ORBITAL STABILITY OF STANDING WAVES OF A CLASS OF FRACTIONAL SCHRÖDINGER EQUATIONS WITH HARTREE-TYPE NONLINEARITY

YONGGEUN CHO †, MOUHAMED M. FALL ‡, HICHEM HAJAIEJ §, PETER A. MARKOWICH ¶, AND SABER TRABELSI ‖

Abstract. his article is devoted to the mathematical analysis of a class of nonlinear fractional Schrödinger equations with a general Hartree-type integrand. We show the well-posedness of the associated Cauchy problem and prove the existence and stability of standing waves under suitable assumptions on the nonlinearity. Our proofs relies mainly on a contraction argument in mixed functional spaces and the concentration compactness method.

Keywords: Porous media; Darcy law, Navier-Stokes equations, Brinkman-Forchheimer model, global regularity, blow-up, determining modes, Volume elements, determining nodes

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1. Introduction

A partial differential equation is referred to as fractional PDE when it involves derivatives or integrals of fractional order. The concept of fractional calculus started with some speculations of Leibnitz in 1695 in a response to De L'Hopital's question about what would be the n^{th} derivative of a function when $n=\frac{1}{2}$. Leibnitz's answer was: *An apparent paradox, from which one day useful consequences will be drawn.* Based on the mathematical studies and the contributions of several mathematicians, L. Euler, P.S. Laplace, S.F. Lacroix, J.B.J Fourier, N.H. Abel, J. Liouville, B. Riemann and many others and the recent development of applied mathematics, Leibnitz's answer appears today at least half right. Indeed, the definitions of fractional derivatives and integrals are no less rigorous than those of their integer order counterparts even though the fractional calculus appears in the modeling of several non paradoxical physical phenomena.

In particular, the class of PDEs

$$(-\Delta)^s u = f(|u|)u, \quad (-\Delta)^s u = -\frac{1}{2}\mathcal{C}_{N,s}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|x-y|^{N+2s}}\,dy, \tag{1.1}$$

whith $N\in\mathbb{N}^\star$ being the space dimension and $0<s<1$, $\mathcal{C}_{N,s}$ a normalization constant and f a given local or nonlocal nonlinearity, received the interest of the mathematical community and has been widely studied in the last decades. This interest is due to

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†Department of Mathematics, and Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju 561-756, South Korea (mail).

‡ African Institute for Mathematical Sciences of Senegal, AIMS-Senegal, KM 2, Route de Joal, B.P. 14 18. Mbour, Sénégal (mail).

§Department of Mathematics, College of Science, King Saud University P.O. Box 2455, Riyadh 11451, Saudi Arabia.(mail).

¶Division of Math and Computer Sci. and Eng., King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia, (peter.markowich@kaust.edu.sa).

‖Division of Math and Computer Sci. and Eng., King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia, (saber.trabelsi@kaust.edu.sa).

the fact that the fractional Laplacian operator arises naturally in many contexts and concrete applications in various fields, such as optimization, phase transition, the thin obstacle Theorem, anomalous diffusion, financial markets, crystal dislocations, membrane and flame propagation, quantum mechanics, quasi-geostrophic flows, minimal surfaces, water waves, elliptic problems with measure data etc. We refer the reader to Ref. [12] for (non exhaustive) list of references about applications. It is worth noticing that applications appear as well in bioengineering and medicine where the equation of motion of heart valve vibrations and stimuli of neural systems are modeled through fractional differential equations, [30, 31, 32, 33, 40]. The mathematical literature dedicated to the analysis of fractional PDEs is too wide to be fully listed and mathematically situated here. Briefly speaking, equation (1.1) usually derives from the time-dependent fractional nonlinear Schrödinger equation

$$i\partial_t u(t,x)+(-\Delta)^s u(t,x)=f(|u|)\,u(t,x), \quad u(t=0,x)=u_0, \tag{1.2}$$

where ∂_t denotes the partial derivative with respect to time variable t. So far, only two cases have been considered in the mathematical literature, namely

$$f(|u|)=|u|^{p-1}, \tag{1.3a}$$

$$f(|u|)=|u|^2*\frac{1}{|x|^\gamma}, \gamma>0 \tag{1.3b}$$

where $*$ denotes the convolution operator on $\mathbb{R}^N$. The nonlinearity (1.3a) arises in applications like ferromagnets materials such spin glasses, iron ores, cobalt and nickel. It appears as well in the modeling of other phenomena like neural networks and Bose-Einstein condensation. In the latter case, it is essential to study the minimization problem associated to (1.2), namely

$$\mathcal{I}_\lambda=\inf_{u\in H^s(\mathbb{R}^N)}\left\{\mathcal{E}(u):=\frac{1}{2}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}u|^2-F(u)\right)dx, \quad \int_{\mathbb{R}^N}|u|^2\,dx=\lambda\right\}, \tag{1.4}$$

where $F(x)=\int_0^x t\,f(|t|)\,dt$. Indeed, in a Bose-Einstein condensate, the particles are so super-cooled (billionths of degree Kelvin) that they all fall in the lowest quantum state (ground state) and exhibit a quantum behavior macroscopically. Therefore, as a first step it is fundamental to study the quantitative and qualitative properties of the minimizers of (1.4). We refer the reader for instance to Refs. [36, 37, 15, 2, 3, 34, 35, 13]. In the case of the Hartree-type nonlinearity (1.3b) ($N=3, \gamma=1$ and $s=\frac{1}{2}$), the most relevant application arises in relativistic physics. Indeed, the nonlinearity describes the short-term interactions between particles. The associated minimization problem is the following

$$\mathcal{I}_\lambda=\inf_{u\in H^{\frac{1}{2}}(\mathbb{R}^3)}\left\{\mathcal{E}(u):=\frac{1}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{\frac{s}{2}}u|^2-\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\,dy\right)dx,\right.$$
$$\left.\int_{\mathbb{R}^3}|u|^2\,dx=\lambda\right\}. \tag{1.5}$$

The above constrained variational problem plays a fundamental role in the mathematical theory of gravitational collapse of boson stars. In Ref. [28], the authors have showed that (1.4) admits a nonnegative radial solution if and only if $\lambda=\lambda^\star$ where $\lambda^\star$

denotes the critical mass. They have also proved that bosons stars with local mass strictly less than the critical mass are gravitationally stable, whereas those with total mass exceeding the critical mass may undergo a gravitational collapse and we refer the reader as well to Refs [11, 16, 18, 26, 27, 10, 1, 22] for more details about this family of models.

In this paper, we present a complete study of equation (1.1) with a generalized version of nonlinearity (1.3b). We start by showing that the associated Cauchy problem is well-posed under suitable assumptions on the nonlinearity and the power s. Next, we prove the existence of standing waves using a variational approach based on the concentration-compactness method [29], thereby extending some known results (see above). Eventually, we show the orbital stability of such standing waves and characterize the orbit following the classical argument of [6].

2. Main results

In this paper, we are concerned with the mathematical analysis of the following Cauchy problem

$$\mathscr{S}: \quad \begin{cases} i\partial_t \phi + (-\Delta)^s \phi = (G(|\phi|) * V(|x|))\, g(\phi), \\ \\ \phi(t=0,x) = \phi_0. \end{cases}$$

In the system $\mathscr{S}$, $\phi(t,x)$ is a complex-valued function on $\mathbb{R}\times\mathbb{R}^N$ and ϕ_0 is a prescribed initial data in $H^s(\mathbb{R}^N)$. The operator $(-\Delta)^s$ is the fractional Laplacian of power $0<s<1$ defined in (1.1). It is worth mentioning that it is a pseudo-differential operator where $\mathcal{F}[(-\Delta)^s\phi](\xi) = |\xi|^{2s}\mathcal{F}[\phi](\xi)$ with $\mathcal{F}$ being the Fourier transform. The potential V is such that $V(|x|) = |x|^{\beta-N}$ where $\beta > \max\{0, N-2s\}$ and $G:\mathbb{R}^+\to\mathbb{R}^+$, differentiable function. Eventually, the function g is such that for all $z\in\mathbb{C}$, $G(z) = \int_0^{|z|} g(\alpha)\, d\alpha$ where $g:\mathbb{R}^+\to\mathbb{R}$ is continuous and extended to the complex plane by setting $g(z) = \frac{z}{|z|} g(|z|)$ for all $z\in\mathbb{C}$ and $z\neq 0$, that is $g = G'$.

In Ref. [20], the authors studied the associated variational problem

$$\mathcal{I}_\lambda^G = \inf\left\{ \||\xi|^s \mathcal{F}[u](\xi)\|^2_{L^2(\mathbb{R}^N)} - \mathcal{D}(G(|\phi|), G(|\phi|)), u\in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} |u(x)|^2\, dx = \lambda \right\}, \tag{2.1}$$

with the notation

$$\mathcal{D}(G(|\phi|), G(|\phi|)) := \int_{\mathbb{R}^N\times\mathbb{R}^N} G(|u(x)|)\, V(|x-y|)\, G(|u(y)|)\, dx\, dy,$$

for a general nonlinearity G and a kernel $V(|x|) = |x|^{\beta-N}$ where the Sobolev space $H^s(\mathbb{R}^N)$ is given by

$$H^s(\mathbb{R}^N) := \left\{ u\in L^2(\mathbb{R}^N),\ \||\xi|^s \mathcal{F}[u](\xi)\|^2_{L^2(\mathbb{R}^N)} < \infty \right\}.$$

In the critical case, $2s = N-\beta$, they were able to extend the results of [28]. Moreover, in the subcritical case, $2s > N-\beta$, they have also proved the existence and symmetry of all minimizers of (2.1) by using rearrangement techniques. More precisely, they showed that under suitable assumptions on G, one can always take a radial and radially decreasing minimizing sequence of problem (2.1).

Beyond the existence and uniqueness facts, a very important issue related to the nonlinear fractional Schrödinger equation $\mathscr{S}$ is the orbital stability of standing waves. For such an issue, it is essential to show that all minimizing sequences are relatively compact in $H^s(\mathbb{R}^N)$. This is the gist of the paper [6]. The line of attach is the following

- Prove the uniqueness of the solutions of $\mathscr{S}$.
- Prove the conservation of mass and energy of the solutions.
- Prove the relative compactness of all minimizing sequences of the problem (2.1).

Our first result concerns the well-posedness of the system $\mathscr{S}$ with G being nonnegative, differentiable with $G(0)=0$ and for all $\psi\in\mathbb{R}_+$, there exists $\kappa>0$ such that

$$\mathcal{A}_0: \qquad \exists\mu\in\left[2,1+\frac{2s+\beta}{N}\right] \quad \text{s.t.} \quad \begin{cases} G(\psi)\le\kappa(|\psi|^2+|\psi|^\mu), \\ |G'(\psi)|\le\kappa(|\psi|+|\psi|^{\mu-1}). \end{cases}$$

More precisely, we have the following

Theorem 2.1. *Let $0<s<1, 0<\beta<N, N-2s\le\beta, \phi_0\in H^s(\mathbb{R}^N)$ and G such that $\mathcal{A}_0$ holds true. Then, there exists a weak global-in-time solution $\phi(t,x)$ to the system $\mathscr{S}$ such that*

$$\phi\in L^\infty(\mathbb{R};H^s(\mathbb{R}^N))\cap W^{1,\infty}(\mathbb{R};H^{-s}(\mathbb{R}^N)).$$

Moreover, the solution is unique if

- $\mu=2$, $N\ge1$ *and* $2s\ge N-\beta$,
- $\mu>2$, $N=1$ *and* $\frac{1}{2}<s<1$,
- $2<\mu<2+\frac{N}{N-2s}\frac{2s-1-2N+2\beta}{2s-1+N}$, $N\ge3$, $\frac{N}{2(N-1)}<s<1$ *and*

$$N-s+\frac{1}{2}<\beta<\min\Big(N,\frac{3N}{2}-s-\frac{N}{4s}\Big).$$

The existence part of Theorem 2.1 will be shown using a classical contraction argument and the conservation laws associated to the dynamics of system $\mathscr{S}$. For the uniqueness when $\mu=2$ one can easily deduce that the L^2 or H^s norm of the difference of two weak solutions vanishes via Hardy-Sobolev inequality (see [10]). If $\mu>2$, then the situation is quite different from the case $\mu=2$. One cannot use the usual energy estimate (except for the case when the embedding $H^s\hookrightarrow L^\infty$ holds) or the usual Strichartz estimates owing to the regularity loss. Fortunately, the uniqueness part for $\mu>2$ and $N=1$ readily follows from the embedding $H^s\hookrightarrow L^\infty$ for all $s>\frac{1}{2}$. The uniqueness in the case $N\ge3$ will be obtained using mixed norms and weighted Strichartz and convolution inequalities, which require $N\ge3$. It would be relevant to find estimates to handle the uniqueness for $N=2$. Let us mention that in Ref. [21] the authors showed the orbital stability of standing waves in the case of power nonlinearities by assuming energy conservation and time continuity without proving uniqueness, which is an inescapable and quite hard step, especially in the fractional setting.

