A complete study of the lack of compactness and existence results of a Fractional Nirenberg Equation via a flatness hypothesis: Part I

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Abstract. In this paper, we consider a nonlinear critical problem involving the fractional Laplacian operator arising in conformal geometry, namely the prescribed σ-curvature problem on the standard n-sphere $n \geq 2$. Under the assumption that the prescribed function is flat near its critical points, we give precise estimates on the losses of the compactness and we provide existence results. In this first part, we will focus on the case $1 < \beta \leq n - 2\sigma$, which was not included in the results of Jin, Li and Xiong [14] and [15].

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1 Introduction and main results

Fractional calculus has attracted a lot of scientists during the last decades. This is essentially due to its numerous applications in various domains: Medicine, modeling populations, biology, earthquakes, optics, signal processing, astrophysics, water waves, porous media, nonlocal diffusion, image reconstruction problems; see [13] and references [1, 2, 6, 7, 13, 14, 19, 22, 25, 36, 38, 41, 43, 45, 46, 58] therein.

Many important properties of the Laplacian are not inherited, or are only partially satisfied, by its fractional powers. This gave birth to many challenging and rich mathematical problems. However, the literature remained quite silent until the publication of the breakthrough paper of Caffarelli and Silvester in 2007, [11]. This seminal work has hugely contributed to unblock a lot of difficult problems and opened the way for the resolution of many other ones. In this paper, we study another important fractional PDE

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whose resolution also requires some novelties because of the nonlocal properties of the operator present in it. More precisely, we investigate the existence of solutions for the following critical fractional nonlinear equation

$$P_\sigma u = c(n,\sigma) K u^{\frac{n+2\sigma}{n-2\sigma}}, \qquad u > 0, \text{ on } S^n. \tag{1.1}$$

where $\sigma \in (0,1)$, K is a positive function defined on (S^n, g),

$$P_\sigma = \frac{\Gamma(B + \frac{1}{2} + \sigma)}{\Gamma(B + \frac{1}{2} - \sigma)}, \qquad B = \sqrt{-\Delta_g + \Big(\frac{n-1}{2}\Big)^2},$$

Γ is the Gamma function, $c(n,\sigma) = \Gamma(\frac{n}{2} + \sigma)/\Gamma(\frac{n}{2} - \sigma)$, and Δ_g is the Laplace-Beltrami operator on (S^n, g). The operator P_σ can be seen more concretely on $\mathbb{R}^n$ using stereographic projection. The stereographic projection from $S^n \setminus \{N\}$ to $\mathbb{R}^n$ is the inverse of $F : \mathbb{R}^n \to S^n \setminus \{N\}$ defined by

$$F(x) = \Big(\frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1}\Big)$$

where N is the north pole of S^n. For all $f \in C^\infty(S^n)$, we have

$$(P_\sigma(f)) \circ F = \Big(\frac{2}{1+|x|^2}\Big)^{\frac{-(n+2\sigma)}{2}} \Big(-\Delta\Big)^\sigma \Big(\Big(\frac{2}{1+|x|^2}\Big)^{\frac{n-2\sigma}{2}} (f \circ F)\Big) \tag{1.2}$$

where $\big(-\Delta\big)^\sigma$ is the fractional Laplacian operator (see, e.g., page 117 of [17]).

Problem (1.1) is heavily connected to the fractional order curvature, usually called the σ-curvature. This challenging problem has been first addressed in [14] and [15]. In these two seminal papers, the authors have been able to show the existence of solutions of (1.1) and to derive some compactness properties. More precisely, thanks to a very subtle approach based on approximation of the solutions of (1.1) by a blowing-up subcritical method, they proved the existence of solutions for the critical fractional Nirenberg problem (1.1), (see Theorem 1.1 and Theorem 1.2 of [14]). Their method is based on tricky variational tools, in particular they have established many interesting fractional functional inequalities. Their main hypothesis is the so-called flatness condition:
Let $K : \mathbb{S}^n \to \mathbb{R}$, be a C^2 positive function. We say that K satisfies a flatness condition $(f)_\beta$: if for each critical point y of K there exist $(b_i)_{i\leq n} \in \mathbb{R}^*$, such that in some geodesic normal coordinate centered at y, we have

$$K(x) = K(y) + \sum_{i=1}^{n} b_i |(x-y)_i|^\beta + R(x-y), \tag{1.3}$$

where $b_i = b_i(y) \in \mathbb{R}^*$, $\sum_{i=1}^n b_i \neq 0$ and $\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta-s} = o(1)$ a y tends to zero. Here ∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β.

However, they have only been able to handle the case $n-2\sigma<\beta<n$ in the flatness hypothesis. This excludes some very interesting functions K. In fact, note that an important class of functions which is worth to include in any results of existence for (1.1) are the Morse functions (C^2 having only non-degenerate critical points). Such functions can be written in the form $(f)_\beta$ with $\beta=2$. Since Jin, Li and Xiong require $n-2\sigma<\beta<n$ ($0<\sigma<1$), their theorems do not apply to this relevant class of functions. Moreover, they require some additional technical assumptions (K antipodally symmetric in Theorem 1.1 and $K\in C^{1,1}$ positive in Theorem 1.2).

Motivated by the breakthrough papers [14] and [15] and aiming to include a larger class of functions K in the existence results for (1.1), we develop in this paper a self-contained approach which enables us to include all the plausible cases ($1<\beta<n$). Our method hinges on a readapted characterization of critical points at infinity techniques of the proof are different for $1<\beta\leq n-2\sigma$ and $n-2\sigma\leq\beta<n$. In this work, we will handle the first case.

