho_0}(y_0)} \frac{\eta_1(x)}{2 \max_{\mathbb{R}^N} \eta_1}$. Therefore, (4.15) and (4.13) imply that

$$\lim_{k \rightarrow \infty} u_k(y) \geq \lim_{k \rightarrow \infty} w_k(y) = \infty, \quad \forall y \in B_{\frac{\varrho_0}{2}}(y_0).$$

Similarly, we can prove

$$\lim_{k \rightarrow \infty} u_k(y) \geq \lim_{k \rightarrow \infty} w_k(y) = \infty, \quad \forall y \in \Omega.$$

The proof ends. \square

5 Nonexistence in the critical case

In this section, we prove the nonexistence in the critical case. To this end, we consider the weak solution to elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + u^{\frac{N+\alpha}{N-\alpha}} &= k \frac{\partial^\alpha \delta_0}{\partial e_N^\alpha} \quad \text{in } \overline{\mathbb{R}_+^N}, \\ u &= 0 \quad \text{in } \mathbb{R}_-^N, \end{aligned} \tag{5.1}$$

where $\mathbb{R}_+^N = \mathbb{R}^{N-1} \times \mathbb{R}_+$ and $e_N = (0, \dots, 0, 1)$.

Definition 5.1 A function $u \in L^1(\mathbb{R}^N, \mu dx)$ is a weak solution of (5.1) if $u^p \in L^1(\mathbb{R}^N, \rho^\alpha \mu dx)$ and

$$\int_{\mathbb{R}_+^N} [u(-\Delta)^\alpha \xi + u^{\frac{N+\alpha}{N-\alpha}} \xi] dx = \frac{\partial^\alpha \xi(0)}{\partial e_N^\alpha}, \quad \forall \xi \in \mathbb{X}_{\alpha, \mathbb{R}_+^N}, \tag{5.2}$$

where $\mu(x) = \frac{1}{1+|x|^{N+2\alpha}}$, $\rho(x) = \min\{1, \rho_{\partial\Omega}(x)\}$ and $\mathbb{X}_{\alpha, \mathbb{R}_+^N} \subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:

- (i) the support of ξ is a compact set in $\overline{\mathbb{R}_+^N}$;
- (ii) $(-\Delta)^\alpha \xi(x)$ exists for any $x \in \mathbb{R}_+^N$ and there exists $c > 0$ such that

$$|(-\Delta)^\alpha \xi(x)| \leq c\mu(x), \quad \forall x \in \mathbb{R}_+^N;$$

- (iii) there exist $\varphi \in L^1(\mathbb{R}_+^N, \rho^\alpha dx)$ and $\varepsilon_0 > 0$ such that $|(-\Delta)^\alpha \xi| \leq \varphi$ a.e. in \mathbb{R}_+^N , for all $\varepsilon \in (0, \varepsilon_0]$.

Let $\mathbb{G}_{\alpha, \mathbb{R}_+^N}$ the Green's function on $\mathbb{R}_+^N \times \mathbb{R}_+^N$ and

$$\Gamma_\alpha(x) = \lim_{t \rightarrow 0} t^{-\alpha} \mathbb{G}_{\alpha, \mathbb{R}_+^N}(x, te_N). \quad (5.3)$$

Lemma 5.1 *Let Γ_α defined in (5.3), then*

$$\begin{aligned} (-\Delta)^\alpha \Gamma_\alpha &= \frac{\partial^\alpha \delta_0}{\partial e_N^\alpha} & \text{in } \bar{\mathbb{R}}_+^N, \\ \Gamma_\alpha &= 0 & \text{in } \mathbb{R}_-^N. \end{aligned} \quad (5.4)$$

Moreover,

$$\Gamma_\alpha(x) = |x|^{-N+\alpha} \Gamma_\alpha\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^N, \quad (5.5)$$

and

$$\Gamma_\alpha\left(\frac{x}{|x|}\right) \begin{cases} > 0 & \text{if } x \in \mathbb{R}_+^N, \\ = 0 & \text{if } x \notin \mathbb{R}_+^N. \end{cases}$$

Proof. We observe that

$$(-\Delta)_x^\alpha t^{-\alpha} \mathbb{G}_{\alpha, \mathbb{R}_+^N}(x, te_N) = t^{-\alpha} \delta_{te_N}$$

and

$$\lim_{t \rightarrow 0^+} \langle t^{-\alpha} \delta_{te_N}, \xi \rangle = \frac{\partial^\alpha \xi(0)}{\partial e_N^\alpha}, \quad \forall \xi \in \mathbb{X}_{\alpha, \mathbb{R}_+^N}.$$

Then (5.4) holds in the weak sense. By the regularity results, Γ_α is a solution of

$$\begin{aligned} (-\Delta)^\alpha \Gamma_\alpha &= 0 & \text{in } \mathbb{R}_+^N, \\ \Gamma_\alpha &= 0 & \text{in } \bar{\mathbb{R}}_-^N \setminus \{0\}. \end{aligned} \quad (5.6)$$

Let $\Gamma_{\alpha, \lambda}(x) = \lambda^{N-\alpha} \Gamma_\alpha(\lambda x)$ and $\xi_\lambda(x) = \xi(x/\lambda)$ for $\xi \in \mathbb{X}_{\alpha, \mathbb{R}_+^N}$, then we have that

$$\begin{aligned} \int_{\mathbb{R}_+^N} \Gamma_{\alpha, \lambda} (-\Delta)^\alpha \xi dx &= \lambda^\alpha \int_{\mathbb{R}_+^N} \Gamma_\alpha(z) (-\Delta)^\alpha \xi_\lambda(x) dx, \\ &= \lambda^\alpha \frac{\partial^\alpha \xi_\lambda(0)}{\partial e_N^\alpha}, \end{aligned}$$

which implies that

$$\int_{\mathbb{R}_+^N} \Gamma_{\alpha, \lambda} (-\Delta)^\alpha \xi dx = \frac{\partial^\alpha \xi(0)}{\partial e_N^\alpha}.$$