A solution of system $\mathscr{S}$ is called standing wave solution if it has the form $\phi(t,x) = e^{i\nu t}u(x)$ with $\nu \in \mathbb{R}$ and $u(x)$ solves the following bifurcation problem

$$\tilde{\mathscr{S}}: \quad (-\Delta)^s u = \nu u + \mathcal{N}(\phi),$$

where

$$\mathcal{N}(\phi) := [V(|x|) \star G(\phi)]G'(\phi).$$

In order to study the existence of a solution (κ, u) to the stationary equation $\tilde{\mathscr{S}}$, we use a variational method based on the minimization problem (2.1), namely

$$\mathcal{I}_\lambda = \inf\left\{\mathcal{E}(u), \quad u \in H^s(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u(x)|^2\,dx = \lambda\right\}, \tag{2.2}$$

where λ denotes a nonnegative prescribed number and

$$\mathcal{E}(u) = \frac{1}{2}\|\nabla_s u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\mathcal{D}(G(|u|), G(|u|)).$$

For all function u in the Schwarz class, the kinetic energy is precisely expressed as follows

$$\|\nabla_s u\|^2_{L^2(\mathbb{R}^N)} = \mathcal{C}_{N,s}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy. \tag{2.3}$$

The standing waves solutions of $\tilde{\mathscr{S}}$ will be obtained as critical points of the functional $\mathcal{E}$ with the following extra assumption of G for all $\psi \in \mathbb{R}_+$,

$$\mathcal{A}_1: \quad \begin{cases} \exists 0 < \alpha < 1 + \frac{2s+\beta}{N} \quad s.t. \quad \forall \psi,\ 0 < \psi \ll 1, \quad \exists \kappa > 0 \quad s.t. \quad G(\psi) \geq \kappa\psi^\alpha, \\ G(\theta\psi) \geq \theta^{1+\frac{2s+\beta}{2N}}\,G(\psi). \end{cases}$$

More precisely, we have

Theorem 2.2. *Let $0 < s < 1, 0 < \beta < N \leq \beta + 2s$ and G such that $\mathcal{A}_0$ and $\mathcal{A}_1$ hold true. Then, for all $\lambda > 0$, problem* (2.2) *has a minimizer $u_\lambda \in H^s(\mathbb{R}^N)$ such that $I_\lambda = \mathcal{E}(u_\lambda)$.*

This Theorem will be proved by showing that any minimizing sequence of problem (2.2) is, up to translation, relatively compact in $H^s(\mathbb{R}^N)$. Our argument is based on the concentration-compactness method of P-L. Lions, [29].

The last part of the paper is devoted to the stability of such standing waves. Before going further, let us introduce

$$\hat{\mathcal{I}}_\lambda = \inf\left\{\mathcal{J}(z), \quad z = u + iv \in H^s(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |z|^2\,dx = \lambda\right\},$$

where

$$\begin{aligned} \mathcal{J}(z) &= \frac{1}{2}\|\nabla_s z\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\mathcal{D}(G(|z|), G(|z|)), \\ &= \frac{1}{2}\|\nabla_s u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|\nabla_s v\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\mathcal{D}(G((u^2+v^2)^{\frac{1}{2}}), G((u^2+v^2)^{\frac{1}{2}})) \\ &:= \mathcal{J}(u,v). \end{aligned}$$

Obviously $\mathcal{E}(u)=\mathcal{J}(u,0)$ holds. Now, we let

$$\hat{\mathcal{O}}_\lambda=\left\{z\in H^s(\mathbb{R}^N),\quad \int_{\mathbb{R}^N}|z|^2\,dx=\lambda\,:\,\mathcal{J}(z)=\hat{\mathcal{I}}_\lambda\right\}.$$

The set $\hat{\mathcal{O}}_\lambda$ is the so called orbit of the standing waves of $\mathscr{S}$ with mass $\sqrt{\lambda}$. We define the stability of $\hat{\mathcal{O}}_\lambda$ as follows

Definition 2.3. *Let $\phi_0\in H^s(\mathbb{R}^N)$ be an initial data and $\phi(t,x)\in H^s(\mathbb{R}^N)$ the associated solution of problem $\mathscr{S}$. We say that $\hat{\mathcal{O}}_\lambda$ is $H^s(\mathbb{R}^N)-$stable with respect to the system $\mathscr{S}$ if*

- $\hat{\mathcal{O}}_\lambda\neq\varnothing$.
- *For all $\varepsilon>0$, there exists $\delta>0$ such that for any $\phi_0\in H^s(\mathbb{R}^N)$ satisfying $\inf_{z\in\hat{\mathcal{O}}_\lambda}|\phi_0-z|<\delta$, we have $\inf_{z\in\hat{\mathcal{O}}_\lambda}|\phi(t,x)-z|<\epsilon$ for all $t\in\mathbb{R}$.*

Obviously, the notion of stability depends intimately on the well-posedness of the Cauchy problem $\mathscr{S}$ and the existence of standing waves. Therefore, having in hand Theorems 2.1 and 2.2, we prove the following

Theorem 2.4. *Let $N\geq 3$, $\frac{N}{2(N-1)}<s<1$, $N-s+\frac{1}{2}<\beta<\min(N,\frac{3N}{2}-s-\frac{N}{4s})$ and let G satisfying $\mathcal{A}_0$ and $\mathcal{A}_1$ with μ (in $\mathcal{A}_0$) such that*

$$2<\mu<2+\frac{N}{N-2s}\,\frac{2s-1-2N+2\beta}{2s-1+N}.$$

Let $\phi_0\in H^s(\mathbb{R}^N)$ and $\phi(t,x)\in H^s(\mathbb{R}^N)$ the associated solution to the problem $\mathscr{S}$. Then $\hat{\mathcal{O}}_\lambda$ is $H^s(\mathbb{R}^N)-$stable with respect to the system $\mathscr{S}$.

The paper is organized as follows. The first section is dedicated to the analysis of the dynamical system $\mathscr{S}$, that is the proof of Theorem 2.1. This is achieved in a classical way by showing the local-in-time existence of solutions using a contraction argument. Next, we prove the uniqueness under extra assumptions. Eventually, the conservation laws allow us to obtain the necessary *a priori* estimates to show the global-in-time existence and uniqueness of solutions. The second section is devoted to the proof of Theorem 2.2 using variational tools, namely the concentration-compactness method, [29]. In the last section, we prove Theorem 2.4, that is the stability of standing waves.

From this point onward, η and κ will denote variant universal constants that may change from line to line (possibly of the same inequality). When η depends on some parameters, we will write $\eta(\cdot)$ instead of η and equivalently for κ. In order to lighten the notation and the calculation, we shall use L^p and H^s instead of $L^p(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N)$ respectively for real or complex valued functions. Also, we shall use $\|\cdot\|_{L^p}$ instead of $\|\cdot\|_{L^p(\mathbb{R}^N)}$ for all $p\in[1,\infty]$. The exponent p' will denotes the conjugate exponent of p, that is $\frac{1}{p}+\frac{1}{p'}=1$. For more details on Sobolev spaces H^s, we refer the reader to any textbook of functional analysis (see [7] for instance). Eventually, we shall use the shorthand notation $s_N:=\frac{2N}{N-2s}$.

3. Well-posedness of the system $\mathscr{S}$

In this section we prove Theorem 2.1 by proceeding in three steps. First, we show the existence of local-in-time weak solutions, then we show their uniqueness. Eventually, we derive *a priori* estimates to obtain the global-in-time existence and uniqueness of such weak solutions. Since, $G'=g$, the following observation holds true

$$|g(z)|+|g'(z)z|\leq C(|z|+|z|^{\mu-1})\quad\text{for all}\quad z\in\mathbb{C}. \tag{3.1}$$

3.1. Weak solutions In this subsection, we show the existence of weak solutions to system $\mathscr{S}$ in H^s. We will prove that $\mathcal{N}$ is a Lipschitz map from $L^{p'}$ to L^r for some $p,r\in[2,s_N)$. The rest of the proof follows using a classical contraction method and we refer the reader to the book [7] for instance.

PROPOSITION 3.1. *Let $N\geq 1$, $0<s<1$ and $0<\beta<N\leq \frac{1}{2}\beta s_N$. If g satisfies (3.1) with $\mu\in\left[2,1+\frac{2s+\beta}{N}\right]$. Then there exists a weak solution ϕ such that for all $t\in(-T_{min},T_{max})$*

$$\phi\in L^\infty(-T_{min},T_{max};H^s)\cap W^{1,\infty}(-T_{min},T_{max};H^{-s}),$$

where $(-T_{min},T_{max})$ is the maximal existence time interval of ϕ for given initial data $\phi_0\in H^s$.

Proof. Let us introduce the following cut-off for the function g. We introduce $g_1(\alpha)=\chi_{\{0\leq\alpha<1\}}g(\alpha)$ and $g_2(\alpha)=\chi_{\{\alpha\geq 1\}}g(\alpha)$ so that $g=g_1+g_2$ with obvious definition of the Euler function χ. Moreover, let $G_i(z)=\int_0^{|z|}g_i(\alpha)\,d\alpha$. Then, we write

$$\mathcal{N}(\phi)=\sum_{i,j=1,2}\mathcal{N}_{ij}(\phi)\qquad\text{where}\qquad \mathcal{N}_{ij}(\phi)=\int_{\mathbb{R}^N}\frac{G_i(|\phi(y)|)}{|x-y|^{N-\beta}}\,dy\,g_j(\phi).$$

Now, we claim that for all $1\leq i,j\leq 2$, there exist $p_{ij},r_{ij}\in[2,s_N)$[1], $a_{ij}>0$ and a positive constant $\eta(K)\leq\kappa K^{a_{i,j}}$ such that

$$\|\mathcal{N}_{ij}(\phi)-\mathcal{N}_{ij}(\psi)\|_{p'_{ij}}\leq\eta(K)\|\phi-\psi\|_{r_{ij}},\tag{3.2}$$

holds true provided that $\|\phi\|_{H^s}+\|\psi\|_{H^s}\leq K$. This would implies that $\mathcal{N}:H^s\to H^{-s}$ is a Lipschitz map on bounded sets of H^s. Indeed, let $\mu_1=2$ and $\mu_2=\mu$, then

$$\begin{aligned}|\mathcal{N}_{ij}(\phi)-\mathcal{N}_{ij}(\psi)|\leq{}&\eta\int_{\mathbb{R}^N}\frac{|\phi|^{\mu_i-1}+|\psi|^{\mu_i-1}}{|x-y|^{N-\beta}}|\phi-\psi|\,dy\,|\phi|^{\mu_j-1}\\&+\eta\int_{\mathbb{R}^N}|\psi|^{\mu_i}\frac{|\phi|^{\mu_j-2}+|\psi|^{\mu_j-2}}{|x-y|^{N-\beta}}dy\,|\phi-\psi|.\end{aligned}$$

Applying Hölder and Hardy-Littlewood-Sobolev inequalities, we obtain

$$\begin{aligned}\|\mathcal{N}_{ij}(\phi)-\mathcal{N}_{ij}(\psi)\|_{p'_{ij}}\leq{}&\eta\left[\|\phi\|_{r_{ij}}^{\mu_i-1}+\|\psi\|_{r_{ij}}^{\mu_i-1}\right]\|\phi\|_{r_{ij}}^{\mu_j-1}\|\phi-\psi\|_{r_{ij}}\\&+\eta\,\|\psi\|_{r_{ij}}^{\mu_i}\left[\|\phi\|_{r_{ij}}^{\mu_j-2}+\|\psi\|_{r_{ij}}^{\mu_j-2}\right]\|\phi-\psi\|_{r_{ij}},\end{aligned}$$

provided that for all $1\leq i,j\leq 2$, we have

$$\frac{1}{p_{ij}}-\frac{\beta}{N}+\frac{\mu_i}{r_{ij}}+\frac{\mu_j-1}{r_{ij}}=1\quad\text{and}\quad\frac{\beta}{N}<\frac{\mu_i}{r_{ij}}.\tag{3.3}$$

Now, we show that there exist $p_{ij},r_{ij}\in[2,s_N)$ satisfying the system (3.3) for all $N\geq 2$, $0<s<1$ and $0<\beta<N\leq\frac{1}{2}\beta s_N$ which infers in particular that the inequality (3.2) holds true using Sobolev inequality. For that purpose, let p_{ij},r_{ij} be on the line

$$\frac{1}{r_{ij}}=\frac{1}{\mu_i+\mu_j-1}\left(1+\frac{\beta}{N}-\frac{1}{p_{ij}}\right).\tag{3.4}$$

[1] If $N=1$ and $\frac{1}{2}\leq s<1$, then s_N is interpreted as ∞.

Obviously, $\frac{1}{r_{ij}}$ can be seen as a decaying function of $\frac{1}{p_{ij}}$ and one has

$$\frac{1}{\mu_i+\mu_j-1}\left(\frac{1}{2}+\frac{\beta}{N}\right)<\frac{1}{2} \quad \text{and} \quad \frac{1}{s_N}<\frac{1}{\mu_i+\mu_j-1}\left(1+\frac{\beta}{N}-\frac{1}{s_N}\right)$$

Therefore, the line (3.4) of $(\frac{1}{p_{ij}},\frac{1}{r_{ij}})$ always passes through the open square $(\frac{1}{s_N},\frac{1}{2})^2$. Thus, all what we need is to find a pair $(\frac{1}{p_{ij}},\frac{1}{r_{ij}})$ of line (3.4) such that $\frac{\mu_i}{r_{ij}}>\frac{\beta}{N}$ for all $1\le i,j\le 2$. Actually, it is rather easy to see that we have

$$\frac{\beta}{N}<2\left(1+\frac{2s+\beta}{N}\right)\left(\frac{1}{2}+\frac{s}{N}\right)<\frac{\mu_i}{\mu_j-1}\left(\frac{1}{2}+\frac{s}{N}\right).$$

This inequality implies that

$$\frac{\beta}{N}<\frac{1}{\mu_i+\mu_j-1}\left(1+\frac{\beta}{N}-\frac{1}{s_N}\right),$$

which infers the existence of a infinite number of pairs $(r_{i,j},p_{ij})\in[2,s_N)^2$ satisfying (3.3). The proof of Proposition 3.1 follows now by a straightforward application of a contraction argument. □

3.2. Uniqueness The cases when $\mu=2$, or $\mu>2$ and $N=1$ can be treated as in [10], [7], respectively, and we omit the details. In the case of $N\ge 3$, the uniqueness of weak solutions can be shown by the mean of weighted Strichartz and convolution estimates. For that purpose, we introduce the following mixed norm for all $1\le m,\widetilde{m}<\infty$

$$\|h\|_{L^m_\rho L^{\widetilde{m}}_\sigma}:=(\int_0^\infty(\int_{S^{N-1}}|h(\rho\sigma)|^{\widetilde{m}}\,d\sigma)^{\frac{m}{\widetilde{m}}}\,\rho^{n-1}d\rho)^{\frac{1}{m}}.$$

The case of $m=\infty$ or $\widetilde{m}=\infty$ can be defined in a usual way. Now, we claim the following.