The spirit of this approach goes back to the work of Bahri [3] and Bahri-Coron [5]. Nevertheless, the nonlocal properties of the fractional Laplacian involve many additional obstacles and require some novelties in the proof. Note that in [1], the two first authors have given an existence result for $n=2,\ 0<\sigma<1$, through an Euler-Hopf type formula. In their paper, they assumed that K is a Morse function satisfying the following non-degeneracy condition:

$$(nd) \qquad \Delta K(y)\neq 0 \ \text{ whenever } \ \nabla K(y)=0.$$

We point out that the criterium of [1] has an equivalent in dimension three (see [2]). However, the method cannot be generalized to higher dimensions $n\geq 4$ under the condition (nd), since the corresponding index-Counting-Criteria, when taking into account all the critical points at infinity is always equal to 1.

Convinced that the non-degeneracy assumption would exclude some interesting class of functions K, we opted for the flatness hypothesis used in [14] and [15]. But again, in order to include all plausible cases (both $1<\beta\leq n-2\sigma$ and $n-2\sigma\leq\beta<n$), we need to develop a new line of attack with new ideas. This leads to an interesting new phenomenon; that is the presence of multiple blow-up points. In fact, looking to the possible formations of blow-up points, it turns out that the strong interaction of the bubbles in the case where $n-2\sigma<\beta<n$ forces all blow-up points to be single, while in the case where $1<\beta<n-2\sigma$ such an interaction of two bubbles is negligible with respect to the self interaction, while if $\beta=n-2\sigma$ there is a phenomenon of balance that is the interaction of two bubbles is of the same order with respect to the self interaction. In order to state our results, we need the following notations and assumptions. Let

$$\mathcal{K}=\{y\in S^n, \nabla K(y)=0\}$$

$$\mathcal{K}^+=\{y\in\mathcal{K}, -\textstyle\sum_{k=1}^n b_k(y)>0\}$$

$$\widetilde{i}(y) = \sharp\Big\{ b_k = b_k(y),\ 1 \le k \le n \text{ such that } b_k < 0 \Big\}.$$

$$\mathcal{K}_{n-2\sigma} = \big\{ y \in \mathcal{K},\ \beta = \beta(y) = n - 2\sigma \big\}.$$

For each p-tuple, $p \ge 1$ of distinct points $\tau_p := (y_{l_1}, ..., y_{l_p}) \in (\mathcal{K}_{n-2\sigma})^p$, we define a p×p symmetric matrix $M(\tau_p) = (m_{ij})$ by

$$m_{ii} = \frac{n-2\sigma}{n}\widetilde{c}_1 \frac{-\sum_{k=1}^n b_k\big(y_{l_i}\big)}{K\big(y_{l_i}\big)^{\frac{n}{2\sigma}}}, \quad m_{ij} = 2^{\frac{n-2\sigma}{2}} c_1 \frac{-G\big(y_{l_i}, y_{l_j}\big)}{\big[K\big(y_{l_i}\big)K\big(y_{l_j}\big)\big]^{\frac{n-2\sigma}{4\sigma}}}, \tag{1.4}$$

where

$$G(y_{l_i}, y_{l_j}) = \frac{1}{(1 - \cos d(y_{l_i}, y_{l_j}))^{\frac{n-2\sigma}{2}}} \tag{1.5}$$

$$c_1 = \int_{\mathbb{R}^n} \frac{dx}{\big(1 + |x|^2\big)^{\frac{n+2\sigma}{2}}} \quad \text{and} \quad \widetilde{c}_1 = \int_{\mathbb{R}^n} \frac{|x_1|^{n-2}}{\big(1 + |x|^2\big)^n} dx.$$

Here x_1 is the first component of x in some geodesic normal coordinates system. Let $\rho(\tau_p)$ be the least eigenvalue of $M\big(\tau_p\big)$.

$\big(A_1\big)$ Assume that $\rho\big(\tau_p\big) \neq 0$ for each distinct points $y_1, ..., y_p \in \mathcal{K}_{n-2\sigma}$.

Now, we introduce the following sets:

$$\mathcal{C}^\infty_{n-2\sigma} := \big\{ \tau_p = (y_{l_1}, ..., y_{l_p}) \in (\mathcal{K}_{n-2\sigma})^p, p \ge 1, s.t.\ y_i \neq y_j\ \ \forall i \neq j,\ \text{and } \rho(\tau_p) > 0 \big\},$$

$$\mathcal{C}^\infty_{<(n-2\sigma)} := \big\{ \tau_p = (y_{l_1}, ..., y_{l_p}) \in (\mathcal{K}^+ \backslash \mathcal{K}_{n-2\sigma})^p, p \ge 1, s.t.\ y_i \neq y_j\ \ \forall i \neq j \big\}.$$

For any $\tau_p = (y_{l_1}, ..., y_{l_p}) \in (\mathcal{K})^p$, we denote $i(\tau_p)_\infty = p - 1 + \sum_{j=1}^{p} [\, n - \widetilde{i}\,(y_{l_j})\,]$.

The main result of this paper is the following.