By the uniqueness, we derive that

$$\lambda^{N-\alpha} \Gamma_\alpha(\lambda x) = \Gamma_\alpha(x),$$

which, choosing $\lambda = \frac{1}{|x|}$, implies (5.5). The last argument is obvious. \square

Theorem 5.1 *Let $k > 0$, then problem (5.1) has no any weak solution.*

Proof. If there exists a weak solution u_k to (5.1), then we observe that

$$u_k > 0 \quad \text{in } \mathbb{R}_+^N.$$

By Maximum Principle, we have that

$$u_k \leq k \Gamma_\alpha \quad \text{in } \mathbb{R}^N. \quad (5.7)$$

Denoting

$$u_\infty = \lim_{k \rightarrow \infty} u_k \quad \text{in } \mathbb{R}^N.$$

We claim that

$$u_\infty(x) = |x|^{\alpha-N} u_\infty\left(\frac{x}{|x|}\right), \quad \forall x \in \mathbb{R}^N \setminus \{0\}. \quad (5.8)$$

Indeed, let

$$\tilde{u}_\lambda(x) = \lambda^{N-\alpha} u_k(\lambda x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

By direct computation, we have that for $x \in \mathbb{R}_+^N$,

$$\begin{aligned} (-\Delta)^\alpha \tilde{u}_\lambda(x) + \tilde{u}_\lambda^{\frac{N+\alpha}{N-\alpha}}(x) &= \lambda^{N+\alpha} [(-\Delta)^\alpha u_k(\lambda x) + u_k^{\frac{N+\alpha}{N-\alpha}}(\lambda x)] \\ &= 0. \end{aligned} \quad (5.9)$$

Moreover, for $f \in C_0^1(\mathbb{R}_+^N)$,

$$\begin{aligned} \langle (-\Delta)^\alpha \tilde{u}_\lambda + \tilde{u}_\lambda^{\frac{N+\alpha}{N-\alpha}}, f \rangle &= \lambda^{N+\alpha} \int_{\mathbb{R}^N} [(-\Delta)^\alpha u_k(\lambda x) + u_k^{\frac{N+\alpha}{N-\alpha}}(\lambda x)] f(x) dx \\ &= \lambda^\alpha \int_{\mathbb{R}^N} [(-\Delta)^\alpha u_k(z) + u_k^{\frac{N+\alpha}{N-\alpha}}(z)] f\left(\frac{z}{\lambda}\right) dz \\ &= \lambda^\alpha k \frac{\partial^\alpha f(0)}{\partial e_N^\alpha}. \end{aligned}$$

Thus,

$$(-\Delta)^\alpha \tilde{u}_\lambda + \tilde{u}_\lambda^{\frac{N+\alpha}{N-\alpha}} = \lambda^\alpha k \frac{\partial^\alpha \delta_0}{\partial e_N^\alpha} \quad \text{in } \mathbb{R}_+^N. \quad (5.10)$$

We observe that $\lim_{|x| \rightarrow \infty} \tilde{u}_\lambda(x) = 0$ and $u_{k\lambda^\alpha}$ is the unique weak solution of (5.1) with k replaced by $\lambda^\alpha k$, then for $x \in \mathbb{R}^N \setminus \{0\}$,

$$u_{k\lambda^\alpha}(x) = \tilde{u}_\lambda(x) = \lambda^{N-\alpha} u_k(\lambda x) \quad (5.11)$$

and letting $k \rightarrow \infty$ we have that

$$u_\infty(x) = \lambda^{N-\alpha} u_\infty(\lambda x), \quad \forall x \in \mathbb{R}^N \setminus \{0\},$$

which implies (5.8) by taking $\lambda = |x|^{-1}$.

Combine (5.5), (5.7) and (5.11), then we have that

$$u_{k\lambda^\alpha}(x) \leq \lambda^{N-\alpha} k \Gamma_\alpha(\lambda x) = k \Gamma_\alpha(x), \quad \forall x \in \mathbb{R}^N.$$

Thus,

$$u_\infty(x) \leq k \Gamma_\alpha(x), \quad \forall x \in \mathbb{R}^N.$$

By arbitrary of k , it implies that

$$u_\infty \equiv 0,$$

then $u_1 \equiv 0$ in \mathbb{R}^N , which is impossible. \square

Proof of Theorem 1.3. Without loss generality, we let $k = 1$, $0 \in \partial\Omega$ and e_N is the unit normal vector pointing inside of Ω at 0. If

$$\begin{aligned} (-\Delta)^\alpha u + u^{\frac{N+\alpha}{N-\alpha}} &= \frac{\partial^\alpha \delta_0}{\partial e_N^\alpha} \quad \text{in } \bar{\Omega}, \\ u &= 0 \quad \text{in } \bar{\Omega}^c \end{aligned}$$

admits a solution weak v_1 , we claim that there is a weak solution of (5.1), then the contradiction is obtained from Theorem 5.1.