PROPOSITION 3.2. *Let $N\ge 3$, $\frac{N}{2(N-1)}<s<1, N-s+\frac{1}{2}<\beta<\min(N,\frac{3N}{2}-s-\frac{N}{4s})$ and g such that condition (3.1) holds true with*

$$2<\mu<2+\frac{N}{N-2s}\,\frac{2s-1-2N+2\beta}{2s-1+N}.$$

Then the H^s-weak solution to the problem $\mathscr{S}$ constructed in proposition 3.1 is unique.

Remark 3.1. *The restriction on the dimension N in Proposition 3.2 is imposed to ensure the conditions $\frac{N}{2(N-1)}<s<1$ and $N-s+\frac{1}{2}<\beta<\frac{3N}{2}-s-\frac{N}{4s}$. These conditions are needed to guarantee the existence of exponents $q,\tilde{q}$ and r satisfying the equality* (3.8) *below.*

Proof. Let $U(t)=e^{it(-\Delta)^s}$, then the solution ϕ constructed in Proposition 3.1 satisfies a.e. for all $t\in(-T_{min},T_{max})$ the integral equation

$$\phi(t)=U(t)\varphi-i\int_0^t U(t-t')\mathcal{N}(\phi(t'))\,dt'. \tag{3.5}$$

Before going further, let us recall the following weighted Strichartz estimate

Lemma 3.3 (Lemma 6.2 of [10], Lemma 2 of [8]). *Let $N\ge 2$ and $2\le q<4s$. Then, for all $\psi\in L^2$, we have*

$$\||x|^{-\delta}U(t)\psi\|_{L^q(-t_1,t_2;L^q_\rho L^{\tilde q}_\sigma)}\le \eta\,\|\psi\|_{L^2},$$

where $\delta=\frac{N+2s}{q}-\frac{N}{2}$, $\frac{1}{\tilde q}=\frac12-\frac{1}{N-1}\left(\frac{2s}{q}-\frac12\right)$ and η is independent of t_1,t_2.

In Ref. [10], it was shown that

$$\||x|^{-\delta}D_\sigma^{\frac{2s}{q}-\frac12}U(t)\psi\|_{L^q(-t_1,t_2;L^q_\rho L^2_\sigma)}\le \eta\,\|\psi\|_{L^2}.$$

Lemma 3.3 can be derived using Sobolev embedding on the unit sphere. Here $D_\sigma=\sqrt{1-\Delta_\sigma}$ where Δ_σ is the Laplace-Beltrami operator on the unit sphere. Furthermore, let us recall the following weighted convolution inequality

Lemma 3.4 (Lemma 4.3 of [9]). *Let $r\in[1,\infty]$ and $0\le\delta\le\gamma<N-1$. If $\frac1r>\frac{\gamma}{N-1}$, then for all f such that $|x|^{-(\gamma-\delta)}f\in L^1$, we have*

$$\||x|^{\delta}(|x|^{-\gamma}*f)\|_{L^\infty_\rho L^r_\sigma}\le \eta\,\||x|^{-(\gamma-\delta)}f\|_1.$$

Now, using Lemma 3.3 one can readily deduces that

$$\||x|^{-\delta}\int_0^t U(t-t')f(t')\|_{L^q(-t_1,t_2;L^q_\rho L^{\tilde q}_\sigma)}\le \eta\,\|f\|_{L^1(-t_1,t_2;L^2)}. \tag{3.6}$$

Thus, if we set $f=\mathcal N(\phi)-\mathcal N(\psi)$ and $\gamma=N-\beta$. Then from (3.5) we infer

$$\begin{aligned}\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)} &+ \||x|^{-\delta}(\phi-\psi)\|_{L^q(-t_1,t_2;L^q_\rho L^{\tilde q}_\sigma)}\\ &\le \eta\sum_{i,j=1}^{2}\int_{-t_1}^{t_2}\|\mathcal N_{ij}(\phi)-\mathcal N_{ij}(\psi)\|_{L^2}\,dt',\\ &\le \eta\sum_{i,j=1}^{2}\int_{-t_1}^{t_2}\|\int_{\mathbb R^N}|x-y|^{-\gamma}(|\phi|^{\mu_i-1}+|\psi|^{\mu_i-1})|\phi-\psi|\,dy|\phi|^{\mu_j-1}\|_{L^2}\,dt',\\ &+\eta\sum_{i,j=1}^{2}\int_{-t_1}^{t_2}\|\int_{\mathbb R^N}|x-y|^{-\gamma}|\psi|^{\mu_i}\,dy(|\phi|^{\mu_j-2}+|\psi|^{\mu_j-2})|\phi-\psi|\|_{L^2}\,dt',\\ &:=\sum_{i,j=1}^{2}(\mathcal T^1_{ij}+\mathcal T^2_{ij}).\end{aligned}$$

We first estimate $\mathcal T^1_{ij}$ using Hölder's and Hardy-Littlewood-Sobolev inequalities via Lemma 3.3 and (3.6). On the one side if $j=2$, we choose r_1,q_1,δ_1 with $q_1=2(\mu-1),\delta_1=\frac{N+2s}{2(\mu-1)}-\frac{N}{2}$ and

$$\frac12=\frac{\mu_i-1}{r_1}+\frac{\mu-1}{\tilde q_1},\quad \frac{1}{\tilde q_1}=\frac12-\frac{1}{N-1}\left(\frac{s}{\mu-1}-\frac12\right),\quad \frac{1}{r_1}>\frac{N-\beta}{N-1},$$

for which we need $\beta>N-s+\frac{1}{2}$ and $\mu<2+\frac{2\beta+2s-2N-1}{N}$, then our choice of s,μ,β enables us to use (3.6) so that

$$\begin{aligned}\mathcal{T}_{i2}^1 \le \eta\int_{-t_1}^{t_2} \||x|^{-\delta_1(\mu-1)}\int |x-y|^{-\gamma}(|\phi|^{\mu_i-1}+|\psi|^{\mu_i-1})|\phi-\psi|\,dy\|_{L_\rho^\infty L_\sigma^{r_1}} \\ \times \||x|^{-\delta_1}\phi\|^{\mu-1}_{L_\rho^{q_1}L_\sigma^{\tilde q_1}}\,dt' \\ \le \eta\int_{-t_1}^{t_2} \||x|^{(-(\gamma-\delta_1(\mu-1))}(|\phi|^{\mu_i-1}+|\psi|^{\mu_i-1})\|_2\|\phi-\psi\|_2\||x|^{-\delta_1}\phi\|^{\mu-1}_{L_\rho^{q_1}L_\sigma^{\tilde q_1}}\,dt'.\end{aligned}$$

Here we use Hardy-Sobolev inequality such that for $0<q<N$ and $2\le p<\infty$

$$\||x|^{-\frac{q}{p}}f\|_p \le \eta\|f\|_{\dot H^{\frac{N}{2}-\frac{N-q}{p}}}. \tag{3.7}$$

Since in our case $q=2(N-\beta-\delta_1(\mu-1)), p=2(\mu_i-1)$ and $\frac{N}{2}-\frac{N-q}{p}=\frac{N}{2}-\frac{N-2(N-\beta-\delta_1(\mu-1))}{2(\mu_i-1)}\le s$ (for this we need $\mu<2+\frac{\beta+2s-N}{N-2s}$), we have

$$\begin{aligned}\mathcal{T}_{i2}^1 \le \eta\int_{-t_1}^{t_2} (\|\phi\|^{\mu_i-1}_{\dot H^{\frac{N}{2}-\frac{N-2(N-\beta-\delta_1(\mu-1))}{2(\mu_i-1)}}}+\|\psi\|^{\mu_i-1}_{\dot H^{\frac{N}{2}-\frac{N-2(N-\beta-\delta_1(\mu-1))}{2(\mu_i-1)}}}) \\ \times\|\phi-\psi\|_2\||x|^{-\delta_1}\phi\|^{\mu-1}_{L_\rho^{q_1}L_\sigma^{\tilde q_1}}\,dt' \\ \le \eta(t_1+t_2)^{\frac{1}{2}}\left(\|\phi\|^{\mu_i-1}_{L^\infty(-t_1,t_2;H^s)}+\|\psi\|^{\mu_i-1}_{L^\infty(-t_1,t_2;H^s)}\right) \\ \times\||x|^{-\delta_1}\phi\|^{\mu-1}_{L^{q_1}(-t_1,t_2;L_\rho^{q_1}L_\sigma^{\tilde q_1})}\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)}.\end{aligned}$$

On the opposite side, if $j=1$, then we can choose $r\in\left[2,\frac{2N}{N-2s}\right]$ such that

$$\frac{\beta}{N}=\frac{\mu_i}{r},\ \frac{\mu_i-1}{r}+\frac{1}{2}>\frac{\beta}{N}.$$

Such a combination is always possible thanks to our conditions on μ,β and s. Therefore, we get as above

$$\begin{aligned}\mathcal{T}_{ij}^1 \le \eta\int_{-t_1}^{t_2}(\|\phi\|_r^{\mu_i-1}+\|\psi\|_r^{\mu_i-1})\|\phi-\psi\|_2\|\phi\|_r\,dt', \\ \le \eta(t_1+t_2)(1+\|\phi\|_{L^\infty(-t_1,t_2;H^s)}+\|\psi\|_{L^\infty(-t_1,t_2;H^s)})^{\mu_1}\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)}.\end{aligned}$$

We are kept with the estimates of $\mathcal{T}_{ij}^2$. If $j=1$, then we can use Hardy-Sobolev inequality (3.7). In fact, we have

$$\begin{aligned}\mathcal{T}_{11}^2 \le \int_{-t_1}^{t_2}\|\int_{\mathbb{R}^N}|x-y|^{-\gamma}|\psi|^2\,dy\|_{L_x^\infty}\|\phi-\psi\|_2\,dt', \\ \le \eta\int_{-t_1}^{t_2}\|\psi\|^2_{\dot H^{\frac{\gamma}{2}}}\|\phi-\psi\|_2\,dt', \\ \le \eta(t_1+t_2)\|\psi\|^2_{L^\infty(-t_1,t_2;H^s)}\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)}.\end{aligned}$$

Since $\frac{N}{2}-\frac{\beta}{\mu}\le s$ we also have

$$\begin{aligned}\mathcal{T}_{21}^2 &\le \int_{-t_1}^{t_2} \| \int_{\mathbb{R}^N} |x-y|^{-\gamma}|\psi|^{\mu}\,dy\|_{L_x^\infty}\|\phi-\psi\|_2\,dt',\\ &\le C\int_{-t_1}^{t_2}\|\psi\|^{\mu}_{\dot H^{\frac{N}{2}-\frac{\beta}{\mu}}}\|\phi-\psi\|_2\,dt',\\ &\le C(t_1+t_2)\|\psi\|^2_{L^\infty(-t_1,t_2;H^s)}\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)}.\end{aligned}$$

When $j=2$, we use the weighted convolution inequality (Lemma 3.4). The hypothesis on β,μ guarantees the existence of exponents $q,\widetilde{q}$ and r satisfying the conditions of Lemmas 3.3, 3.4 and also the following combination

$$\frac{1}{2}=\frac{(\mu-2)(N-2s)}{2N}+\frac{1}{q}=\frac{1}{r}+\frac{(\mu-2)(N-2s)}{2N}+\frac{1}{\widetilde{q}}. \tag{3.8}$$

For this we actually need $\mu<2+\frac{N}{N-2s}\frac{2\beta+2s-2N-1}{N+2s-1}$, which is less than $2+\min\left(\frac{\beta+2s-N}{N-2s},\frac{2\beta+2s-N-1}{N}\right)$. Hence, using the Hardy-Sobolev inequality (3.7) we write

$$\begin{aligned}\mathcal{T}_{i,2}^2 &\le \int_{-t_1}^{t_2}\||x|^{\delta}\int_{\mathbb{R}^N}|x-y|^{-\gamma}|\psi|^{\mu_i}\,dy\|_{L_\rho^\infty L_\sigma^r}\left(\|\phi\|^{\mu-2}_{\frac{2N}{N-2s}}+\|\psi\|^{\mu-2}_{\frac{2N}{N-2s}}\right)\times\\ &\qquad\qquad\times\||x|^{-\delta}(\phi-\psi)\|_{L_\rho^q L_\sigma^{\widetilde{q}}}\,dt',\\ &\le \eta\int_{-t_1}^{t_2}\||x|^{(-\gamma-\delta)}|\psi|^{\mu_i}\|_1\left(\|\phi\|^{\mu-2}_{H^s}+\|\psi\|^{\mu-2}_{H^s}\right)\||x|^{-\delta}(\phi-\psi)\|_{L_\rho^q L_\sigma^{\widetilde{q}}}\,dt',\\ &\le \eta\int_{-t_1}^{t_2}\|\psi\|^{\mu_i}_{\dot H^{\frac{N}{2}-\frac{\beta+\delta}{\mu_i}}}\left(\|\phi\|^{\mu-2}_{H^s}+\|\psi\|^{\mu-2}_{H^s}\right)\||x|^{-\delta}(\phi-\psi)\|_{L_\rho^q L_\sigma^{\widetilde{q}}}\,dt',\\ &\le \eta(t_1+t_2)^{1-\frac{1}{q}}\left(\|\phi\|^{\mu_i+\mu-2}_{L^\infty(-t_1,t_2;H^s)}+\|\psi\|^{\mu_i+\mu-2}_{L^\infty(-t_1,t_2;H^s)}\right)\times\\ &\qquad\qquad\times\||x|^{-\delta}(\phi-\psi)\|_{L^q(-t_1,t_2;L_\rho^q L_\sigma^{\widetilde{q}})}.\end{aligned}$$

Now, if $(-t_1,t_2)\subset[-T_1,T_2]$ and $\|\phi\|_{L^\infty(-T_1,T_2;H^s)}+\|\psi\|_{L^\infty(-T_1,T_2;H^s)}\le K$, then gathering all the estimates above we infer

$$\begin{aligned}&\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)}+\||x|^{-\delta}(\phi-\psi)\|_{L^q(-t_1,t_2;L_\rho^q L_\sigma^{\widetilde{q}})}\le\eta\,(K^2+K^{2\mu-2})\times\\ &\qquad\times(t_1+t_2)^{1-\frac{1}{q}}\left(\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)}+\||x|^{-\delta}(\phi-\psi)\|_{L^q(-t_1,t_2;L_\rho^q L_\sigma^{\widetilde{q}})}\right).\end{aligned}$$