Theorem 1.1 *Assume that K satisfies (A_1) and $(f)_\beta$, with $1 < \beta \le n - 2\sigma$. If*

$$\sum_{\tau_p \in \mathcal{C}^\infty_{n-2\sigma}} (-1)^{i(\tau_p)_\infty} + \sum_{\tau'_p \in \mathcal{C}^\infty_{<(n-2\sigma)}} (-1)^{i(\tau'_p)_\infty}$$

$$- \sum_{(\tau_p, \tau'_p) \in \mathcal{C}^\infty_{n-2\sigma} \times C^\infty_{<(n-2\sigma)}} (-1)^{i(\tau_p)_\infty + i(\tau'_p)_\infty} \neq 1,$$

then (1.1) has at least one solution.

In part 2, we will address the case $n - 2\sigma \le \beta < n$, following another approach and recovering the main results of [14] and [15]. More precisely, we will prove:

Theorem 1.2 *Assume that K satisfies (A_1) and $(\mathfrak{f})_\beta$, with $n-2\sigma \leq \beta < n$. If*

$$\sum_{y \in \mathcal{K}^+ \backslash \mathcal{K}_{n-2\sigma}} (-1)^{i(y)_\infty} + \sum_{\tau_p \in \mathcal{C}^\infty_{n-2\sigma}} (-1)^{i(\tau_p)_\infty} \neq 1$$

then (1.1) has at least one solution.

We organize the remainder of our paper as follows. The second section is devoted to recall some preliminary results ralated to the Caffarelli-Silvestre method (see [11]). In section three, we characterize the critical points at infinity of the associated variational problem. In the fourth section, we give the proof of the main results. The characterization of critical points at infinity requires some technical results which for the convenience of the reader, are given in the appendix.

2 Preliminary results

In this section, we recall some preliminary results ralated to the Caffarelli-Silvestre extension (see [11]), which provides a variational structure to the fractional problem.
We say that $u \in H^\sigma(S^n)$ is a solution of (1.1) if the identity

$$\int_{S^n} P_\sigma u \varphi dx = c(n,\sigma) \int_{S^n} K u^{\frac{n+2\sigma}{n-2\sigma}} \varphi dx, \tag{2.1}$$

holds for all $\varphi \in H^\sigma(S^n)$, where $H^\sigma(S^n) = \{u \in L^2(S^n), \|u\|^2_{H^\sigma(S^n)} \in L^2(S^n)\}$, equipped with the norm,

$$\|u\|_{H^\sigma(S^n)} = \Big(\int_{S^n} P_\sigma u u\Big)^{1/2}. \tag{2.2}$$

We recall that the set of smooth functions $C^\infty(S^n)$ is dense in $H^\sigma(S^n)$. Observe also that for $u \in H^\sigma(S^n)$, we have $u^{\frac{n+2\sigma}{n-2\sigma}} \in L^{\frac{2n}{n+2\sigma}}(S^n) \hookrightarrow H^{-\sigma}(S^n)$.

We associate to problem (1.1), the functional

$$I(u) = \frac{1}{2}\int_{S^n} u P_\sigma u - \frac{n-2\sigma}{2n}\int_{S^n} K u^{\frac{2n}{n-2\sigma}}, \tag{2.3}$$

defined in $H^\sigma(S^n)$.

Motivated by the work of Caffarelli and Silvestre [11], several authors have considered an equivalent definition of the operator P_σ by means of an auxiliary variable, see [11], (see also [8], [9], [10], [12] and [16]). In fact, we handle problem (1.1), through a localization method introduced by Caffarelli and Silvestre on the Euclidean space $\mathbb{R}^n$, through which (1.1) is connected to a degenerate elliptic differential equation in one dimension higher by a Dirichlet to Neumann map. This provides a good variational structure to the problem. By studying this problem with classical local techniques, we establish existence of positive

solutions. Here the Sobolev trace embedding comes into play, and its critical exponent $2^* = \frac{2n}{n-2\sigma}$.
Namely, let $D_n = S^n \times [0,\infty)$. Given $u \in H^\sigma(S^n)$, we define its harmonic extension $U = E_\sigma(u)$ to D_n as the solution to the problem

$$\begin{cases} -div(t^{1-2\sigma}\nabla U) & = 0 \text{ in } D_n \\ U & = u \text{ on } S^n \times \{t=0\}. \end{cases} \tag{2.4}$$

The extension belongs to the space $H^1(D_n)$ defined as the completion of $C^\infty(D_n)$ with the norm

$$\|U\|_{H^1(D_n)} = \Big(\int_{D_n} t^{1-2\sigma} |\nabla U|^2 dx dt \Big)^{1/2}. \tag{2.5}$$

Observe that this extension is an isometry in the sense that

$$\|E_\sigma(u)\|_{H^1(D_n)} = \|u\|_{H^\sigma(S^n)}, \qquad \forall u \in H^\sigma(S^n). \tag{2.6}$$

Moreover, for any $\varphi \in H^1(D_n)$, we have the following trace inequality

$$\|\varphi\|_{H^1(D_n)} \geq \|\varphi(.,0)\|_{H^\sigma(S^n)}. \tag{2.7}$$

The relevance of the extension function $U = E_\sigma(u)$ is that it is related to the fractional Laplacian of the original function u through the formula

$$-\lim_{t\to 0^+} t^{1-2\sigma} \frac{\partial U}{\partial t}(x,t) = P_\sigma u(x). \tag{2.8}$$