In fact, we may assume that

$$\Omega = B_1(e_N) \quad \text{and} \quad B_m = B_m(me_N).$$

Then

$$\Omega \subset B_m \subset B_{m+1} \quad \text{and} \quad \lim_{m \rightarrow \infty} B_m = \mathbb{R}_+^N.$$

Let

$$v_m(x) = m^{\alpha-N} v_1\left(\frac{x}{m}\right), \quad x \in \mathbb{R}^N.$$

By direct computation, v_m is a weak solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^{\frac{N+\alpha}{N-\alpha}} &= \frac{\partial^\alpha \delta_0}{\partial e_N^\alpha} \quad \text{in} \quad \bar{B}_m, \\ u &= 0 \quad \text{in} \quad \bar{B}_m^c, \end{aligned} \tag{5.12}$$

We next show that $v_m \leq v_{m+1}$ in \mathbb{R}^N . From Proposition 3.1,

$$\begin{aligned} (-\Delta)^\alpha u + u^{\frac{N+\alpha}{N-\alpha}} &= t^{-\alpha} \delta_{te_N} \quad \text{in} \quad B_m, \\ u &= 0 \quad \text{in} \quad B_m^c \end{aligned} \tag{5.13}$$

admits a unique weak solution, denoting $v_{m,t}$. Choose a sequence nonnegative functions $\{f_{m,i}\}_{i \in \mathbb{N}} \subset C^1(\mathbb{R}^N)$ with support $B_1(e_N)$ such that $f_{m,i} \rightarrow t^{-\alpha} \delta_{te_N}$ as $i \rightarrow \infty$ in the distribution sense. Let $v_{m,i,t}$ be the unique solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^{\frac{N+\alpha}{N-\alpha}} &= f_{m,i} \quad \text{in} \quad B_m, \\ u &= 0 \quad \text{in} \quad B_m^c \end{aligned} \tag{5.14}$$

and by Maximum Principle, see [15, Lemma 2.3], derive that

$$v_{m,i,t} \leq \tilde{v}_{m+1,i,t} \quad \text{in} \quad \mathbb{R}^N.$$

Together with the facts that $v_{m,i,t} \rightarrow v_{m,t}$ a.e. in \mathbb{R}^N and $v_{m+1,i,t} \rightarrow v_{m+1,t}$ a.e. in \mathbb{R}^N as $i \rightarrow \infty$, we obtain that

$$v_{1,t} \leq v_{m,t} \leq v_{m+1,t} \quad \text{a.e. in} \quad \mathbb{R}^N \tag{5.15}$$

and

$$\int_{B_m} v_{m,t}^{\frac{N+\alpha}{N-\alpha}} \rho^\alpha dx < \|\mathbb{G}_{\alpha, B_m}[f_{m,i}]\|_{L^1(\Omega, \rho^\alpha dx)},$$

which implies that

$$\begin{aligned} (-\Delta)^\alpha u + u^{\frac{N+\alpha}{N-\alpha}} &= \frac{\partial^\alpha \delta_0}{\partial e_N^\alpha} \quad \text{in} \quad \bar{B}_m, \\ u &= 0 \quad \text{in} \quad \bar{B}_m^c \end{aligned} \tag{5.16}$$

admits a solution v_m for any $m \in \mathbb{N}$ and

$$v_m \leq v_{m+1} \quad \text{a.e. in} \quad \mathbb{R}^N. \tag{5.17}$$

We observe that

$$0 \leq v_m \leq \mathbb{G}_{\alpha, B_m}\left[\frac{\partial^\alpha \delta_0}{\partial e_N^\alpha}\right] \leq \frac{c_5}{|x|^{N-\alpha}} \quad \text{a.e. in} \quad \mathbb{R}^N \tag{5.18}$$

and

$$\int_{B_m} v_m^{\frac{N+\alpha}{N-\alpha}} \rho^\alpha dx < \|\mathbb{G}_{\alpha, B_m}\left[\frac{\partial^\alpha \delta_0}{\partial e_N^\alpha}\right]\|_{L^1(B_m, \rho^\alpha dx)}.$$

By (5.17) and (5.18), we see that the limit of $\{v_m\}$ exists, denoted it by w_1 . Hence,

$$0 \leq w_1 \leq \mathbb{G}_{\alpha, \mathbb{R}_+^N} \left[\frac{\partial^\alpha \delta_0}{\partial e_N^\alpha} \right] \quad \text{a.e. in } \mathbb{R}^N \quad (5.19)$$

and

$$\int_{\mathbb{R}_+^N} w_1^{\frac{N+\alpha}{N-\alpha}} \rho^\alpha dx < \left\| \mathbb{G}_{\alpha, \mathbb{R}_+^N} \left[\frac{\partial^\alpha \delta_0}{\partial e_N^\alpha} \right] \right\|_{L^1(\mathbb{R}_+^N, \rho^\alpha \mu dx)},$$

which implies that $w_1 \in L^1(\mathbb{R}^N, \mu dx)$. Thus, $v_m \rightarrow w_1$ in $L^1(\mathbb{R}^N, \rho^\alpha \mu dx)$ as $m \rightarrow \infty$.

For $\xi \in \mathbb{X}_{\alpha, \mathbb{R}_+^N}$, there exists $N_0 > 0$ such that for any $m \geq N_0$,

$$\text{supp}(\xi) \subset \bar{B}_m,$$

which implies that $\xi \in \mathbb{X}_{\alpha, B_m}$ and then

$$\int_{\mathbb{R}_+^N} [v_m (-\Delta)^\alpha \xi + v_m^{\frac{N+\alpha}{N-\alpha}} \xi] dx = \frac{\partial^\alpha \xi(0)}{\partial e_N^\alpha}. \quad (5.20)$$

By [15, Lemma 3.1],

$$|(-\Delta)^\alpha \xi(x)| \leq \frac{c_9 \|\xi\|_{L^\infty(\Omega)}}{1 + |x|^{N+2\alpha}}, \quad \forall x \in \mathbb{R}_+^N.$$

Thus,

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}_+^N} v_m(x) (-\Delta)^\alpha \xi(x) dx = \int_{\mathbb{R}_+^N} w_1(x) (-\Delta)^\alpha \xi(x) dx. \quad (5.21)$$