Thus, $\phi=\psi$ on $[-t_1,t_2]$ for sufficiently small t_1,t_2. Let $I=(-a,b)$ be the maximal interval of $[-T_1,T_2]$ with

$$\|\phi-\psi\|_{L^\infty(-c,d;L^2)}+\||x|^{-\delta}(\phi-\psi)\|_{L^q(-c,d;L_\rho^q L_\sigma^{\widetilde{q}})}=0,\quad c<a\quad\text{and}\quad d<b.$$

Assume that $a<T_1$ or $b<T_2$. Without loss of generality, we may also assume that $a<T_1$ and $b<T_2$. Then for a small $\varepsilon>0$ we can find $a<t_1<T_1$ and $b<t_2<T_2$ such that

$$\begin{aligned}&\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)}+\||x|^{-\delta}(\phi-\psi)\|_{L^q(-t_1,t_2;L_\rho^q L_\sigma^{\widetilde{q}})}\le(K^2+K^{2\mu-2})\times\\ &\times(t_1+t_2-a-b)^{1-\frac{1}{q}}\left(\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)}+\||x|^{-\delta}(\phi-\psi)\|_{L^q(-t_1,t_2;L_\rho^q L_\sigma^{\widetilde{q}})}\right),\\ &\le(1-\varepsilon)\left(\|\phi-\psi\|_{L^\infty(-t_1,t_2;L^2)}+\||x|^{-\delta}(\phi-\psi)\|_{L^q(-t_1,t_2;L_\rho^q L_\sigma^{\widetilde{q}})}\right).\end{aligned}$$

This contradicts the maximality of I. Thus $I=[-T_1,T_2]$. Actually, since $[-T_1,T_2]$ is arbitrarily taken in $(-T_{min},T_{max})$, we get the uniqueness o the whole interval $(-T_{min},T_{max})$ and the Proposition 3.2 is now proved. □

3.3. Global well-posedness Using the argument of Ref. [7], one can show that the uniqueness implies actually well-posedness and conservation laws. That is

- $\phi\in C(-T_{min},T_{max};H^s)\cap C^1(-T_{min},T_{max};H^{-s})$,
- ϕ depends continuously on ϕ_0 in H^s,
- $\|\phi(t)\|_{L^2}=\|\phi_0\|_{L^2}$ and $\mathcal{E}(\phi(t))=\mathcal{E}(\phi_0)$ for all $t\in(-T_{min},T_{max})$.

The proofs of these points are standard (see [7]) and we omit them. The global well-posedness is a consequence of the uniform bound on the H^s norm of $\phi(t)$ we shall obtain, for all $t\in(-T_{min},T_{max})$, in the sequel.

Indeed, we first consider the global existence of weak solutions. Let us assume that ϕ is a weak solution on $(-T_{min},T_{max})$ as in Proposition 3.1. We show that $\|\phi(t)\|_{H^s}$ is bounded for all $t\in(-T_{min},T_{max})$. For this purpose, let us introduce the following notation for all $1\le i,j\le 2$

$$\mathcal{D}(G(|\phi|),G(|\phi|))=\sum_{i,j=1}^{2}\mathcal{D}_{i,j}(|\phi|),\quad \mathcal{D}_{i,j}(|\phi|):=\mathcal{D}(G_i(|\phi|),G_j(|\phi|)). \tag{3.9}$$

Using Hardy-Littlewood-Sobolev and the fractional Gagliardo-Nirenberg inequalities and the assumption $\mathcal{A}_0$, we can write the following estimates

$$\mathcal{D}_{1,1}(|\phi|)\le\eta\,\|\phi\|^4_{\frac{4N}{N+\beta}}\le\eta\,\|\phi\|_2^{4-\frac{N-\beta}{s}}\,\|\phi\|_{\dot{H}^s}^{\frac{N-\beta}{s}}, \tag{3.10}$$

$$\mathcal{D}_{2,2}(|\phi|)\le\eta\,\|u\|^{2\mu}_{\frac{2N\mu}{N+\beta}}\le\eta\,\|\phi\|_2^{2\mu-\frac{N(\mu-1)-\beta}{s}}\,\|\phi\|_{\dot{H}^s}^{\frac{N(\mu-1)-\beta}{s}}, \tag{3.11}$$

$$\mathcal{D}_{1,2}(|\phi|),\mathcal{D}_{2,1}(|\phi|)\le\eta\,\|\phi\|_2^{\mu+2-\frac{N\mu-2\beta}{2s}}\,\|\phi\|_{\dot{H}^s}^{\frac{N\mu-2\beta}{2s}}. \tag{3.12}$$

Since $N-\beta\le 2s$, then $0<\frac{N-\beta}{s}\le 2$ and $4-\frac{N-\beta}{s}\ge 2$. Also, since $2\le\mu<1+\frac{2s+\beta}{N}$, then $0<\frac{N-\beta}{s}\le\frac{N(\mu-1)-\beta}{s}<2$ and $2\mu-\frac{N(\mu-1)-\beta}{s}>2$. Eventually, it is rather easy to see that $\frac{N\mu-2\beta}{2s}\ge 2$ so that $\mu+2-\frac{N\mu-2\beta}{2s}\ge\mu\ge 2$. The estimates above can be summarized as follows with $\mu_1=2$ and $\mu_2=\mu$.

$$\begin{aligned}\mathcal{D}_{i,j}(|\phi|)&\le\eta\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\phi(x)|^{\mu_i}|\phi(y)|^{\mu_j}}{|x-y|^{N-\beta}}\,dxdy,\\ &\le\eta\,\|\phi\|_2^{\mu_i+\mu_j-\gamma_{i,j}}\,\|\phi\|_{\dot{H}^s}^{\gamma_{i,j}}\end{aligned} \tag{3.13}$$

where

$$\gamma_{i,j}=\frac{N}{s}\left(1+\frac{\beta}{N}\right)-\left(\frac{N}{2s}-1\right)(\mu_i+\mu_j).$$

Thus, we have clearly with the conservation laws

$$\begin{aligned}\frac{1}{2}\|\phi\|_{H^s}^2 &= \frac{1}{2}\|\phi\|_{L^2}^2 + \mathcal{E}(\phi) + \mathcal{D}(G(|\phi|), G(|\phi|)),\\ &\le \frac{1}{2}\|\phi_0\|_{L^2}^2 + \mathcal{E}(\phi_0) + \eta \sum_{i,j=1,2} \|\phi_0\|_{L^2}^{\frac{2\gamma_{ij}}{2-\mu_i-\mu_j+\gamma_{ij}}} + \frac{1}{4}\|\phi\|_{H^s}^2.\end{aligned}$$

Thus

$$\|\phi\|_{H^s} \le \eta\left(\|\phi_0\|_{H^s}\right), \quad \text{for all}\, t \in (-T_{min}, T_{max}).$$

Therefore $T_{min} = T_{max} = \infty$. Eventually, the proof of Theorem 2.1 follows by combining this fact with the Proposition 3.1.

4. Existence of standing waves

In this section we study the minimization problem $\tilde{\mathscr{S}}$. We prove the existence of solutions to $\tilde{\mathscr{S}}$ using a variational approach via the concentration-compactness method of P-L. Lions [29]. Indeed, we aim to prove the existence of critical points to the energy functional

$$\mathcal{E}(u) = \frac{1}{2}\|\nabla_s u\|_{L^2}^2 - \frac{1}{2}\mathcal{D}(G(|u|), G(|u|)).$$

In other words, we look for a function u_λ such that

$$\mathcal{E}(u_\lambda) = \mathcal{I}_\lambda = \inf\left\{\mathcal{E}(u), \quad u \in H^s(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u(x)|^2\,dx = \lambda\right\}.$$

As noticed in the introduction of this paper, this problem has been studied in various situations depending on the value of s and the conditions on β and the integrand G in Ref. [10, 20]. In order to prove the existence of critical points to the functional $\mathcal{E}$, we start with the following claim

Proposition 4.1. *For all $\lambda > 0$ and G such that $\mathcal{A}_0$ and $\mathcal{A}_1$ hold true, we have*

- *The functional $\mathcal{E} \in C^1(H^s, \mathbb{R})$ and there exists a constant $\eta > 0$ such that*

$$\|\mathcal{E}'(u)\|_{H^{-s}} \le \eta\left(\|u\|_{H^s} + \|u\|_{H^s}^{\frac{2s+\beta}{N}}\right).$$

- $-\infty < \mathcal{I}_\lambda < 0$.

- *Each minimizing sequence for the problem $\mathcal{I}_\lambda$ is bounded in H^s.*

Proof. Let us mention that only assumption $\mathcal{A}_0$ is needed to prove the C^1 property of the energy functional $\mathcal{E}$. The proof of this assertion is standard and we refer the reader to Ref. [19] for details. Now, we prove the second assertion. Let $u \in H^s(\mathbb{R}^N)$ such that $\|u\|_{L^2} = \sqrt{\lambda}$ and assume that $\mathcal{A}_0$ holds true. Then, on the one hand, thanks to (3.10–3.12), it is rather easy to show using Young's inequality that for all ϵ_1, ϵ_2

and ϵ_3, there exist $C_{\epsilon_1}, C_{\epsilon_2}, C_{\epsilon_3} > 0$ such that

$$\mathcal{D}_{1,1} \leq \eta\left(\epsilon_1 \|u\|_{\dot{H}^s}^2 + C_{\epsilon_1}\lambda^{e_1}\right), \quad e_1 := \frac{4s+\beta-N}{2s+\beta-N}. \tag{4.1}$$

$$\mathcal{D}_{1,2} \leq \eta\left(\epsilon_2 \|u\|_{\dot{H}^s}^2 + C_{\epsilon_2}\lambda^{e_2}\right), \quad e_2 := \frac{2s\mu+\beta-N(\mu-1)}{2s+\beta-N(\mu-1)}. \tag{4.2}$$

$$\mathcal{D}_{1,2}, \mathcal{D}_{2,1} \leq \eta\left(\epsilon_1 \|u\|_{\dot{H}^s}^2 + C_{\epsilon_3}\lambda^{e_3}\right), \quad e_3 := 1 + \frac{2s\mu}{4s-N\mu+2\beta}. \tag{4.3}$$

Observe that $0 < 2s+\beta-N < 4s+\beta-N$ so that $e_1 > 1$. Also, $0 < 2s+\beta-N(\mu-1) < 2s\mu+\beta-N(\mu-1)$ so that $e_2 > 1$. Eventually, $4s-N\mu+2\beta > 2s+\beta-N > 0$ so that $\frac{2s\mu}{4s-N\mu+2\beta} > 0$ and $e_3 > 1$. Therefore, for sufficiently small ϵ_1, ϵ_2 and ϵ_3, one has

$$\begin{aligned}
\mathcal{E}(u) &\geq \left(\frac{1}{2} - \eta\,(\epsilon_1+\epsilon_2+\epsilon_3)\right)\|u\|_{H^s}^2 - \frac{1}{2} - \eta\,(C_{\epsilon_1}\lambda^{e_1} + C_{\epsilon_2}\lambda^{e_2} + C_{\epsilon_3}\lambda^{e_3}), \\
&\geq -\frac{1}{2}\lambda - \eta\,(C_{\epsilon_1}\lambda^{e_1} + C_{\epsilon_2}\lambda^{e_2} + C_{\epsilon_3}\lambda^{e_3}).
\end{aligned}$$

Thus, we obtain $\mathcal{I}_\lambda > -\infty$. On the other hand, let us introduce for all $\delta \in \mathbb{R}$, the rescaled function $u_\delta = \delta^{\frac{1}{2}} u(\delta^{\frac{1}{N}}\cdot)$. Obviously, one has $\int_{\mathbb{R}^N} |u_\delta|^2 = \lambda$ and using $\mathcal{A}_1$

$$\mathcal{E}(u_\delta) \leq \frac{1}{2}\delta^{\frac{2s}{N}} \int_{\mathbb{R}^N} |(-\Delta)^s u(x)|^2 dx - \frac{\delta^{\alpha-\left(1+\frac{\beta}{N}\right)}}{2} \mathcal{D}(|u(x)|^\alpha, |u(y)|^\alpha).$$

We have $0 < \alpha - \left(1+\frac{\beta}{N}\right) < \frac{2s}{N}$, therefore we can take δ small enough to get $\mathcal{E}(u_\delta) < 0$. Thus, $\mathcal{I}_\lambda \leq \mathcal{E}(u_\delta) < 0$. We are kept with the proof of the third assertion. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for the problem $\mathcal{I}_\lambda$. Therefore, thanks to (4.1–4.3), we have for all $u \in H^s$

$$\mathcal{D}(G(|u|), G(|u|)) \leq \eta\,(\epsilon_1+\epsilon_2+\epsilon_3)\|u\|_{\dot{H}^s}^2 + \eta\,(C_{\epsilon_1}\lambda^{e_1} + C_{\epsilon_2}\lambda^{e_2} + C_{\epsilon_3}\lambda^{e_3}).$$

Hence

$$\begin{aligned}
\|u_n\|_{H^s}^2 &= 2\mathcal{E}(u_n) + \|u_n\|_{L^2}^2 + \mathcal{D}(G(|u_n|), G(|u_n|)), \\
&\leq 2\mathcal{I}_\lambda + \lambda + \eta\,(\epsilon_1+\epsilon_2+\epsilon_3)\|u_n\|_{H^s}^2 + \eta\,(C_{\epsilon_1}\lambda^{e_1} + C_{\epsilon_2}\lambda^{e_2} + C_{\epsilon_3}\lambda^{e_3}).
\end{aligned}$$

Eventually, we pick ϵ_1, ϵ_2 and ϵ_3 such that $\eta\,(\epsilon_1+\epsilon_2+\epsilon_3) < 1$, we get immediately that the minimizing sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in H^s. □

Before going further, let us introduce the so called Lévy concentration function

$$\mathcal{Q}_n(r) = \sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n(x)|^2 dx.$$

It is known that each $\mathcal{Q}_n$ is nondecreasing on $(0,+\infty)$. Also, with the Helly's selection Theorem, the sequence $(\mathcal{Q}_n)_{n\in\mathbb{N}}$ has a subsequence that we still denote $(\mathcal{Q}_n)_{n\in\mathbb{N}}$ by abuse of notation, such that there is a nondecreasing function $\mathcal{Q}(r)$ satisfying

$$\mathcal{Q}_n(r) \xrightarrow[n\to+\infty]{} \mathcal{Q}(r), \quad \text{for all} \quad r > 0.$$

Since $0 \leq \mathcal{Q}_n(r) \leq \lambda$, there exists $\beta \in \mathbb{R}$ such that $0 \leq \beta \leq \lambda$ and

$$\mathcal{Q}(r) \xrightarrow[r \to +\infty]{} \gamma.$$

Briefly speaking, a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ for the problem $\mathcal{I}_\lambda$ can only be in one of the following situations:

- Vanishing, i.e. $\gamma = 0$.
- Dichotomy, i.e. $0 < \gamma < \lambda$.
- Compactness, i.e. $\gamma = \lambda$.