Thus, we can reformulate (1.1) to the following

$$\begin{cases} div(t^{1-2\sigma}\nabla U(x,t)) & = 0 & \text{and } U > 0 \text{ in } D_n \\ -\lim_{t\to 0^+} t^{1-2\sigma}\frac{\partial U}{\partial t}(x,t) & = KU^{\frac{n+2\sigma}{n-2\sigma}}(x,0) & \text{on } S^n \times \{0\}. \end{cases} \tag{2.9}$$

The functional associated to (2.9), is given by

$$I_1(U) = \frac{1}{2}\int_{D_n} t^{1-2\sigma}|\nabla U|^2 dx dt - \frac{n-2\sigma}{2n}\int_{S^n} KU^{\frac{2n}{n-2\sigma}} dx, \tag{2.10}$$

defined in $H^1(D_n)$.
Note that critical points of I_1 in $H^1(D_n)$ correspond to critical points of I in $H^\sigma(S^n)$. That is, if U satisfies (2.9), then the trace u on $S^n \times 0$ of the function U will be a solution of problem (1.1). Let also define the functional

$$J(U) = \frac{\|U\|^2_{H^1(D_n)}}{\Big(\int_{S^n} KU^{\frac{2n}{n-2\sigma}} dx \Big)^{\frac{n-2\sigma}{n}}}, \tag{2.11}$$

defined on Σ the unit sphere of $H^1(D_n)$. We set, $\Sigma^+ = \{U \in \Sigma / U \geq 0\}$. Problem (1.1) will be reduced to finding the critical points of J under the constraint $U \in \Sigma^+$. The exponent $\frac{2n}{n-2\sigma}$ is critical for the Sobolev trace embedding $H^1(D_n) \to L^q(\mathbb{S}^n)$. This embedding is continuous and not compact. The functional J does not satisfy the Palais-Smale condition, which leads to the failure of the standard critical point theory. This means that there exists a sequence (u_n) belonging to the constraint such that $J(u_n)$ is bounded, its gradient goes to zero and does not converge. The analysis of sequences failing PS condition can be analyzed along the ideas introduced in [5] and [18].

In order to describe such a characterization in our case, we need to introduce some notations.
For $a \in \partial\mathbb{R}^{n+1}_+$ and $\lambda > 0$, define the function:

$$\tilde{\delta}_{a,\lambda}(x) = \bar{c}\frac{\lambda^{\frac{n-2\sigma}{2}}}{\Big((1+\lambda x_{n+1})^2 + \lambda^2|x'-a'|^2\Big)^{\frac{n-2\sigma}{2}}}$$

where $x \in \mathbb{R}^{n+1}_+$, and $\bar{c}$ is chosen such that $\tilde{\delta}_{a,\lambda}$ satisfies the following equation,

$$\begin{cases} \Delta U & = 0 \quad \text{and } u > 0 \text{ in } \mathbb{R}^{n+1}_+ \\ -\dfrac{\partial U}{\partial x_{n+1}} & = u^{\frac{n+2\sigma}{n-2\sigma}} \qquad\qquad \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

Set

$$\delta_{a,\lambda} = i^{-1}(\tilde{\delta}_{a,\lambda}).$$

where i is an isometry from $H^1(D^n)$ to $D^{1,2}(\mathbb{R}^{n+1}_+)$.
In the sequel, we will identify $\delta_{a,\lambda}$ and its composition with i. We will also identify the function u and its extension U. These facts will be assumed as understood in the sequel.
For $\varepsilon > 0$, $p \in \mathbb{N}^*$, we define

$$V(p,\varepsilon) = \begin{cases} u \in \Sigma \text{ s. t } \exists a_1,\dots,a_p \in S^n, \exists \alpha_1,\dots,\alpha_p > 0 \text{ and} \\ \exists \lambda_1,\dots,\lambda_p > \varepsilon^{-1} \text{ with } \Big\|u - \displaystyle\sum_{i=1}^p \alpha_i \delta_{a_i,\lambda_i}\Big\| < \varepsilon, \;\; \varepsilon_{ij} < \varepsilon \;\forall\, i \neq j, \\ \text{ and } \Big|J(u)^{\frac{n}{n-2\sigma}} \alpha_i^{\frac{2}{n-2\sigma}} K(a_i) - 1\Big| < \varepsilon \;\; \forall i,j = 1,\dots,p, \end{cases}$$

where

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{\frac{2\sigma-n}{2}}.$$

3 Characterization of the critical points at infinity for $1 < \beta \leq n - 2\sigma$

This section is devoted to the characterization of the critical points at infinity in $V(p, \varepsilon), p \geq 1$, under β-flatness condition with $1 < \beta \leq n - 2\sigma$. This characterization is obtained through the construction of a suitable pseudo-gradient at infinity for which the Palais-Smale condition is satisfied along the decreasing flow-lines as long as these flow-lines do not enter in the neighborhood of finite number of critical points $y_i, i = 1, ..., p$ of K such that

$$(y_1, ..., y_p) \in \mathcal{P}^\infty := C^\infty_{<(n-2\sigma)} \cup C^\infty_{n-2\sigma} \cup C^\infty_{<(n-2\sigma)} \times C^\infty_{n-2\sigma}.$$

More precisely we have:

Theorem 3.1 *Assume that K satisfies (A_1) and $(f)_\beta$, $1 < \beta \leq n - 2\sigma$. Let $\beta := \max\{\beta(y)/y \in \mathcal{K}\}$. For $p \geq 1$, there exists a pseudo-gradient W in $V(p, \varepsilon)$ so that the following holds.*
There exist a constant $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon)$ such that

$$(i) \Big\langle \partial J(u), W(u) \Big\rangle \leq -c \bigg(\sum_{i=1}^{p} \frac{1}{\lambda_i^\beta} + \sum_{i=1}^{p} \frac{| \nabla K(a_i) |}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} \bigg).$$

$$(ii) \Big\langle \partial J(u + \overline{v}), W(u) + \frac{\partial \overline{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W(u)) \Big\rangle \leq -c \bigg(\sum_{i=1}^{p} \frac{1}{\lambda_i^\beta} + \sum_{i=1}^{p} \frac{| \nabla K(a_i) |}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} \bigg).$$

Furthermore $|W|$ is bounded and the only case where the maximum of the λ_i's is not bounded is when $a_i \in B(y_{l_i}, \rho)$ with $y_{l_i} \in \mathcal{K}$, $\forall i = 1, ..., p$, $(y_{l_1}, ..., y_{l_p}) \in \mathcal{P}^\infty$.

In order to prove Theorem 3.1, we state the following two results which deal with two specific cases of Theorem 3.1. Let,

$$V_1(p, \varepsilon) = \Big\{ u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon) \text{ s.t }, a_i \in B(y_{l_i}, \rho), y_{l_i} \in \mathcal{K} \setminus \mathcal{K}_{n-2\sigma} \ \forall i = 1, ..., p \Big\}.$$

$$V_2(p, \varepsilon) = \Big\{ u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon) \text{ s.t }, a_i \in B(y_{l_i}, \rho), y_{l_i} \in \mathcal{K}_{n-2\sigma}, \ \forall i = 1, ..., p \Big\}.$$

We then have:

Proposition 3.2 *For $p \geq 1$, there exist a pseudo-gradient W_1 in $V_1(p,\varepsilon)$ such that the following holds:*
Theres exist $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V_1(p,\varepsilon)$ such that

$$\Big\langle \partial J(u), W_1(u) \Big\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \Big).$$

Furthermore $|W_1|$ is bounded in $H^1(D^n)$ and the only case where the maximum of the λ_i's is not bounded is when $a_i \in B\left(y_{l_i},\ \rho\right)$ with $y_{l_i} \in \mathcal{K}^+$, $\forall i = 1,...,p$, with $(y_{l_1},...,y_{l_p}) \in \mathcal{C}^{\infty}_{<n-2\sigma}$.

Proposition 3.3 *For $p \geq 1$ there exists a pseudo-gradient W_2 in $V_2(p,\varepsilon)$ such that $\forall u = \sum_{i=1}^{p} \alpha_i \delta_i \in V_2(p,\varepsilon)$, we have*

$$\Big\langle \partial J(u), W_2(u) \Big\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \Big).$$

Where c is a positive constant independent of u. Furthermore, we have $|W_2|$ is bounded and the only case where the maximum of $\lambda_i's$ is not bounded is when $a_i \in B(y_{l_i},\rho)$, $y_{l_i} \in \mathcal{K}^+$, $\forall\, i = 1,...,p$, with $(y_{l_1},...,y_{l_p}) \in \mathcal{C}^{\infty}_{n-2\sigma}$

In our construction of the pseudogradien W, we will use the following notations.
Let $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p,\varepsilon)$, such that $a_i \in B(y_{l_i},\rho)$, $y_{l_i} \in \mathcal{K}$, $\forall i = 1,...,p$.
For simplicity, if a_i is close to a critical point y_{l_i}, we will assume that the critical point is at the origin, so we will confuse a_i with $(a_i - y_{l_i})$. Now, let $i \in \{1,...,p\}$ and let M_1 be a positive large constant. We will say that

$$i \in L_1 \text{ if } \lambda_i|a_i| \leq M_1$$

and we will say that

$$i \in L_2 \text{ if } \lambda_i|a_i| > M_1.$$

For each $i \in \{1,...,p\}$, we define the following vector fields:

$$Z_i(u) = \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \tag{3.1}$$

$$X_i = \alpha_i \sum_{k=1}^{n} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_k} \int_{\mathbb{R}^n} b_k \frac{|x_k + \lambda_i (a_i)_k|^{\beta}}{(1 + \lambda_i|(a_i)_k|)^{\beta-1}} \frac{x_k}{(1+|x|^2)^{n+1}} dx, \tag{3.2}$$

where $(a_i)_k$ is the k^{th} component of a_i in some geodesic normal coordinates system.
We claim that X_i is bounded. Indeed, the claim is trivial if $i \in L_1$. If $i \in L_2$, by elementary computation, we have the following estimate:

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\left|x_k + \lambda_i (a_i)_k\right|^\beta x_k}{(1+|x|^2)^{n+1}} dx &= (\lambda_i|(a_i)_k|)^\beta \int_{\mathbb{R}^n} \left|1 + \frac{x_k}{\lambda_i((a_i)_k)}\right|^\beta \frac{x_k}{(1+\mid x\mid^2)^{n+1}} dx \\ &= c(\text{sign}\lambda_i(a_i)_k)(\lambda_i|(a_i)_k|)^{\beta-1}(1+o(1)), \end{aligned} \tag{3.3}$$

for any k, $1 \leq k \leq n$ such that $\lambda_i|(a_i)_k| > \frac{M_1}{\sqrt{n}}$. Hence our claim is valid.
Let k_i be an index such that

$$|(a_i)_{k_i}| = \max_{1 \leq j \leq n} |(a_i)_j|. \tag{3.4}$$

It easy to see that if $i \in L_2$ then $\lambda_i|(a_i)_{k_i}| > \frac{M_1}{\sqrt{n}}$.