By (5.19) and increasing monotonicity of v_m , for any $n \geq N_0$,

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}_+^N} v_m^{\frac{N+\alpha}{N-\alpha}} \xi(x) dx = \int_{\mathbb{R}_+^N} w_1^{\frac{N+\alpha}{N-\alpha}} \xi(x) dx. \quad (5.22)$$

Combining (5.21), (5.22) and taking $m \rightarrow \infty$ in (5.20), we obtain that

$$\int_{\mathbb{R}_+^N} \left[w_1 (-\Delta)^\alpha \xi + w_1^{\frac{N+\alpha}{N-\alpha}} \xi \right] dx = \frac{\partial^\alpha \xi(0)}{\partial e_N^\alpha}. \quad (5.23)$$

Since $\xi \in \mathbb{X}_{\alpha, \mathbb{R}_+^N}$ is arbitrary, w_1 is a weak solution of (5.1). \square

6 Forcing nonlinearity

This section is devoted to consider problem (1.5) when $\epsilon = -1$, we call it as forcing case. In order to derive the existence of weak solution to (1.5) with forcing nonlinearity, we first introduce the following propositions.

Proposition 6.1 [11, Proposition 2.2] *Let $\alpha \in (0, 1]$, $\beta \in [0, \alpha]$ and $\nu \in \mathfrak{M}(\Omega, \rho_{\partial\Omega}^\beta)$, then there exists $c_{44} > 0$ such that*

$$\|\mathbb{G}_\alpha[\nu]\|_{M^{p_\beta^*}(\Omega, \rho_{\partial\Omega}^\beta dx)} \leq c_{44} \|\nu\|_{\mathfrak{M}(\Omega, \rho_{\partial\Omega}^\beta)}, \quad (6.1)$$

where $p_\beta^* = \frac{N+\beta}{N-2\alpha+\beta}$.

Proposition 6.2 [11, Proposition 2.3] *Let $\alpha \in (0, 1]$ and $\beta \in [0, \alpha]$, then the mapping $f \mapsto \mathbb{G}_\alpha[f]$ is compact from $L^1(\Omega, \rho_{\partial\Omega}^\beta dx)$ into $L^q(\Omega)$ for any $q \in [1, \frac{N}{N+\beta-2\alpha})$. Moreover, for $q \in [1, \frac{N}{N+\beta-2\alpha})$, there exists $c_{45} > 0$ such that for any $f \in L^1(\Omega, \rho_{\partial\Omega}^\beta dx)$*

$$\|\mathbb{G}_\alpha[f]\|_{L^q(\Omega)} \leq c_{45} \|f\|_{L^1(\Omega, \rho_{\partial\Omega}^\beta dx)}. \quad (6.2)$$

For $\nu \in \mathfrak{M}_{\partial\Omega}^b(\bar{\Omega})$, ν_t is given in section 2.2 for $t \in (0, \sigma_0)$. Let $t_j = \frac{1}{j} \in (0, \sigma_0/4)$ if $j \geq j_0$ for some $j_0 > 0$. Choose $\{\tilde{\nu}_n\}_n \subset C_0^1(\Omega)$ a sequence of nonnegative functions such that $\text{supp}(\tilde{\nu}_n) \subset \Omega_{t_{j_0-2^{-n}}} \setminus \Omega_{t_{j_0+2^{-n}}}$ and $\tilde{\nu}_n \rightarrow \nu_{t_{j_0}}$ in the duality sense with $C(\bar{\Omega})$. Denote

$$\nu_{n,j}(x) = \begin{cases} \tilde{\nu}_n(x + t_j \vec{n}_x), & \text{if } x \in \Omega_{t_{j_0-2^{-n}}} \setminus \Omega_{t_{j_0+2^{-n}}}, \\ 0, & \text{if not.} \end{cases}$$

Lemma 6.1 *Up to subsequence, we have that $\nu_{n,j_n} \rightarrow \nu$ in the duality sense with $C(\bar{\Omega})$, that is,*

$$\lim_{n \rightarrow \infty} \int_{\bar{\Omega}} \zeta \nu_{n,j_n} dx = \int_{\bar{\Omega}} \zeta d\nu, \quad \forall \zeta \in C(\bar{\Omega}). \quad (6.3)$$

Moreover,

$$\text{supp}(\nu_n) \subset \Omega_{\frac{t_n}{2}} \setminus \Omega_{2t_n}.$$

Proof. For any fixed j and $\zeta \in C(\bar{\Omega})$, we observe that

$$\lim_{n \rightarrow \infty} \int_{\bar{\Omega}} \zeta \nu_{n,j} dx = \int_{\Omega} \zeta d\nu_{t_j}$$

and pass $j \rightarrow \infty$, we derive that

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\bar{\Omega}} \zeta \nu_{n,j} dx = \int_{\Omega} \zeta d\nu.$$

The second argument is obvious by the definition of $\nu_{n,j}$. \square

6.1 Sub-linear

In this subsection, we are devoted to prove the existence of weak solution to (1.5) when the source nonlinearity is sub-linear.

Proof of Theorem 1.4 (i). Let $\{\nu_n\}$ be a sequence of nonnegative functions such that $\nu_n \rightarrow \nu$ in sense of duality with $C(\bar{\Omega})$, see Lemma 6.1. By the Banach-Steinhaus Theorem, we may assume that $\|\nu_n\|_{L^1(\Omega)} \leq \|\nu\|_{\mathfrak{M}^b(\Omega)} = 1$ for all n . We consider a sequence $\{g_n\}$ of C^1 nonnegative functions defined on \mathbb{R}_+ such that $g_n(0) = g(0)$,

$$g_n \leq g_{n+1} \leq g, \quad \sup_{s \in \mathbb{R}_+} g_n(s) = n \quad \text{and} \quad \lim_{n \rightarrow \infty} \|g_n - g\|_{L_{loc}^\infty(\mathbb{R}_+)} = 0. \quad (6.4)$$

We set

$$M(v) = \|v\|_{L^1(\Omega)}.$$

Step 1. To prove that for $n \geq 1$,

$$\begin{aligned} (-\Delta)^\alpha u &= g_n(u) + kt_n^{-\alpha} \nu_n & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \end{aligned} \quad (6.5)$$

admits a nonnegative solution u_n such that

$$M(u_n) \leq \bar{\lambda},$$

where $\bar{\lambda} > 0$ independent of n .