In the sequel, we shall proceed by elimination and show that vanishing and dichotomy do not occur. Therefore, compactness is the only possible scenario. Our starting point is the following

PROPOSITION 4.2. *Let $\lambda > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence of problem $\mathcal{I}_\lambda$ with G such that $\mathcal{A}_0$ and $\mathcal{A}_1$ hold true. Then $\gamma > 0$.*

The proposition claims then that the situation of vanishing does not occur. In the proof of Proposition 4.2, we shall use, for all subset of $A \subset \mathbb{R}^N$, the notation

$$\mathcal{D}|_A(G(|u|), G(|u|)) := \int_{A\times A} G(|u(x)|)\, V(|x-y|)\, G(|u(y)|)\, dx\, dy.$$

Proof. Let us first prove that $\mathcal{D}(G(|u_n|), G(|u_n|))$ is lower bounded. In other words, we show that for $n \in \mathbb{N}$ large enough there exists $\delta > 0$ such that

$$\delta < \mathcal{D}(G(|u_n|), G(|u_n|)). \tag{4.4}$$

On the one hand, we argue by contradiction and assume that there exist no such δ. Therefore $\liminf_{n\to+\infty} \mathcal{D}(G(|u_n|), G(|u_n|)) \leq 0$, thus

$$\begin{aligned} \mathcal{I}_\lambda = \lim_{n\to+\infty} \mathcal{E}(u_n) &= \lim_{n\to+\infty} \left(\frac{1}{2} \|\nabla_s u_n\|_{L^2}^2 - \frac{1}{2} \mathcal{D}(G(|u_n|), G(|u_n|)) \right) \\ &\geq -\frac{1}{2} \lim_{n\to+\infty} \mathcal{D}(G(|u_n|), G(|u_n|)) \geq 0. \end{aligned}$$

The inequality above is in contradiction with the fact that $\mathcal{I}_\lambda < 0$. On the other hand, arguing by contradiction and assuming that the minimizing sequence $(u_n)_{n\in\mathbb{N}}$ vanishes, i.e. assuming that $\gamma = 0$. Then, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ and a radius $\tilde{r} > 0$ such that

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,\tilde{r})} |u_{n_k}(x)|^2 dx \xrightarrow[k\to+\infty]{} 0.$$

Next, since the sequence $(u_{n_k})_{k\in\mathbb{N}}$ is bounded in H^s, then one can find $r_\epsilon > 0$ such that

$$\mathcal{D}|_{\{|x-y| \geq r_\epsilon\}}(G(|u_{n_k}|), G(|u_{n_k}|)) \leq \frac{\epsilon}{2}.$$

Now, we cover $\mathbb{R}^N$ by balls of radius r and centers c_i for $i=1,2,\dots$ such that each point of $\mathbb{R}^N$ is contained in at most $N+1$ ball. Therefore, there exists N_ϵ ball and a subsequence $(c_{i_l})_{l=1,\dots,N_\epsilon}$ such that

$$\begin{aligned}
&\mathcal{D}|_{\{|x-y|\geq r_\epsilon\}}(G(|u_{n_k}|),G(|u_{n_k}|)) \leq \eta \sum_{p,q=1}^{2} \mathcal{D}_{p,q}|_{|x-y|\geq r_\epsilon}(|u_{n_k}|),\\
&\leq \eta \sum_{p,q=1}^{2}\sum_{l=1}^{\infty}\sum_{i=1}^{N_\epsilon} \int_{B_x(c_l,r)}\int_{B_y(c_{i_l},r)} \frac{|u_{n_k}(x)|^{\mu_p}|u_{n_k}(y)|^{\mu_q}}{|x-y|^{N-\beta}}dxdy,\\
&\leq \eta \sum_{p,q=1}^{2}\sum_{l=1}^{\infty}\sum_{i=1}^{N_\epsilon} \|u_{n_k}\|_{L^2(B_x(c_l,r))}\| \int_{B_y(c_{l_i},r)} \frac{|u_{n_k}(y)|^{\mu_q}}{|x-y|^{N-\beta}}dy\,|u_{n_k}|^{\mu_p-1}\|_{L^2(B_x(c_l,r))},\\
&\leq N_\epsilon \eta \left(\sum_{l=1}^{\infty}\|u_{n_k}\|_{L^2(B_x(c_l,r))}\right)\|u_{n_k}\|_r \|u_{n_k}\|^{\mu-1}_{\frac{2N}{N-2s}} \sup_{y\in\mathbb{R}^N}\|u_{n_k}\|_{L^2(B(y,r))}\\
&+N_\epsilon\eta \sum_{(p,q)\neq(1,2),p,q=1}^{2}\left(\sum_{l=1}^{\infty}\|u_{n_k}\|_{L^2(B_x(c_l,r))}\right)\|u_{n_k}\|^{\mu_q-1}_{r_{pq}}\|u_{n_k}\|^{\mu_p-1}_{\frac{r_{pq}}{\mu_p-1}}\times\\
&\qquad\qquad\times \sup_{y\in\mathbb{R}^N}\|u_{n_k}\|_{L^2(B(y,r))},
\end{aligned}$$

where r and r_{pq} are such that

$$\frac{\beta}{N}=\frac{1}{r}+\frac{\mu-1}{s_N},\quad \frac{1}{r}+\frac{1}{2}>\frac{\beta}{N},$$

$$\frac{\mu_p-1}{r_{pq}}+\frac{\mu_q-1}{r_{pq}}=\frac{\beta}{N},\quad \frac{\mu_p-1}{r_{pq}}+\frac{1}{2}>\frac{\beta}{N},\quad (p,q)\neq(1,2).$$

Since $1+\frac{2\beta-N}{N-2s}<\mu<1+\frac{\beta+2s}{N}$, $0<\beta<N$ and $s>\frac{N-\beta}{2}$, it is rather clear that one can find (as in section 2) $r,r_{p,q}\in[2,s_N]$ satisfying the algebraic system above. Consequently, we have obviously

$$\begin{aligned}
\mathcal{D}|_{\{|x-y|\geq r_\epsilon\}}(G(|u_{n_k}|),G(|u_{n_k}|)) &\leq (N+1)N_\epsilon\eta\|u_{n_k}\|_{L^2}\times\\
&\times\left(\|u_{n_k}\|^{\mu}_{H^s}+\|u_{n_k}\|^2_{H^s}+\|u_{n_k}\|^{2(\mu-1)}_{H^s}\right)\left(\sup_{y\in\mathbb{R}^N}\int_{B(y,r)}|u_{n_k}|^2\right)^{\frac{1}{2}}.\\
&\xrightarrow[n\to+\infty]{} 0.
\end{aligned}$$

This shows that if the minimizing sequence $(u_n)_{n\in\mathbb{N}}$ vanishes, then

$$\mathcal{D}(G(|u_n|),G(|u_n|))\xrightarrow[n\to+\infty]{}0.$$

This is in contradiction with the property (4.4), namely for $n\in\mathbb{N}$ large enough there exists $\gamma>0$ such that $\mathcal{D}(G(|u_n|),G(|u_n|))>\gamma$. Thus, vanishing does not occur. □

Now, we show the following

PROPOSITION 4.3. *Let $0<\pi<\lambda$ and G such that $\mathcal{A}_0$ and $\mathcal{A}_1$ hold true. Then the mapping $\lambda\mapsto\mathcal{I}_\lambda$ is continuous and $\mathcal{I}_\lambda<\mathcal{I}_\pi+\mathcal{I}_{\lambda-\pi}$.*

Proof. Let $\lambda > 0$ and $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\lambda_k \xrightarrow[k\to+\infty]{} \lambda$. Let $\epsilon > 0$ and $u \in H^s(\mathbb{R}^N)$ such that $\|u\|_{L^2} = \sqrt{\lambda}$ and

$$\mathcal{I}_\lambda \leq \mathcal{E}(u) \leq \mathcal{I}_\lambda + \frac{\epsilon}{2}.$$

For all $k\in\mathbb{N}$, let $u_k = \sqrt{\frac{\lambda_k}{\lambda}}u$. Obviously $u_k \in H^s(\mathbb{R}^N)$ and $\|u_k\|_{L^2}^2 = \lambda_k$ so that for all $k\in\mathbb{N}$ we have $\mathcal{I}_{\lambda_k} \leq \mathcal{E}(u_k)$. Now, we show that $\mathcal{E}(u_k) \xrightarrow[k\to+\infty]{} \mathcal{E}(u)$. First, for all $k\in\mathbb{N}$

$$\|u_k - u\|_{\dot{H}^s} \leq \|u_k\|_{\dot{H}^s}\left|1 - \sqrt{\frac{\lambda_k}{\lambda}}\right|.$$

Since any sequence of $\mathcal{I}_\lambda$ is bounded in $H^s(\mathbb{R}^N)$ and $\lambda_k \xrightarrow[k\to+\infty]{} \lambda$, then we have obviously $\frac{1}{2}\|\nabla_s u_k\|_{L^2}^2 \xrightarrow[k\to+\infty]{} \frac{1}{2}\|\nabla_s u\|_{L^2}^2$. Next, following the first assertion of Proposition 4.1, we have $\mathcal{E}(u) \in C^1(H^s(\mathbb{R}^N),\mathbb{R})$. In particular, one can easily see from the proof of this point (see Ref. [19] for details) that $D(u) := \mathcal{D}(G(|u|),G(|u|)) \in C^1(H^s(\mathbb{R}^N),\mathbb{R})$ and

$$|\mathcal{D}'(u)| \leq \eta\left(\|u\|_{H^s} + \|u\|_{H^s}^{\frac{2s+\beta}{N}}\right). \tag{4.5}$$

Therefore, we have

$$\begin{aligned}
|\mathcal{D}(u_k) - \mathcal{D}(u)| &= \left|\int_0^t \frac{d}{dt}\mathcal{D}(tu_k + (1-t)u)dt\right|, \\
&\leq \eta \sup_{u\in H^s, \|u\|_{H^s}\leq \eta} \|\mathcal{D}'(u)\|_{H^{-s}}\|u_k - u\|_{H^s}, \\
&\leq \eta\|u_k\|_{H^s}\left|1 - \sqrt{\frac{\lambda_k}{\lambda}}\right| \xrightarrow[k\to+\infty]{} 0.
\end{aligned}$$

Thus, we have $\mathcal{E}(u_k) \xrightarrow[k\to+\infty]{} \mathcal{E}(u)$. Consequently, we have $\mathcal{I}_{\lambda_k} \leq \mathcal{I}_\lambda + \epsilon$ for k large enough. Next, for all $k\in\mathbb{N}$, let us choose $\tilde{u}_k \in H^s(\mathbb{R}^N)$ such that $\|\tilde{u}_k\|_{L^2} = \sqrt{\lambda_k}$ and $\mathcal{E}(\tilde{u}_k) \leq \mathcal{I}_{\lambda_k} + \frac{1}{k}$. Moreover, for all $k\in\mathbb{N}$, we set $\bar{u}_k = \sqrt{\frac{\lambda}{\lambda_k}}\tilde{u}_k$. Obviously, since $\bar{u}_k \in H^s(\mathbb{R}^N)$ and $\|\bar{u}_k\|_{L^2}^2 = \lambda$, we have $\mathcal{I}_\lambda \leq \mathcal{E}(\bar{u}_k)$. The same argument as above shows that $\mathcal{E}(\tilde{u}_k) \xrightarrow[k\to+\infty]{} \mathcal{E}(\bar{u})$ so that for k large enough, we have $\mathcal{I}_\lambda \leq \mathcal{I}_{\lambda_k} + \epsilon$. Whence, $\lambda \mapsto \mathcal{I}_\lambda$ is continuous on $\mathbb{R}_+^\star$. Eventually, using the energy estimates (3.10-3.12) or (3.13), it is rather easy to show that $I_\lambda \xrightarrow[\lambda\to 0^+]{} 0$. This shows that the mapping $\lambda \mapsto \mathcal{I}_\lambda$ is continuous.