Proof of Theorem 3.1 In order to complete the construction of the pseudo-gradient W suggested in Theorem 3.1, it only remains (using proposition 3.3 and 3.2) to focus attention at the two following subsets of $V(p,\varepsilon)$.

Subset 1. We consider here the case of $u = \sum_{i=1}^{p} \alpha_i\delta_i = \sum_{i\in I_1} \alpha_i\delta_i + \sum_{i\in I_2} \alpha_i\delta_i$ such that

$$I_1 \neq \emptyset, I_2 \neq \emptyset, \sum_{i\in I_1} \alpha_i\delta_i \in V_1(\sharp I_1, \varepsilon) \text{ and} \sum_{i\in I_2} \alpha_i\delta_i \in V_2(\sharp I_2, \varepsilon).$$

Without loss of generality, we can assume that $\lambda_1 \leq \cdots \leq \lambda_p$. We distinguish three cases.

case 1. $u_1 := \sum_{i\in I_1} \alpha_i\delta_i \notin V_1^1(\sharp I_1, \varepsilon) = \{u = \sum_{j=1}^{\sharp I_1} \alpha_j\delta_j, a_j \in B(y_{l_j}, \rho), y_{l_j} \in \mathcal{K}^+ \ \forall\ j = 1, ..., \sharp I_1$ and $y_{l_j} \neq y_{l_k} \forall j \neq k\}$.
Let $\widetilde{W_1}$ be the pseudo-gradient on $V(p,\varepsilon)$ defined by $\widetilde{W_1}(u) = W_1(u_1)$, where W_1 is the vector filed defined by proposition 3.2 in $V_1(\sharp I_1, \varepsilon)$. Note that if $u_1 \notin V_1^1(\sharp I_1, \varepsilon)$, then the pseudo-gradient $W_1(u_1)$ does not increase the maximum of the λ_i's, $i \in I_1$. Using proposition 3.2, we have

$$\left\langle \partial J(u), \widetilde{W_1}(u)\right\rangle \leq -c\left(\sum_{i\in I_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{j\neq i, i,j\in I_1} \varepsilon_{ij} + \sum_{i\in I_1} \frac{|\nabla K(a_i)|}{\lambda_i}\right) + O\left(\sum_{i\in I_1, j\in I_2} \varepsilon_{ij}\right) \tag{3.5}$$

An easy calculation implies that

$$\varepsilon_{ij} = o\left(\frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\frac{1}{\lambda_j^{\beta_j}}\right), \forall\ i \in I_1 \text{ and } \forall\ j \in I_2. \tag{3.6}$$

Fix $i_0 \in I_1$, we denote by

$$J_1 = \{i \in I_2, \text{ s.t, } \lambda_i^{n-2} \geq \frac{1}{2}\lambda_{i_0}^{\beta_{i_0}}\} \text{ and } J_2 = I_2 \setminus J_1.$$

Using (3.5) and (3.6), we find that

$$\Big\langle \partial J(u), \widetilde{W_1}(u) \Big\rangle \leq -c\Big(\sum_{i\in I_1\cup J_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j\neq i\in I_1} \varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}}\Big). \quad (3.7)$$

From another part, by Lemma 3.4 we have

$$\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u) \Big\rangle \leq c \sum_{j\neq i\ , i\in J_1} 2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\Big(\sum_{i\in J_1} \frac{1}{\lambda_i^{\beta_i}}\Big) + O\Big(\sum_{i\in J_1\cap L_2} \frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 2}}{\lambda_i^2}\Big) \quad (3.8)$$

Observe that for $i < j$, we have

$$2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + 2^j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c\, \varepsilon_{ij}. \quad (3.9)$$

In addition for $i \in J_1$ and $j \in J_2$ we have $\lambda_j \leq \lambda_i$, so by (3.18) we obtain $\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq -c\, \varepsilon_{ij}$. These estimates yield

$$\begin{aligned}\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u) \Big\rangle \leq\ & -c \sum_{j\neq i\ , i\in J_1,\, j\in J_1\cup J_2} \varepsilon_{ij} + O\Big(\sum_{i\in J_1} \frac{1}{\lambda_i^{\beta_i}}\Big) \\ & + O\Big(\sum_{i\in J_1\cap L_2} \frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2}\Big) + O\Big(\sum_{i\in J_1,\, j\in I_1} \varepsilon_{ij}\Big).\end{aligned}$$

Let $m_1 > 0$ small enough, using Lemma 3.5 (3.21) and (3.16) we get

$$\begin{aligned}\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u) + m_1 \sum_{i\in J_1\cap L_2} X_i(u) \Big\rangle \leq\ & -c\Big(\sum_{j\neq i\ , i\in J_1,\, j\in J_1\cup J_2} \varepsilon_{ij} + \sum_{i\in J_1} \frac{|\nabla K(a_i)|}{\lambda_i}\Big) \\ & + O\Big(\sum_{i\in J_1} \frac{1}{\lambda_i^{\beta_i}}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}}\Big),\end{aligned}$$

and by (3.7) we obtain

$$\begin{aligned}&\Big\langle \partial J(u), \widetilde{W_1}(u) + m_1\Big(\sum_{i\in J_1} -2^i Z_i(u) + m_1 \sum_{i\in J_1\cap L_2} X_i(u)\Big) \Big\rangle \\ &\leq -c\Big(\sum_{i\in I_1\cup J_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i\neq j\in I_1} \varepsilon_{ij} + \sum_{j\neq i\ , i\in J_1,\, j\in J_1\cup J_2} \varepsilon_{ij} \sum_{i\in I_1\cup J_1} \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}}\Big) \quad (3.10)\end{aligned}$$