To this end, we define the operators $\{\mathcal{T}_n\}$ by

$$\mathcal{T}_n u = \mathbb{G}_\alpha [g_n(u) + kt_n^{-\alpha} \nu_n], \quad \forall u \in L_+^1(\Omega),$$

where $L_+^1(\Omega)$ is the positive cone of $L^1(\Omega)$. By (6.2) and (1.13), we have that

$$\begin{aligned}
M(\mathcal{T}_n u) &\leq c_{45} \|g_n(u) + kt_n^{-\alpha} \nu_n\|_{L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)} \\
&\leq c_3 c_{45} \int_{\Omega} u^{p_0} \rho^\alpha(x) dx + c_{46}(k + \epsilon) \\
&\leq c_3 c_{47} \int_{\Omega} u^{p_0} dx + c_{46}(k + \epsilon) \\
&\leq c_3 c_{48} (\int_{\Omega} u dx)^{p_0} + c_{46}(k + \epsilon) \\
&= c_3 c_{48} M(u)^{p_0} + c_{46}(k + \epsilon),
\end{aligned} \tag{6.6}$$

where $c_{47}, c_{48} > 0$ independent of n . Therefore, we derive that

$$M(\mathcal{T}_n u) \leq c_3 c_{48} M(u)^{p_0} + c_{45}(k + \epsilon).$$

If we assume that $M(u) \leq \lambda$ for some $\lambda > 0$, it implies

$$M(\mathcal{T}_n u) \leq c_3 c_{48} \lambda^{p_0} + c_{45}(k + \epsilon).$$

In the case of $p_0 < 1$, the equation

$$c_3 c_{48} \lambda^{p_0} + c_{45}(k + \epsilon) = \lambda$$

admits a unique positive root $\bar{\lambda}$. In the case of $p_0 = 1$, for $c_3 > 0$ satisfying $c_3 c_{48} < 1$, the equation

$$c_3 c_{48} \lambda + c_{45}(k + \epsilon) = \lambda$$

admits a unique positive root $\bar{\lambda}$. For $M(u) \leq \bar{\lambda}$, we obtain that

$$M(\mathcal{T}_n u) \leq c_3 c_{48} \bar{\lambda}^{p_0} + c_{45}(k + \epsilon) = \bar{\lambda}. \tag{6.7}$$

Thus, \mathcal{T}_n maps $L^1(\Omega)$ into itself. Clearly, if $u_m \rightarrow u$ in $L^1(\Omega)$ as $m \rightarrow \infty$, then $g_n(u_m) \rightarrow g_n(u)$ in $L^1(\Omega)$ as $m \rightarrow \infty$, thus \mathcal{T}_n is continuous. For any fixed $n \in \mathbb{N}$, $\mathcal{T}_n u_m = \mathbb{G}_\alpha [g_n(u_m) + k\nu_n]$ and $\{g_n(u_m) + k\nu_n\}_m$ is uniformly bounded in $L^1(\Omega, \rho_{\partial\Omega}^\beta dx)$, then it follows by Proposition 6.2 that $\{\mathbb{G}_\alpha [g_n(u_m) + kt_n^{-\alpha} \nu_n]\}_m$ is pre-compact in $L^1(\Omega)$, which implies that \mathcal{T}_n is a compact operator.

Let

$$\mathcal{G} = \{u \in L_+^1(\Omega) : M(u) \leq \bar{\lambda}\},$$

which is a closed and convex set of $L^1(\Omega)$. It infers by (6.7) that

$$\mathcal{T}_n(\mathcal{G}) \subset \mathcal{G}.$$

It follows by Schauder's fixed point theorem that there exists some $u_n \in L_+^1(\Omega)$ such that $\mathcal{T}_n u_n = u_n$ and $M(u_n) \leq \bar{\lambda}$, where $\bar{\lambda} > 0$ independent of n .

We observe that u_n is a classical solution of (6.5). Let open set O satisfy $O \subset \bar{O} \subset \Omega$. By [27, Proposition 2.3], for $\theta \in (0, 2\alpha)$, there exists $c_{49} > 0$ such that

$$\|u_n\|_{C^\theta(O)} \leq c_{49} \{ \|g(u_n)\|_{L^\infty(\Omega)} + kt_n^{-\alpha} \|\nu_n\|_{L^\infty(\Omega)} \},$$

then applied [27, Corollary 2.4], u_n is $C^{2\alpha+\epsilon_0}$ locally in Ω for some $\epsilon_0 > 0$. Then u_n is a classical solution of (6.5). Moreover, from [13, Lemma 2.2], we derive that

$$\int_{\Omega} u_n (-\Delta)^\alpha \xi dx = \int_{\Omega} g(u_n) \xi dx + k \int_{\Omega} \xi t_n^{-\alpha} \nu_n dx, \quad \forall \xi \in \mathbb{X}_\alpha. \tag{6.8}$$

Step 2. Convergence. We observe that $\{g_n(u_n)\}$ is uniformly bounded in $L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)$, so is $\{\nu_n\}$. By Proposition 6.2, there exist a subsequence $\{u_{n_k}\}$ and u such that $u_{n_k} \rightarrow u$ a.e. in Ω and in $L^1(\Omega)$, then by (1.13), we derive that $g_{n_k}(u_{n_k}) \rightarrow g(u)$ in $L^1(\Omega)$. Pass the limit of (6.8) as $n_k \rightarrow \infty$ to derive that

$$\int_{\Omega} u (-\Delta)^\alpha \xi = \int_{\Omega} g(u) \xi dx + k \int_{\Omega} \frac{\partial^\alpha \xi}{\partial \vec{n}^\alpha} d\nu, \quad \forall \xi \in \mathbb{X}_\alpha,$$

thus u is a weak solution of (1.5) and u is nonnegative since $\{u_n\}$ are nonnegative. \square

6.2 Integral subcritical

In this subsection, we prove the existence of weak solution to (1.5) when the nonlinearity is integral subcritical.