Let us now prove the strict sub–additivity inequality. For that purpose, we introduce $u_\theta = \theta^\kappa u(\theta^{\frac{\kappa}{N}})$ for all $\kappa > \frac{N}{N+2s}$. Obviously $u_\kappa \in H^s(\mathbb{R}^N)$ and $\|u_\theta\|_{L^2(\mathbb{R}^N)} = \sqrt{\theta\lambda}$. Moreover, using $\mathcal{A}_1$, we have

$$\begin{aligned}
\mathcal{E}(u_\theta) &= \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_\theta|^2 dx - \frac{1}{2}\mathcal{D}(G(|u_\theta|),G(|u_\theta|)), \\
&\leq \frac{\theta^{\kappa\left(1+\frac{2s}{N}\right)}}{2}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx - \mathcal{D}(G(|u|),G(|u|))\right) = \theta^{\kappa\left(1+\frac{2s}{N}\right)}\mathcal{E}(u).
\end{aligned}$$

Thus, we deduce that $\mathcal{I}_{\theta\lambda} \leq \theta^{\kappa\left(1+\frac{2s}{N}\right)}\mathcal{I}_\lambda$ for all $\theta > 0$. Now, we let $0 < \pi < \lambda$. Therefore, since $\kappa\left(1+\frac{2s}{N}\right) > 1$ we have

$$\begin{aligned}\mathcal{I}_\lambda &\leq \lambda^{\kappa\left(1+\frac{2s}{N}\right)}\mathcal{I}_1 < \pi^{\kappa\left(1+\frac{2s}{N}\right)}\mathcal{I}_1 + (\lambda-\pi)^{\kappa\left(1+\frac{2s}{N}\right)}\mathcal{I}_1,\\ &\leq \pi^{\kappa\left(1+\frac{2s}{N}\right)}\pi^{-\kappa\left(1+\frac{2s}{N}\right)}\mathcal{I}_\pi + (\lambda-\pi)^{\kappa\left(1+\frac{2s}{N}\right)}(\lambda-\pi)^{-\kappa\left(1+\frac{2s}{N}\right)}\mathcal{I}_{\lambda-\pi},\\ &= \mathcal{I}_\pi + \mathcal{I}_{\lambda-\pi}.\end{aligned}$$

In summary, for all $0 < \pi < \lambda$, we have $\mathcal{I}_\lambda < \mathcal{I}_\pi + \mathcal{I}_{\lambda-\pi}$. □

Now, we are able to claim the following

Proposition 4.4. *Let $\lambda > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence of problem $\mathcal{I}_\lambda$ with G such that $\mathcal{A}_0$ and $\mathcal{A}_1$ hold true. Then dichotomy does not occur for $(u_n)_{n\in\mathbb{N}}$.*

Proof. Let us introduce ξ^1 and ξ^2 in C^∞ such that $0 \leq \xi^1, \xi^2 \leq 1$ and

$$\xi^1(x) = \begin{cases} 1 \text{ if } |x| \leq 1 \\ \\ 0 \text{ if } |x| \geq 2 \end{cases}, \xi^2(x) = 1 - \xi^1(x), \|\nabla\xi^1\|_{L^\infty}, \|\nabla\xi^2\|_{L^\infty} \leq 2.$$

For all $r > 0$, let $\xi^1_r(\cdot) = \xi^1(\frac{\cdot}{R})$ and $\xi^2_r(\cdot) = \xi^2(\frac{\cdot}{R})$. We will show that dichotomy does not occur by contradicting the fact that for all $0 < \pi < \lambda$, we have $\mathcal{I}_\lambda < \mathcal{I}_\pi + \mathcal{I}_{\lambda-\pi}$ proved in Proposition 4.3. Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence of problem $\mathcal{I}_\lambda$ and assume that dichotomy holds. Then, using the construction of [29], there exist

- $0 < \pi < \lambda$,
- a sequence $(y_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^N$,
- two increasing sequences of positive real number $(r_{1,n})_{n\in\mathbb{N}}$ and $(r_{2,n})_{n\in\mathbb{N}}$ such that

$$r_{1,n} \xrightarrow[n\to+\infty]{} +\infty \quad \text{and} \quad \frac{r_{2,n}}{2} - r_{1,n} \xrightarrow[n\to+\infty]{} +\infty,$$

such that the sequences $u_{1,n} = \xi^1_{r_{1,n}}(\cdot - y_n)u_n$ and $u_{2,n} = \xi^2_{r_{2,n}}(\cdot - y_n)u_n$ satisfy for all $p \in [2, s_N]$ the following properties

$$\begin{cases} u_n = u_{1,n} \quad \text{on} \quad B(y_n, r_{1,n}), \quad u_n = u_{2,n} \quad \text{on} \quad B^c(y_n, r_{2,n}) = \mathbb{R}^N \setminus B(y_n, r_{2,n}) \\ \\ \int_{\mathbb{R}^N} |u_{1,n}|^2 dx \xrightarrow[n\to+\infty]{} \pi, \quad \int_{\mathbb{R}^N} |u_{1,n}|^2 dx \xrightarrow[n\to+\infty]{} \lambda - \pi, \\ \\ \|u_n - (u_{1,n} + u_{2,n})\|_{L^p} \xrightarrow[n\to+\infty]{} 0, \quad \|u_n\|_{L^p(B(y_n,r_{2,n})\setminus B(y_n,r_{1,n}))} \xrightarrow[n\to+\infty]{} 0, \\ \\ \operatorname{dist}(\operatorname{Supp}(u_{1,n}), \operatorname{Supp}(u_{2,n})) \xrightarrow[n\to+\infty]{} +\infty. \end{cases}$$

We obviously have

$$\begin{aligned}\mathcal{E}(u_n) = \mathcal{E}(u_{1,n}) + \mathcal{E}(u_{2,n}) + \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 - \frac{1}{2}\mathcal{D}(G(|u_n|), G(|u_n|))dx \\ - \frac{1}{2}\sum_{i=1}^2 \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_{i,n}|^2 dx - \mathcal{D}(G(|u_{i,n}|), G(|u_{i,n}|))\right).\end{aligned}$$

Now we show the existence of $\epsilon>0$ such that for sufficiently large radius $r_{1,n}$ and $r_{1,n}$ we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}u_n|^2-\sum_{i=1}^{2}|(-\Delta)^{\frac{s}{2}}u_{i,n}|^2\right)dx\geq -\eta\,\epsilon. \tag{4.6}$$

First of all, it is rather easy to show that by construction , we have

$$\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}u_n|^2-\sum_{i=1}^{2}|(-\Delta)^{\frac{s}{2}}u_{i,n}|^2\right)dx \geq -\sum_{i=1}^{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi^i_{r_{i,n}}(x-y_n)-\xi^i_{r_{i,n}}(y-y_n)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dx\,dy.$$

Indeed, the estimate above is justified using the definition (2.3) combined with the following basic fact for $i=1,2$

$$\begin{aligned}|u_{i,n}(x)-u_{i,n}(y)|^2&=|\xi^i_{r_{i,n}}(x-y_n)u_n(x)-\xi^i_{r_{i,n}}(y-y_n)u_n(y)|^2\\&\leq\frac{1}{2}|\xi^i_{r_{i,n}}(x-y_n)-\xi^i_{r_{i,n}}(y-y_n)|^2\left(|u_{i,n}(x)|^2+|u_{i,n}(y)|^2\right)\\&+\frac{1}{2}\left(|\xi^i_{r_{i,n}}(x-y_n)|^2+|\xi^i_{r_{i,n}}(y-y_n)|^2\right)|u_{i,n}(x)-u_{i,n}(y)|^2.\end{aligned}$$

In order to show (4.6), it suffices to show that there exists $\epsilon>0$ such that for large radius $r_{1,n}$ and $r_{2,n}$, we have

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi^i_{r_{i,n}}(x-y_n)-\xi^i_{r_{i,n}}(y-y_n)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dx\,dy\leq\eta\epsilon,\quad i=1,2.$$

For that purpose, we consider the case $i=1$ (the case $i=2$ follows similarly) and we split the sum in two parts as follows

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi^1_{r_{1,n}}(x-y_n)-\xi^1_{r_{1,n}}(y-y_n)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy\\&=\int_{|x-y|\leq r_{1,n}}\frac{|\xi^1_{r_{1,n}}(x-y_n)-\xi^1_{r_{1,n}}(y-y_n)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy\\&+\int_{|x-y|>r_{1,n}}\frac{|\xi^1_{r_{1,n}}(x-y_n)-\xi^1_{r_{1,n}}(y-y_n)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy:=\mathcal{T}_1+\mathcal{T}_2\end{aligned}$$

Now, we write

$$\begin{aligned}\mathcal{T}_1&\leq r_{1,n}^{-2}\int_{|x-y|\leq r_{1,n}}\frac{|u_n(x)|^2}{|x-y|^{N+2s-2}}dxdy\\&\leq r_{1,n}^{-2}\int_{\mathbb{R}^N}|u_n(x)|^2dx\int_{|x|\leq r_{1,n}}\frac{1}{|x|^{N+2s-2}}dx\leq\eta\,r_{1,n}^{-2s}\int_{\mathbb{R}^N}|u_n(x)|^2dx.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{T}_2&\leq r_{1,n}^{-s}\int_{|x-y|>r_{1,n}}\frac{|\xi^1_{r_{1,n}}(x-y_n)-\xi^1_{r_{1,n}}(y-y_n)|^2|u_n(x)|^2}{|x-y|^{N+s}}dxdy\\&\leq\eta\,r_{1,n}^{-s}\int_{\mathbb{R}^N}|u_n(x)|^2dx\int_{|x-y|>r_{1,n}}\frac{1}{|x-y|^{N+s}}dy\leq\eta\,r_{1,n}^{-s}\int_{\mathbb{R}^N}|u_n(x)|^2dx.\end{aligned}$$

Eventually summing up $\mathcal{T}_1$ and $\mathcal{T}_2$ and use the same argument in order to handle the term $\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi^2_{r_{2,n}}(x-y_n)-\xi^2_{r_{2,n}}(y-y_n)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy$, one ends with

$$\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}u_n|^2-\sum_{i=1}^{2}|(-\Delta)^{\frac{s}{2}}u_{i,n}|^2\right)dx\geq -\eta\left(\sum_{i=1}^{2}r_{i,n}^{-2s}+r_{i,n}^{-s}\right)\int_{\mathbb{R}^N}|u_n(x)|^2dx.$$

The estimate (4.6) follows for $r_{1,n}$ and $r_{2,n}$ large enough. Next, observe that $|u_n-u_{1,n}-u_{2,n}|\leq 3\mathbb{1}_{(B(y_n,r_{2,n})\setminus B(y_n,r_{1,n}))}$ where $\mathbb{1}_{(B(y_n,r_{2,n})\setminus B(y_n,r_{1,n}))}$ denotes the characteristic function of $B(y_n,r_{2,n})\setminus B(y_n,r_{1,n})$. Now, we have

$$\begin{aligned}
&|\mathcal{D}(G(|u_n|),G(|u_n|))-\mathcal{D}(G(|v_n|),G(|v_n|))-\mathcal{D}(G(|w_n|),G(|w_n|))|\\
&\leq\int_{B(y_n,2r)\setminus\bar{B}(y_n,2r)}\left(\left|\frac{G(|u_n|)G(|u_n|)}{|x-y|^{N-\beta}}\right|+\left|\frac{G(|v_n|)G(|v_n|)}{|x-y|^{N-\beta}}\right|\right.\\
&\qquad\qquad\left.+\left|\frac{G(|w_n|)G(|w_n|)}{|x-y|^{N-\beta}}\right|\right)dxdy,\\
&\leq\eta\left(\|u\|^{4-\frac{N-\beta}{s}}_{L^2(B(y_n,r_{2,n})\setminus B(y_n,r_{1,n}))}\|u\|^{\frac{N-\beta}{s}}_{H^s}+\|u\|^{2\mu-\frac{N(\mu-1)-\beta}{s}}_{L^2(B(y_n,r_{2,n})\setminus B(y_n,r_{1,n}))}\|u\|^{\frac{N(\mu-1)-\beta}{s}}_{H^s}\right)\\
&+\eta\,\|u\|^{\mu+2-\frac{N\mu-2\beta}{2s}}_{L^2(B(y_n,r_{2,n})\setminus B(y_n,r_{1,n}))}\|u\|^{\frac{N\mu-2\beta}{2s}}_{H^s}\xrightarrow[n\to+\infty]{}0.
\end{aligned}$$

Here, we used the estimates (3.10–3.12). Thus, for $r_{1,n}$ and $r_{2,n}$ large enough, we have

$$-\frac{1}{2}(\mathcal{D}(G(|u_n|),G(|u_n|))-\mathcal{D}(G(|v_n|),G(|v_n|))-\mathcal{D}(G(|w_n|),G(|w_n|)))\geq-\eta\epsilon. \quad (4.7)$$

Summing up (4.6) and (4.7), we end up for large $r_{1,n}$ and $r_{2,n}$ with

$$\mathcal{E}(u_n)-\mathcal{E}(u_{1,n})-\mathcal{E}(u_{2,n})\geq-\eta\epsilon. \quad (4.8)$$

Since we have $\int_{\mathbb{R}^N}|u_{1,n}|^2dx\xrightarrow[n\to+\infty]{}\pi$ and $\int_{\mathbb{R}^N}|u_{1,n}|^2dx\xrightarrow[n\to+\infty]{}\lambda-\pi$, there exist two positive real sequences $(\mu_{1,n})_{n\in\mathbb{N}}$ and $(\mu_{2,n})_{n\in\mathbb{N}}$ such that $|\mu_{1,n}-1|,|\mu_{2,n}-1|<\epsilon$ and

$$\int_{\mathbb{R}^N}|\mu_{1,n}u_{1,n}|^2dx=\pi,\qquad\int_{\mathbb{R}^N}|\mu_{2,n}u_{2,n}|^2dx=\lambda-\pi,$$

so that

$$\begin{aligned}
\mathcal{I}_\pi&\leq\mathcal{E}(\mu_{1,n}u_{1,n})\leq\mathcal{E}(u_{1,n})+\frac{\eta\epsilon}{2},\\
\mathcal{I}_{\lambda-\pi}&\leq\mathcal{E}(\mu_{2,n}u_{2,n})\leq\mathcal{E}(u_{2,n})+\frac{\eta\epsilon}{2}.
\end{aligned}$$

Thus, with (4.8), we have the continuity of the mapping $\lambda\mapsto\mathcal{I}_\lambda$ for all $\lambda>0$ and we have

$$\mathcal{I}_\pi+\mathcal{I}_{\lambda-\pi}-3\eta\epsilon\leq\mathcal{E}(u_{1,n})+\mathcal{E}(u_{2,n})-\eta\epsilon\leq\mathcal{E}(u_n)\xrightarrow[n\to+\infty]{}\mathcal{I}_\lambda.$$

In summary, we proved that for all $0<\pi<\lambda$, we have $\mathcal{I}_\pi+\mathcal{I}_{\lambda-\pi}\leq\mathcal{I}_\lambda$ contradicting the strict sub–additivity inequality proved above. Then, the dichotomy does not occur. ∎