We need to add the remainder indices $i \in J_2$. Note that $\widetilde{u} := \sum_{j\in J_2} \alpha_j\delta_j \in V_2(\sharp J_2, \varepsilon)$. Thus using proposition 3.3, we apply the associated vector field which we will denote $\widetilde{W_2}$. We then have the following estimate

$$\begin{aligned}\Big\langle \partial J(u), \widetilde{W_2}(u)\Big\rangle \;\leq\; & -c\Big(\sum_{j\in J_2}\frac{1}{\lambda_j^{\beta_j}} + \sum_{i\neq j,\, i,j\in J_2}\varepsilon_{ij} + \sum_{j\in J_2}\frac{|\nabla K(a_j)|}{\lambda_j}\Big) \\ & + O\Big(\sum_{j\in J_2,\, i\in J_1}\varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big),\end{aligned} \tag{3.11}$$

since $|a_i - a_j| \geq \rho$ for $j \in J_2$ and $i \in I_1$.

In this case $W = \widetilde{W_1} + m_1\Big(\widetilde{W_2} + \sum_{i\in J_1} -2^i Z_i + m_1 \sum_{i\in J_1\cap L_2} X_i\Big)$.

From (3.10) and (3.11) we obtain

$$\Big\langle \partial J(u), W(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

<u>case 2.</u> $u_1 := \sum_{i\in I_1}\alpha_i\delta_i \in V_1^1(\sharp I_1, \varepsilon)$ and $u_2 := \sum_{i\notin I_2}\alpha_i\delta_i \notin V_2^1(\sharp I_2, \varepsilon) := \{u = \sum_{j=1}^{\sharp I_2}\alpha_j\delta_j, a_j \in B(y_{l_j}, \rho), y_{l_j} \in \mathcal{K}^+, \forall j = 1, ..., \sharp I_2$ and $\rho(y_{l_1}, ..., y_{\sharp I_2}) > 0\}$.

Since $u_2 \in V_2(\sharp I_2, \varepsilon)$, by proposition 3.3, we can apply the associated vector field which we will denote V_1. We get

$$\Big\langle \partial J(u), V_1(u)\Big\rangle \;\leq\; -c\Big(\sum_{i\in I_2}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in I_2}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j,\, i,j\in I_2}\varepsilon_{ij}\Big) + O\Big(\sum_{i\in I_2,\, j\in I_1}\varepsilon_{ij}\Big) \tag{3.12}$$

Observe that $V_1(u)$ does not increase the maximum of the λ_i's, $i \in I_2$, since $u_2 \notin V_2^1(\sharp I_2, \varepsilon)$. Fix $i_0 \in I_2$ and let

$$\widetilde{J_1} = \{i \in I_1,\ s.t,\ \lambda_i^{\beta_i} \geq \frac{1}{2}\lambda_{i_0}^{n-2}\} \text{ and } \widetilde{J_2} = I_1 \setminus \widetilde{J_1}.$$

Using (3.12) and (3.6), we get

$$\Big\langle \partial J(u), V_1(u)\Big\rangle \;\leq\; -c\Big(\sum_{i\in I_2\cup\widetilde{J_1}}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in I_2}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j,\, i,j\in I_2}\varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big) \tag{3.13}$$

We need to add the indices i, $i \in \widetilde{J_2}$. Let $\widetilde{u} := \sum_{j\in\widetilde{J_2}}\alpha_j\delta_j$, since $\widetilde{u} \in V_1(\sharp\widetilde{J_2}, \varepsilon)$, we can apply the associated vector field giving by proposition 3.3. Let V_2 this vector field. By proposition 3.2 we have

$$\Big\langle \partial J(u), V_2(u)\Big\rangle \leq -c\Big(\sum_{j\in\widetilde{J_2}}\frac{1}{\lambda_j^{\beta_j}}+\sum_{j\in\widetilde{J_2}}\frac{|\nabla K(a_j)|}{\lambda_j}+\sum_{i\neq j,\, i,j\in\widetilde{J_2}}\varepsilon_{ij}\Big)+O\Big(\sum_{j\in\widetilde{J_2},\, i\notin\widetilde{J_2}}\varepsilon_{ij}\Big).$$

Observe that $I_1=\widetilde{J}_1\cup\widetilde{J}_2$ and we are in the case where $\forall\ i\neq j\in I_1$, we have $|a_i-a_j|\geq\rho$. Thus by (3.16) and (3.6), we get

$$O\Big(\sum_{j\in\widetilde{J_2}, i\notin\widetilde{J_2}}\varepsilon_{ij}\Big)=o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big),$$

and hence

$$\Big\langle \partial J(u), V_1(u)+V_2(u)\Big\rangle\leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2\cup\widetilde{J_2}}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Let in this case $W=V_1+V_2+m_1\sum_{i\in\widetilde{J_1}}X_i(u)$, m_1 small enough.
Using the above estimate and Lemma 3.5, we find that

$$\Big\langle \partial J(u), W(u)\Big\rangle\leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