Proof of Theorem 1.4 (ii). Let $\{\nu_n\} \subset C^1(\bar{\Omega})$ be a sequence of nonnegative functions given as the above and $\|\nu_n\|_{L^1(\Omega)} \leq 2\|\nu\|_{\mathfrak{M}^b(\bar{\Omega})} = 1$ for all n . We consider a sequence $\{g_n\}$ of C^1 nonnegative functions defined on \mathbb{R}_+ satisfying $g_n(0) = g(0)$ and (6.4). We set

$$M_1(v) = \|v\|_{M^{\frac{N+\alpha}{N-\alpha}}(\Omega, \rho_{\partial\Omega}^\alpha dx)} \quad \text{and} \quad M_2(v) = \|v\|_{L^{p_*}(\Omega)},$$

where p_* is (1.16). We may assume that $p_* \in (1, \frac{N}{N-\alpha})$. In fact, if $p_* \geq \frac{N}{N-\alpha}$, then for any given $p \in (1, \frac{N}{N-\alpha})$, (1.16) implies that

$$g(s) \leq c_4 s^p + \epsilon, \quad \forall s \in [0, 1].$$

Step 1. To prove that for $n \geq 1$,

$$\begin{aligned} (-\Delta)^\alpha u &= g_n(u) + kt_n^{-\alpha} \nu_n & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \end{aligned} \tag{6.9}$$

admits a nonnegative solution u_n such that

$$M_1(u_n) + M_2(u_n) \leq \bar{\lambda},$$

where $\bar{\lambda} > 0$ independent of n .

To this end, we define the operators $\{\mathcal{T}_n\}$ by

$$\mathcal{T}_n u = \mathbb{G}_\alpha [g_n(u) + kt_n^{-\alpha} \nu_n], \quad \forall u \in L^1_+(\Omega).$$

By Proposition 6.1, we have

$$\begin{aligned} M_1(\mathcal{T}_n u) &\leq c_{44} \|g_n(u) + kt_n^{-\alpha} \nu_n\|_{L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)} \\ &\leq c_{44} [\|g_n(u)\|_{L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)} + k]. \end{aligned} \tag{6.10}$$

In order to deal with $\|g_n(u)\|_{L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)}$, for $\lambda > 0$ we set

$$\begin{aligned} S_\lambda &= \{x \in \Omega : u(x) > \lambda\} \quad \text{and} \quad \omega(\lambda) = \int_{S_\lambda} \rho_{\partial\Omega}^\alpha dx, \\ \|g_n(u)\|_{L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)} &\leq \int_{S_1^c} g(u) \rho_{\partial\Omega}^\alpha dx + \int_{S_1} g(u) \rho_{\partial\Omega}^\alpha dx. \end{aligned} \tag{6.11}$$

We first deal with $\int_{S_1} g(u) \rho_{\partial\Omega}^\alpha dx$. In fact, we observe that

$$\int_{S_1} g(u) \rho_{\partial\Omega}^\alpha dx = \omega(1)g(1) + \int_1^\infty \omega(s)dg(s),$$

where

$$\int_1^\infty g(s)d\omega(s) = \lim_{T \rightarrow \infty} \int_1^T g(s)d\omega(s).$$

It infers by Proposition 3.1 and Proposition 6.1 that there exists $c_{50} > 0$ such that

$$\omega(s) \leq c_{50} M_1(u) s^{\frac{N+\alpha}{N-\alpha}} s^{-\frac{N+\alpha}{N-\alpha}} \tag{6.12}$$

and by (1.16) and Lemma 3.4 with $p = \frac{N+\alpha}{N-\alpha}$, there exist a sequence of increasing numbers $\{T_j\}$ such that $T_1 > 1$ and $T_j^{-\frac{N+\alpha}{N-\alpha}} g(T_j) \rightarrow 0$ when $j \rightarrow \infty$, thus

$$\begin{aligned} \omega(1)g(1) + \int_1^{T_j} \omega(s)dg(s) &\leq c_{50}M_1(u)^{\frac{N+\alpha}{N-\alpha}}g(1) + c_{50}M(u)^{\frac{N+\alpha}{N-\alpha}} \int_1^{T_j} s^{-\frac{N+\alpha}{N-\alpha}} dg(s) \\ &\leq c_{50}M_1(u)^{\frac{N+\alpha}{N-\alpha}}T_j^{-\frac{N+\alpha}{N-\alpha}}g(T_j) + \frac{c_{50}M_1(u)^{\frac{N+\alpha}{N-\alpha}}}{\frac{N+\alpha}{N-\alpha} + 1} \int_1^{T_j} s^{-1-\frac{N+\alpha}{N-\alpha}} g(s)ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{S_1} g(u)\rho^\alpha dx &= \omega(1)g(1) + \int_1^\infty \omega(s) dg(s) \\ &\leq \frac{c_{50}M_1(u)^{\frac{N+\alpha}{N-\alpha}}}{\frac{N+\alpha}{N-\alpha} + 1} \int_1^\infty s^{-1-\frac{N+\alpha}{N-\alpha}} g(s)ds \\ &\leq c_{50}g_\infty M_1(u)^{\frac{N+\alpha}{N-\alpha}}, \end{aligned} \tag{6.13}$$

where $c_{50} > 0$ independent of n .