Now, we finish the proof of Theorem 2.2. Since vanishing and dichotomy do not occur for any minimizing sequence $(u_n)_{n\in\mathbb{N}}$ for the problem $\mathcal{I}_\lambda$, then the compactness certainly occurs. Following the concentration-compactness principle [29], we know that every minimizing sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{I}_\lambda$ satisfies (up to extraction if necessary)

$$\lim_{r\to+\infty}\lim_{n\to+\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,r)}|u_n(x)|^2dx=\lambda.$$

That is, for all $\epsilon>0$, there exist $r_\epsilon>0$ and $n_\epsilon\in\mathbb{N}^\star$ and $\{y_n\}\subset\mathbb{R}^N$ such that for all $r>r_\epsilon$ and $n\geq n_\epsilon$, we have

$$\int_{B(y_n,r)}|u_n(x)|^2dx=\lambda-\epsilon.$$

Now, let $w_n=u_n(x+y_n)$, we have obviously that $\|w_n\|_{H^s}=\|u_n\|_{H^s}$ is bounded in $H^s(\mathbb{R}^N)$, therefore $(w_n)_{n\in\mathbb{N}}$ (up to extraction if necessary) converges weakly to w in $H^s(\mathbb{R}^N)$. In particular $(w_n)_{n\in\mathbb{N}}$ converges weakly to w in $L^2(\mathbb{R}^N)$ and $\|w_n\|_{L^2}=\sqrt{\lambda}$. Now, let $\tilde{r}_\epsilon>r_\epsilon$ such that $\|w\|_{L^2(B^c(0,\tilde{r}_\epsilon))}<\frac{\epsilon}{2}$. Thus, there exists $\tilde{n}_\epsilon\in\mathbb{N}^\star$, $\tilde{n}_\epsilon>n_\epsilon$ such that for all $n\geq\tilde{n}_\epsilon$, we have $\|w_n-w\|_{L^2(B(0,\tilde{r}_\epsilon))}<\frac{\epsilon}{2}$. Therefore, with the triangle inequality, we have

$$\begin{aligned}\|w\|_{L^2}&\geq\|u_n\|_{L^2}-\|w_n-w\|_{L^2(B(0,\tilde{r}_\epsilon))}-\|w_n-w\|_{L^2(B^c(0,\tilde{r}_\epsilon))},\\&\geq\|u_n\|_{L^2(B(y_n,\tilde{r}_\epsilon))}-\|w_n-w\|_{L^2(B(0,\tilde{r}_\epsilon))}-\|w\|_{L^2(B^c(0,\tilde{r}_\epsilon))}\geq\sqrt{\lambda-\epsilon}-\epsilon.\end{aligned}$$

Passing to the limit we get $\|w\|_{L^2}\geq\sqrt{\lambda}$. Since the L^2 norm is lower semi continuous, we obtain that $\|w\|_{L^2}\leq\liminf_{n\to+\infty}\|w_n\|_{L^2}=\sqrt{\lambda}$. Eventually, we get $\|w\|_{L^2}=\sqrt{\lambda}$, therefore the sequence $(w_n)_{n\in\mathbb{N}}$ converges strongly in $L^2(\mathbb{R}^N)$ to w.

Also, we have

$$\begin{aligned}|\mathcal{D}(G(|w_n|),G(|w_n|))-D(G(|w|),G(|w|))|&\leq\left|\int_0^t\frac{d}{dt}\mathcal{D}(tG(|w_n|)+(1-t)G(|w|))\,dt\right|,\\&\leq\eta\sup_{u\in H^s,\|u\|_{H^s}\leq\eta}\|\mathcal{D}'(u)\|_{H^{-s}}\|w_n-w\|_{H^s},\\&\leq\eta\,\|w_n-w\|_{L^2}+\eta\,\|w_n-w\|_{\frac{2s+\beta}{N}}\xrightarrow[n\to+\infty]{}0.\end{aligned}$$

In the last line we used (4.5)-kind inequality and again we refer to [19] for a proof. Using the lower semi-continuity of the $-s$ norm, we have $\|w\|_{H^s}\leq\liminf_{n\to+\infty}\|w_n\|_{H^s}$. Summing up, we get clearly

$$\mathcal{I}_\lambda\leq\mathcal{E}(w)\leq\liminf_{n\to+\infty}\mathcal{E}(w_n)=\mathcal{I}_\lambda.$$

This shows that w is a minimizer of $\mathcal{I}_\lambda$ and $w_n\xrightarrow[n\to+\infty]{}w$ in $H^s(\mathbb{R}^N)$. Theorem 2.1 is now proved.

5. Stability of standing waves

In this section, we prove Theorem 2.4 by showing the orbital stability of standing waves in the sense of Definition 2.3. We argue par contradiction. Assume that $\hat{\mathcal{O}}_\lambda$

is not stable, then either $\hat{\mathcal{O}}_\lambda$ is empty or there exist $w \in \hat{\mathcal{O}}_\lambda$ and a sequence $\phi_0^n \in H^s$ such that $\|\phi_0^n - w\|_{H^s} \xrightarrow[n\to+\infty]{} 0$ as $n \to \infty$ but

$$\inf_{z\in\hat{\mathcal{O}}_\lambda} \|\phi^n(t_n,.) - z\|_{H^s} \geq \varepsilon, \tag{5.1}$$

for some sequence $t_n \subset \mathbb{R}$, where $\phi^n(t_n,.)$ is the solution of the Cauchy problem $\mathscr{S}$ corresponding to the initial condition ϕ_0^n.

Now let $w_n = \phi^n(t_n,.)$, since $\mathcal{J}(w) = \hat{\mathcal{I}}_\lambda$, it follows from the continuity of the L^2 norm and $\mathcal{J}$ in H^s that $\|\phi_0^n\|_{L^2} \xrightarrow[n\to+\infty]{} \sqrt{\lambda}$ and $\mathcal{J}(w_n) = \mathcal{J}(\phi_0^n) = \hat{\mathcal{I}}_\lambda$. With the conservation of mass and energy associated to the dynamics of the system $\mathscr{S}$, we deduce that

$$\|w_n\|_{L^2} = \|\phi_0^n\|_{L^2} \xrightarrow[n\to+\infty]{} \sqrt{\lambda} \quad \text{and} \quad \mathcal{J}(w_n) = \mathcal{J}(\phi_0^n) \xrightarrow[n\to+\infty]{} \hat{\mathcal{I}}_\lambda.$$

Therefore if $(w_n)_{n\in\mathbb{N}}$ has a subsequence converging to an element $w \in H^s$, then $\|w\|_{L^2} = \sqrt{\lambda}$ and $\mathcal{J}(w) = \hat{\mathcal{I}}_c$. This shows that $w \in \hat{\mathcal{O}}_\lambda$, but

$$\inf_{z\in\hat{\mathcal{O}}_\lambda} \|\phi^n(t_n,.) - z\|_{H^s} \leq \|w_n - w\|_{H^s}$$

contradicting (5.1).

In summary, to show the orbital stability of $\hat{\mathcal{O}}_\lambda$, one has to prove that $\hat{\mathcal{O}}_\lambda$ is not empty and that any sequence $(w_n)_{n\in\mathbb{N}} \subset H^s$ such that

$$\|w_n\|_{L^2} \xrightarrow[n\to+\infty]{} \sqrt{\lambda} \quad \text{and} \quad \mathcal{J}(w_n) \xrightarrow[n\to+\infty]{} \hat{\mathcal{I}}_\lambda, \tag{5.2}$$

is relatively compact in H^s (up to a translation). From now on, we consider a sequence $(w_n)_{n\in\mathbb{N}}$ satisfying (5.2). Our aim is to prove that it admits a convergent subsequence to an element $w \in H^s$.

If $(w_n)_{n\in\mathbb{N}} = (u_n, v_n)_{n\in\mathbb{N}} \subset H^s$, it is easy to see that $(|w_n|)_{n\in\mathbb{N}} \subset H^s$. Thanks to $\mathcal{A}_0$, we have that $(w_n)_{n\in\mathbb{N}}$ is bounded in H^s and hence by passing to a subsequence, there exists $w = (u,v) \in H^s$ such that

$$\begin{cases} u_n \text{ converges weakly to } u \text{ in } H^s, \quad v_n \text{ converges weakly to } v \text{ in } H^s, \\ \lim_{n\to+\infty} \|\nabla_s u_n\|_{L^2} + \|\nabla_s v_n\|_{L^2} \text{ exists .} \end{cases} \tag{5.3}$$

Now, a straightforward calculation shows that

$$\mathcal{J}(w_n) - \mathcal{E}(|w_n|) = \frac{1}{2}\|\nabla_s w_n\|_{L^2}^2 - \frac{1}{2}\|\nabla_s |w_n|\|_{L^2}^2 \geq 0. \tag{5.4}$$

Thus we have $\hat{\mathcal{I}}_\lambda = \lim_{n\to+\infty} \mathcal{J}(w_n) \geq \limsup_{n\to+\infty} \mathcal{E}(|w_n|)$. However, since $\||w_n|\|_{L^2}^2 = \|w_n\|_{L^2}^2 = \lambda_n \xrightarrow[n\to+\infty]{} \lambda$, by the continuity of the mapping $\lambda \mapsto \mathcal{I}_\lambda$ (see Proposition 4.3), we obtain

$$\liminf_{n\to+\infty} \mathcal{E}(|w_n|) \geq \liminf_{n\to+\infty} \mathcal{I}_{\lambda_n} = \mathcal{I}_\lambda \geq \hat{\mathcal{I}}_\lambda. \tag{5.5}$$

Hence $\lim_{n\to+\infty}\mathcal{J}(w_n)=\lim_{n\to+\infty}\mathcal{E}(|w_n|)=\mathcal{I}_\lambda=\hat{\mathcal{I}}_\lambda$. The properties (5.3) and the inequalities (5.4) and (5.5) imply that

$$\lim_{n\to+\infty}\|\nabla_s u_n\|_{L^2}^2-\|\nabla_s v_n\|_{L^2}^2-\|\nabla_s(u_n^2+v_n^2)^{1/2}\|_{L^2}^2=0, \tag{5.6}$$

which is equivalent to say that

$$\lim_{n\to+\infty}\|\nabla_s w_n\|^2=\lim_{n\to+\infty}\|\nabla_s|w_n|\|_{L^2}^2. \tag{5.7}$$

The convergence $\||w_n|\|_{L^2}^2=\|w_n\|_{L^2}^2=\lambda_n\xrightarrow[n\to+\infty]{}\lambda$, the inequality (5.5) and Theorem 2.2 imply that $|w_n|$ is relatively compact in H^s (up to a translation). Therefore, there exists $\varphi\in H^s$ such that $(u_n^2+v_n^2)^{1/2}\to\varphi$ in H^s and $\|\varphi\|_{L^2}=\sqrt{\lambda}$ with $\mathcal{E}(\varphi)=I_\lambda$. Let us prove that $\varphi=|w|=(u^2+v^2)^{1/2}$. Using (5.3), it follows that $u_n\xrightarrow[n\to+\infty]{}u$ and $v_n\xrightarrow[n\to+\infty]{}v$ in $L^2(B(0,R))$. Now, using the fact that $|(u_n^2+v_n^2)^{1/2}-(u^2+v^2)^{1/2}|^2\le |u_n-u|^2+|v_n-v|^2$, we get $(u_n^2+v_n^2)^{1/2}\xrightarrow[n\to+\infty]{}(u^2+v^2)^{1/2}$ in $L^2(B(0,R))$ which leads to the desired result. On the other hand $\||w_n|\|_{L^2}=\|w_n\|_{L^2}\xrightarrow[n\to+\infty]{}\sqrt{\lambda}=\|w\|_{L^2}=\||w|\|_{L^2}$. Therefore, all what we need is to prove that $\lim_{n\to\infty}\|\nabla_s w_n\|_{L^2}^2=\|\nabla_s w\|_{L^2}^2$. Thanks to (5.7), we have that $\lim_{n\to+\infty}\|\nabla_s w_n\|_{L^2}^2=\lim_{n\to+\infty}\|\nabla_s|w_n|\|_{L^2}^2$ and $\lim_{n\to+\infty}\|\nabla_s|w_n|\|_{L^2}^2=\|\nabla_s|w|\|_{L^2}^2$. Hence by the lower semi-continuity of $\|\nabla_s\cdot\|_{L^2}$, we get that $\|\nabla_s w\|_{L^2}^2\le\lim_{n\to+\infty}\|\nabla_s|w_n|\|_{L^2}^2=\|\nabla_s|w|\|_{L^2}^2$. Eventually, using (5.4), it follows that $\|\nabla_s w\|_{L^2}^2\ge\|\nabla_s|w|\|_{L^2}^2$. Since by (5.3), we know that w_n converges weakly to w in H^s, it follows that $w_n\xrightarrow[n\to+\infty]{}w$ in H^s and the proof is now completed.

Now, we turn to the characterization of the Orbit $\hat{\mathcal{O}}_\lambda$. We show the following

Proposition 5.1. *With the same assumptions of Theorem 2.4, we have*

$$\hat{\mathcal{O}}_\lambda=\left\{e^{i\sigma}w(.+y),\quad \sigma\in\mathbb{R},\quad y\in\mathbb{R}^N\right\},$$

and w is a minimizer of Problem $\mathcal{I}_\lambda$.

Proof. Let $z=(u,v)\in\hat{\mathcal{O}}_\lambda$ and set $\varphi=(u^2+v^2)^{1/2}$. By the previous section, we know that $\mathcal{E}(\varphi)=I_\lambda$, thus φ satisfies $\tilde{\mathscr{S}}$. with ν being a Lagrange multiplier. Furthermore the equality $\|\nabla_s w\|_2=\|\nabla_s|w|\|_2$ implies that

$$u(x)v(y)-v(x)u(y)=0. \tag{5.8}$$

By Proposition 5.2, it is plain that $\varphi\in C(\mathbb{R}^N)$ and $V(|x|)\star G(|\varphi|)\in C(\mathbb{R}^N)$. We can write $(-\Delta)^s\varphi=\nu\varphi+V(|x|)\star G(|\varphi|)\frac{G'(\varphi)}{\varphi}\chi_{\{\varphi\neq0\}}\varphi$ with χ_A being the characteristic function of the set A. Since φ is nontrivial and $V(|x|)\star G(|\varphi|)\frac{G'(\varphi)}{\varphi}\chi_{\{\varphi\neq0\}}\in L^\infty_{loc}(\mathbb{R}^N)$, we conclude that $\varphi>0$ in $\mathbb{R}^N$ by the Harnack inequality (see Lemma 4.9 in [5]) and a standard argument of intersecting balls. There is three cases, namely $u\equiv0$, $v\equiv0$ and the third case corresponds to $u\neq0$ and $v\neq0$. For simplicity, we investigate the latter case, the other cases can be treated easily. The equality (5.8) implies that $\frac{u(x)}{v(x)}=\frac{u(y)}{v(y)}$ for all $x,y\in\mathbb{R}^N$. Thus, there exists α such that $u(x)=\alpha v(x)$ and therefore $z=(\alpha+i)v=e^{i\sigma}w$ with $w=|z|$.

Let us now prove (5.8). By the fact that $\mathcal{J}(z)=\hat{\mathcal{I}}_\lambda$, we can find a Lagrange multiplier $\alpha\in\mathbb{C}$ such that $\mathcal{J}'(z)(\xi)=\dfrac{\alpha}{2}\displaystyle\int_{\mathbb{R}^N} z\bar{\xi}+\xi\bar{z}$ for all $\xi\in H^s$. Putting $\xi=z$, it follows immediately that $\alpha\in\mathbb{R}$ and $V:=V(|x-y|)$

$$\begin{cases}\displaystyle\int_{\mathbb{R}^N}\nabla_s u\nabla_s f-\int_{\mathbb{R}^N\times\mathbb{R}^N}G(u^2+v^2)^{1/2}(y)V\,dy\,G'(f(x))dx=\alpha\int_{\mathbb{R}^N}u(x)f(x)dx,\\[2ex] \displaystyle\int_{\mathbb{R}^N}\nabla_s v\nabla_s f-\int_{\mathbb{R}^N\times\mathbb{R}^N}G(u^2+v^2)^{1/2}(y)V\,dy\,G'(f(x))dx=\alpha\int_{\mathbb{R}^N}v(x)f(x)dx,\end{cases}$$

for all $f\in H^s$. It follows that u and v solve the following system

$$\begin{cases}(-\Delta)^s u-\displaystyle\int G(u^2+v^2)^{1/2}(y)V(|x-y|)dy\,G'(u(x))+\alpha u(x)=0,\\ (-\Delta)^s v-\displaystyle\int G(u^2+v^2)^{1/2}(y)V(|x-y|)dy\,G'(v(x))+\alpha v(x)=0.\end{cases}$$

By Proposition 5.2 (see the Appendix), we have that u and $v\in C(\mathbb{R}^N)$ because $(u^2+v^2)^{1/2}\in H^s(\mathbb{R}^N)$. Let $\Omega=\{x\in\mathbb{R}^N \quad \text{such that} \quad u(x)=0\}$, obviously Ω is closed since u is continuous. Let us prove that it is also open. Suppose that $x_0\in\Omega$. Since $\varphi(x_0)>0$, we can find a ball B centered in x_0 such that $v(x)\neq 0$ for any $x\in B$. Replacing u and v in (5.6), we certainly have $u(x)v(y)-v(x)u(y)=0$ for all $x,y\in B$. This proves the result. □

Appendix. In this appendix, we prove the following

Proposition 5.2. *Let $s\in(0,1), N-2s\le\beta<N, \beta>0, u,\varphi\in H^s(\mathbb{R}^N)$, G such that $\mathcal{A}_0$ holds true and ν is a real number such that*

$$(-\Delta)^s u=\nu u+[V\star G(\varphi)]G'(u). \tag{5.9}$$

Then, there exists $\alpha\in(0,1)$ depending only on N,ν,s,β such that $u\in C^{0,\alpha}_{loc}(\mathbb{R}^N)$. Moreover, if $\varphi\in L^\infty_{loc}(\mathbb{R}^N)$, then $u\in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ if $\beta\le 1$ and $u\in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ if $\beta>1$ and in addition $V\star G(\varphi)\in C^{0,\alpha}_{loc}(\mathbb{R}^N)$.

Proof. We start by recalling the Gagliardo-Nirenberg inequality

$$\|\varphi\|_{L^p}\le c_{N,s,p}\|\varphi\|_{H^s} \quad \text{for all } \varphi\in H^s(\mathbb{R}^N)\ ,$$

for $p\in[2,s_N]$ if $N>2s$ and for all $p\in[2,s_N)$ and $2s\ge N$ (here we put $s_N\equiv+\infty$). Also we recall the Hardy-Littlewood-Sobolev inequality:

$$\|V\star g\|_{L^{\frac{qN}{N-q\beta}}}\le C_{N,\beta,q}\|g\|_{L^q} \quad \text{for every } g\in L^q,$$

for $N-q\beta>0$.

First of all we focus on the case $N>2s$. We have

$$\|G(\varphi)\|_{L^q}\le\kappa\|\varphi^2\|_{L^q}+\kappa\||\varphi|^\mu\|_{L^q}=\kappa\|\varphi\|^2_{L^{2q}}+\kappa\|\varphi\|^\mu_{L^{\mu q}}.$$

Hence, since $\varphi\in H^s$, we infer that $G(\varphi)\in L^q$ provided that $1\le q\le\frac{s_N}{2}$ and $\frac{2}{\mu}\le q\le\frac{s_N}{\mu}$, that is $1\le q\le\frac{s_N}{2}$ and $1\le q\le\frac{Ns_N}{N+2s+\beta}$. Now, thanks to the fact that $N-2s\le\beta<N$,

we get $1 < \frac{N}{\beta} \le \frac{s_N}{2}$ and $1 < \frac{N}{\beta} \le \frac{N s_N}{N+2s+\beta}$. In particular, we deduce that $G(\varphi) \in L^q$ for all $q \in \left[1, \frac{N}{\beta}\right]$. Now, for all $\epsilon > 0$ we let $q_\epsilon = \frac{N}{\beta} - \epsilon > 1$. Using the Hardy-Littelwood-Sobolev inequality, we get $V \star G(\varphi) \in L^{\frac{N q_\epsilon}{\epsilon\beta}}$ which in turns with the fact that $\beta \ge N - 2s$ shows that $V \star G(\varphi) \in L^r$ for all $r > \frac{N}{N-\beta} \ge \frac{N}{2s}$. Now, using the notation $b(x) = \frac{G'(u)}{1+|u|}$ and $sign(u) = \frac{u}{|u|}$, we reformulate the equation (5.9) as follows

$$\begin{aligned}(-\Delta)^s u(x) &= \nu u(x) + [V \star G(\varphi)]\, b(x)\, (1 + |u|)),\\ &= \int_{\mathbb{R}^N} V(|x-y|) G(\varphi(y)) dy\, b(x)\, (1 + sign(u) u).\end{aligned}$$

Observing that $\mu - 2 < s_N - 2 = \frac{4s}{N-2s}$, then for all $r > \frac{N}{2s}$, we can write

$$\begin{aligned}\|[V \star G(\varphi)] b\|_{L^r} &= \|[V \star G(\varphi)] \frac{G'(u)}{1+|u|}\|_{L^r},\\ &\le \kappa \|[V \star G(\varphi)] \frac{|u| + |u|^{\mu-1}}{1+|u|}\|_{L^r},\\ &\le \kappa \|V \star G(\varphi)\|_{L^r} + \kappa \|[V \star G(\varphi)] |u|^{\mu-2}\|_{L^r}.\end{aligned}$$

In order to deduce that the right hand side of this estimate is finite, we use Hölder's inequality to get

$$\|[V \star G(\varphi)]\, |u|^{\mu-2}\|_{L^r}^r \le \|V \star G(\varphi)]\|_{L^{\frac{r\theta}{\theta-1}}} \||u|\|_{L^{r(\mu-2)\theta}}^{\mu-2},$$

for all $\theta > 1$. Therefore, we can choose $r > \frac{N}{2s}$ and $\theta > 1$ respectively close to $\frac{N}{2s}$ and 1 so that $1 < r(\mu-2)\theta < s_N$. Hence, using the Gagliardo-Nirenberg inequality and the fact that $V \star G(\varphi) \in L^r$ for all $r > \frac{N}{N-\beta} \ge \frac{N}{2s}$ and $u \in H^s$, we end up with $[V \star G(\varphi)] b \in L^r$ for some $r > \frac{N}{2s}$. Thus $u \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$.

Now, we write the equation (5.9) as follows

$$\begin{aligned}&(-\Delta)^s u(x) = c(x) u(x) + d(x):\\ &c(x) = \nu + [V \star G(\varphi)]\, b(x)\, sign(u) \in L^r,\\ &d(x) = [V \star G(u)]\, b(x) \in L^r,\end{aligned}$$

for some $r > \frac{N}{2s}$. From now on $[\cdot]$ will stands for the integer part of $\cdot$. Using the regularity result of Ref. [39], we conclude that $u \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ provided $\frac{N}{2s} > 1$. If $N = 1$ and $s > \frac{1}{2}$, then it is well-known that H^s is embedded in $C^{0,\alpha}_{loc}(\mathbb{R}^N)$ with $\alpha = s - \frac{1}{2} - \left[s - \frac{1}{2}\right]$ so that $u \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$. Moreover, if $N = 1$ and $s = \frac{1}{2}$, we have obviously $u \in L^p$ for every $p \ge 2$ and the classical elliptic regularity yields $u \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.

Let us introduce a cutoff function $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $\eta \equiv 1$ in the closed ball B_R of center 0 and radius $R > 0$ and $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_{2R}$. To alleviate the notation, we denote $f = G(\varphi)$ which belongs to $L^\infty_{loc} \cap L^q$ with $1 < q \le \frac{N}{\beta}$. We define $V_1(\varphi) := V \star (\eta f)$ and $V_2(\varphi) := V \star ((1-\eta) f)$. Then, using Fourier transform, we get $(-\Delta)^{\frac{\beta}{2}} V_1(\varphi) = f$ in the sense of distributions. Now, if $\frac{\beta}{2} \in \mathbb{N}^\star$, then it is rather easy to show using the classical regularity theory that $V \star G(\varphi) \in C^\beta(\mathbb{R}^N)$. Next, if $0 < \frac{\beta}{2} < 1$, then we apply Proposition 2.1.9 of Ref. [38] to show that $V_1(\varphi) \in C^{0,\alpha}(\mathbb{R}^N)$ for $\beta \le 1$ and $V_1(\varphi) \in$

$C^{[\beta],\alpha}(\mathbb{R}^N)$ for $\beta>1$ and some $\alpha\in(0,1)$. Now, $V_2(\varphi)$ is smooth on B_R since it is $\frac{\beta}{2}-$harmonic in such a ball (see Ref. [4]). Hence, $V\star G(\varphi)\in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for $\beta\le 1$ and $V\star G(\varphi)\in C^{[\beta],\alpha}_{loc}(\mathbb{R}^N)$ for $\beta>1$ and some $\alpha\in(0,1)$. Let us now turn to the case of $\frac{\beta}{2}>1$ and $\frac{\beta}{2}\notin\mathbb{N}$. We let $\sigma=\frac{\beta}{2}-\left[\frac{\beta}{2}\right]$. Using Fourier transform, we have

$$(-\Delta)^{\left[\frac{\beta}{2}\right]}V_1(\varphi)=(-\Delta)^{\left[\frac{\beta}{2}\right]}((-\Delta)^{\sigma}V_1(\varphi))=\eta\, f,$$

in the sense of distributions. Again, the classical regularity theory implies that $(-\Delta)^{\sigma}V_1(\varphi)\in C^{[\beta]}(\mathbb{R}^N)$ and so $V_1(\varphi)\in C^{[\beta]}(\mathbb{R}^N)$. Similarly, we have

$$(-\Delta)^{\frac{\beta}{2}}V_2(\varphi)=(-\Delta)^{\sigma}\left((-\Delta)^{\left[\frac{\beta}{2}\right]}V_2(\varphi)\right)=(1-\eta)\, f,$$

in the sense of distributions. Therefore the function $g:=(-\Delta)^{\left[\frac{\beta}{2}\right]}V_2(\varphi)$ is given by

$$(-\Delta)^{\left[\frac{\beta}{2}\right]}V_2(\varphi)(x)=\int_{\mathbb{R}^N}\frac{(1-\eta(y))\,f(y)}{|x-y|^{N-\sigma}}\,dy.$$

Also, using the Hardy-Littelwood-Sobolev inequality, it is rather straightforward to see that $g\in L^p$ for some $p>1$. Thus, g belongs to the set $\left\{u,\int_{\mathbb{R}^N}\frac{|u(x)|}{1+|x|^{N+2\sigma}}\,dx<+\infty\right\}$. Again, since g is $\sigma-$harmonic in B_R, we deduce that g is smooth on B_R by Ref. [4]. The radius R being arbitrary, it follows that $V_2(\varphi)$ is smooth on $\mathbb{R}^N$. In particular, we have $V_2(\varphi)\in C^{[\beta]}(\mathbb{R}^N)$ because $\left[\frac{\beta}{2}\right]$ is a positive integer. Recalling that we showed $V_1(\varphi)\in C^{[\beta]}(\mathbb{R}^N)$, we conclude $V\star G(\varphi)\in C^{[\beta]}(\mathbb{R}^N)$.

Let us now summarize and conclude the proof. We considered the partial differential equation (5.9) and proved that for some $\alpha\in(0,1)$, $V\star G(\varphi)\in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for $\beta\le 1$ and $V\star G(\varphi)\in C^{[\beta],\alpha}_{loc}(\mathbb{R}^N)$ for $\beta>1$. Since G' is locally Lipschitz, we deduce that $u\in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for $\beta\le 1$ and $u\in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ for $\beta>1$ by adapting the proof of Lemma 3.3 of Ref. [14] for $N>2s$. If $N=1$ and $2s\ge 1$, we have that $[V\star G(\varphi)]\,G'(u)\in C^{0,\gamma}_{loc}(\mathbb{R}^N)$ for some $\gamma\in(0,1)$, thus using Proposition 2.1.8 of Ref. [38], we get $u\in C^{1,\alpha}_{loc}(\mathbb{R}^N)$. □

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