We next deal with $\int_{S_1^c} g(u)\rho_{\partial\Omega}^\alpha dx$. For $p_* \in (1, \frac{N}{N-2\alpha+\beta})$, we have that

$$\begin{aligned} \int_{S_1^c} g(u)\rho_{\partial\Omega}^\alpha dx &\leq c_4 \int_{S_1^c} u^{p_*} \rho_{\partial\Omega}^\alpha dx + \epsilon \int_{S_1^c} \rho_{\partial\Omega}^\alpha dx \\ &\leq c_4 c_{51} \int_\Omega u^{p_*} dx + c_{51}\epsilon \\ &\leq c_4 c_{51} M_2(u)^{p_*} + c_{51}\epsilon, \end{aligned} \tag{6.14}$$

where $c_{51} > 0$ independent of n .

Along with (6.10), (6.11), (6.13) and (6.14), we derive

$$M_1(\mathcal{T}_n u) \leq c_{44}c_{50}g_\infty M_1(u)^{\frac{N+\alpha}{N-\alpha}} + c_{44}c_4c_{51}M_2(u)^{p_*} + c_{44}c_{51}\epsilon + c_{44}k. \tag{6.15}$$

By [18, Theorem 6.5] and (6.2), we derive that

$$M_2(\mathcal{T}_n u) \leq c_{45}\|g_n(u) + k\nu_n\|_{L^1(\Omega, \rho_{\partial\Omega}^\alpha dx)},$$

which along with (6.11), (6.13) and (6.14), implies that

$$M_2(\mathcal{T}_n u) \leq c_{45}c_{50}g_\infty M_1(u)^{\frac{N+\alpha}{N-\alpha}} + c_{45}c_4c_{51}M_2(u)^{p_*} + c_{45}c_{51}\epsilon + c_{45}k. \tag{6.16}$$

Therefore, inequality (6.15) and (6.16) imply that

$$M_1(\mathcal{T}_n u) + M_2(\mathcal{T}_n u) \leq c_{52}g_\infty M_1(u)^{\frac{N+\alpha}{N-\alpha}} + c_{53}c_4M_2(u)^{p_*} + c_{54}\epsilon + c_{54}k,$$

where $c_{52} = (c_{44}+c_{45})c_{50}$, $c_{21} = (c_{44}+c_{45})c_{51}$ and $c_{54} = c_{44}+c_{45}$. If we assume that $M_1(u)+M_2(u) \leq \lambda$, implies

$$M_1(\mathcal{T}_n u) + M_2(\mathcal{T}_n u) \leq c_{52}g_\infty \lambda^{\frac{N+\alpha}{N-\alpha}} + c_{21}\lambda^{p_*} + c_{21}\epsilon + c_{54}k.$$

Since $\frac{N+\alpha}{N-\alpha}$, $p_* > 1$, then there exist $k_0 > 0$ and $\epsilon_0 > 0$ such that for any $k \in (0, k_0]$ and $\epsilon \in (0, \epsilon_0]$, the equation

$$c_{52}g_\infty \lambda^{\frac{N+\alpha}{N-\alpha}} + c_{21}\lambda^{p_*} + c_{21}c_3\epsilon + c_{54}k = \lambda$$

admits the largest root $\bar{\lambda} > 0$.

We redefine $M(u) = M_1(u) + M_2(u)$, then for $M(u) \leq \bar{\lambda}$, we obtain that

$$M(\mathcal{T}_n u) \leq c_{52}g_\infty \bar{\lambda}^{\frac{N+\alpha}{N-\alpha}} + c_{21}\bar{\lambda}^{p_*} + c_{21}\epsilon + c_{54}k = \bar{\lambda}. \tag{6.17}$$

Especially, we have that

$$\|\mathcal{T}_n u\|_{L^1(\Omega)} \leq c_8 M_1(\mathcal{T}_n u) |\Omega|^{\frac{2\alpha}{N+\alpha}} \leq c_{23}\bar{\lambda} \quad \text{if } M(u) \leq \bar{\lambda}.$$

Thus, \mathcal{T}_n maps $L^1(\Omega)$ into itself. Clearly, if $u_m \rightarrow u$ in $L^1(\Omega)$ as $m \rightarrow \infty$, then $g_n(u_m) \rightarrow g_n(u)$ in $L^1(\Omega)$ as $m \rightarrow \infty$, thus \mathcal{T}_n is continuous. For any fixed $n \in \mathbb{N}$, $\mathcal{T}_n u_m = \mathbb{G}_\alpha [g_n(u_m) + k\nu_n]$ and $\{g_n(u_m) + k\nu_n\}_m$ is uniformly bounded in $L^1(\Omega, \rho^\alpha dx)$, then it follows by Proposition 6.2 that $\{\mathbb{G}_\alpha [g_n(u_m) + k\nu_n]\}_m$ is pre-compact in $L^1(\Omega)$, which implies that \mathcal{T}_n is a compact operator.

Let

$$\mathcal{G} = \{u \in L^1_+(\Omega) : M(u) \leq \bar{\lambda}\}$$

which is a closed and convex set of $L^1(\Omega)$. It infers by (6.17) that

$$\mathcal{T}_n(\mathcal{G}) \subset \mathcal{G}.$$

It follows by Schauder's fixed point theorem that there exists some $u_n \in L^1_+(\Omega)$ such that $\mathcal{T}_n u_n = u_n$ and $M(u_n) \leq \bar{\lambda}$, where $\bar{\lambda} > 0$ independent of n .

In fact, u_n is a classical solution of (6.9). Let O an open set satisfying $O \subset \bar{O} \subset \Omega$. By [27, Proposition 2.3], for $\theta \in (0, 2\alpha)$, there exists $c_{55} > 0$ such that

$$\|u_n\|_{C^\theta(O)} \leq c_{55} \{\|g(u_n)\|_{L^\infty(\Omega)} + kt_n^{-\alpha} \|\nu_n\|_{L^\infty(\Omega)}\},$$

then applied [27, Corollary 2.4], u_n is $C^{2\alpha+\epsilon_0}$ locally in Ω for some $\epsilon_0 > 0$. Then u_n is a classical solution of (6.9). Moreover,

$$\int_{\Omega} u_n (-\Delta)^\alpha \xi dx = \int_{\Omega} g(u_n) \xi dx + k \int_{\Omega} \xi \nu_n dx, \quad \forall \xi \in \mathbb{X}_\alpha. \quad (6.18)$$

Step 2. Convergence. Since $\{g_n(u_n)\}$ and $\{\nu_n\}$ are uniformly bounded in $L^1(\Omega, \rho^\beta dx)$, then by Propostion 6.2, there exist a subsequence $\{u_{n_k}\}$ and u such that $u_{n_k} \rightarrow u$ a.e. in Ω and in $L^1(\Omega)$, and $g_{n_k}(u_{n_k}) \rightarrow g(u)$ a.e. in Ω .

Finally we prove that $g_{n_k}(u_{n_k}) \rightarrow g(u)$ in $L^1(\Omega, \rho^\beta dx)$. For $\lambda > 0$, we set $S_\lambda = \{x \in \Omega : |u_{n_k}(x)| > \lambda\}$ and $\omega(\lambda) = \int_{S_\lambda} \rho^\alpha dx$, then for any Borel set $E \subset \Omega$, we have that

$$\begin{aligned} \int_E |g_{n_k}(u_{n_k})| \rho^\beta dx &= \int_{E \cap S_\lambda^c} g(u_{n_k}) \rho^\beta dx + \int_{E \cap S_\lambda} g(u_{n_k}) \rho^\beta dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^\beta dx + \int_{S_\lambda} g(u_{n_k}) \rho^\beta dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^\beta dx + \omega(\lambda) g(\lambda) + \int_\lambda^\infty \omega(s) dg(s), \end{aligned} \quad (6.19)$$

where $\tilde{g}(\lambda) = \max_{s \in [0, \lambda]} g(s)$.

On the other hand,

$$\int_\lambda^\infty g(s) d\omega(s) = \lim_{T_m \rightarrow \infty} \int_\lambda^{T_m} g(s) d\omega(s).$$

where $\{T_m\}$ is a sequence increasing number such that $T_m^{-\frac{N+\alpha}{N-\alpha}} g(T_m) \rightarrow 0$ as $m \rightarrow \infty$, which could obtained by assumption (1.16) and Lemma 3.4 with $p = \frac{N+\alpha}{N-\alpha}$.

It infers by (6.12) that

$$\begin{aligned} \omega(\lambda) g(\lambda) + \int_\lambda^{T_m} \omega(s) dg(s) &\leq c_{50} g(\lambda) \lambda^{-\frac{N+\alpha}{N-\alpha}} + c_{56} \int_\lambda^{T_m} s^{-\frac{N+\alpha}{N-\alpha}} dg(s) \\ &\leq c_{56} T_m^{-\frac{N+\alpha}{N-\alpha}} g(T_m) + \frac{c_{56}}{\frac{N+\alpha}{N-\alpha} + 1} \int_\lambda^{T_m} s^{-1-\frac{N+\alpha}{N-\alpha}} g(s) ds, \end{aligned}$$

where $c_{56} = c_{50} \frac{N+\alpha}{N-\alpha}$. Pass the limit of $m \rightarrow \infty$, we have that

$$\omega(\lambda) g(\lambda) + \int_\lambda^\infty \omega(s) dg(s) \leq \frac{c_{56}}{\frac{N+\alpha}{N-\alpha} + 1} \int_\lambda^\infty s^{-1-\frac{N+\alpha}{N-\alpha}} g(s) ds.$$

Notice that the above quantity on the right-hand side tends to 0 when $\lambda \rightarrow \infty$. The conclusion follows: for any $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\frac{c_{56}}{\frac{N+\alpha}{N-\alpha} + 1} \int_{\lambda}^{\infty} s^{-1-\frac{N+\alpha}{N-\alpha}} g(s) ds \leq \frac{\epsilon}{2}.$$

Since λ is fixed, together with (6.11), there exists $\delta > 0$ such that

$$\int_E \rho_{\partial\Omega}^{\alpha} dx \leq \delta \implies g(\lambda) \int_E \rho_{\partial\Omega}^{\alpha} dx \leq \frac{\epsilon}{2}.$$

This proves that $\{g \circ u_{n_k}\}$ is uniformly integrable in $L^1(\Omega, \rho_{\partial\Omega}^{\beta} dx)$. Then $g \circ u_{n_k} \rightarrow g \circ u$ in $L^1(\Omega, \rho_{\partial\Omega}^{\beta} dx)$ by Vitali convergence theorem.

Pass the limit of (6.18) as $n_k \rightarrow \infty$ to derive that

$$\int_{\Omega} u(-\Delta)^{\alpha} \xi = \int_{\Omega} g(u) \xi dx + k \int_{\Omega} \frac{\partial^{\alpha} \xi}{\partial \vec{n}^{\alpha}} d\nu, \quad \forall \xi \in \mathbb{X}_{\alpha},$$

thus u is a weak solution of (1.5) and u is nonnegative since $\{u_n\}$ are nonnegative. \square

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