case 3. $u_1\in V_1^1(\sharp I_1,\varepsilon)$ and $u_2\in V_2^1(\sharp I_2,\varepsilon)$.
Let $\widetilde{V_1}$ (respectively $\widetilde{V_2}$) be the pseudo-gradient in $V(p,\varepsilon)$ defined by $\widetilde{V_1}(u)=W_1(u_1)$ (respectively $\widetilde{V_2}(u)=W_2(u_2)$) where W_1 (respectively W_2) is the vector field defined by proposition 3.2 (respectively 3.3) in $V_1^1(\sharp I_1,\varepsilon)$ (respectively $V_2^1(\sharp I_2,\varepsilon)$) and let in this case $W=\widetilde{V_1}+\widetilde{V_2}$.
Using proposition 3.3, proposition 3.2 and (3.6) we get

$$\Big\langle \partial J(u), W(u)\Big\rangle\leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Notice that in the first and second cases, the maximum of the λ_i's, $1\leq i\leq p$ is a bounded function and hence the Palais-Smail condition is satisfied along the flow-lines of W. However in the third case all the λ_i's, $1\leq i\leq p$, will increase and goes to $+\infty$ along the flow-lines generated by W.

Subset 2. We consider the case of $u=\sum_{i=1}^{p}\alpha_i\delta_i\in V(p,\varepsilon)$, such that there exist a_i satisfying $a_i\notin\cup_{y\in\mathcal{K}}B(y,\rho)$. We order the λ_is in an increasing order, without loss of generality, we suppose that $\lambda_1 \leq \cdots \leq \lambda_p$. Let i_1 be such that for any $i < i_1$, we have $a_i \in B(y_{\ell_i}, \rho), y_{\ell_i} \in \mathcal{K}$ and $a_{i_1} \notin \cup_{y\in\mathcal{K}} B(y, \rho)$. Let us define

$$u_1 = \sum_{i<i_1} \alpha_i \delta_i.$$

Observe that u_1 has to satisfy one of three cases above that is $u \in V_1(i_1 - 1, \varepsilon)$ or $u_1 \in V_2(i_1 - 1, \varepsilon)$ or u_1 satisfies the condition of subset 1. Thus we can apply the associated vector field which we will denote by Y and we have then the following estimate.

$$\Big\langle \partial J(u), Y(u) \Big\rangle \leq -c\Big(\sum_{i<i_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i<i_1} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j, ij<i_1} \varepsilon_{ij} \Big) + O\Big(\sum_{i<i_1, j\geq i_1} \varepsilon_{ij} \Big).$$

Now we define the following vector field

$$Y' = \frac{1}{\lambda_{i_1}} \frac{\partial \delta_{i_1}}{\partial a_{i_1}} \frac{\nabla K(a_{i_1})}{|\nabla K(a_{i_1})|} - c' \sum_{i\geq i_1} 2^i Z_i.$$

Using Propositions 3.3, 3.2 and the fact that $|\nabla K(a_{i_1})| \geq c > 0$, we derive

$$\Big\langle \partial J(u), Y'(u) \Big\rangle \leq -c \frac{1}{\lambda_{i_1}} + O\Big(\sum_{i\neq i_1} \varepsilon_{ij} \Big) - c' \sum_{j\neq i, i\geq i_1} \varepsilon_{ij} + o\Big(\sum_{i\geq i_1} \frac{1}{\lambda_i} \Big).$$

Taking c' positive large enough, we find

$$\Big\langle \partial J(u), Y'(u) \Big\rangle \leq -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=i_1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j, i\geq i_1} \varepsilon_{ij} \Big).$$

Now, let $W := Y' + m_1 Y$ where m_1 is a small positive constant, then we have

$$\Big\langle \partial J(u), W(u) \Big\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij} \Big).$$

Finally, observe that our pseudo-gradient W in $V(p, \varepsilon)$ satisfies claim (i) of Theorem 3.1 and it is bounded, since $||\lambda_i \frac{\partial \delta_{(a_i, \lambda_i)}}{\partial \lambda_i}||$ and $||\frac{1}{\lambda_i} \frac{\partial \delta_{(a_i, \lambda_i)}}{\partial a_i}||$ are bounded. From the definition of W, the λ_i's, $1 \leq i \leq p$ decrease along the flow-lines of W as long as these flow-lines do not enter in the neighborhood of finite number of critical points y_{l_i}, $i = 1, ..., p$, of $\mathcal{K}$ such that $(y_{l_1}, ..., y_{l_p}) \in \mathcal{P}^\infty$.
Now, arguing as in Appendix 2 of [4], claim (ii) of Theorem 3.3 follows from (i) and proposition A.3. This complete the proof of Theorem 3.1.

Proof of Proposition 3.2. In our construction of the pseudo gradient W_1, we need the following lemmas:

Lemma 3.4 *Let* $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon)$*, such that* $a_i \in B(y_{l_i}, \rho)$*,* $y_{l_i} \in \mathcal{K}$*,* $\forall\ i = 1, ..., p$*. We then have*

$$\begin{aligned} < \partial J(u), Z_i(u) > &= -2c_2 J(u) \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\Big(\frac{1}{\lambda_i^{\beta_i}}\Big) \\ &+ \Big[O\Big(\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 2}}{\lambda_i^2} \Big), \textit{if } i \in L_2 \Big] + o\Big(\sum_{j \neq i} \varepsilon_{ij} \Big) + o\Big(\sum_{j=1}^{p} \frac{1}{\lambda_j^{\beta_j}} \Big), \end{aligned}$$

where k_i *is defined in* (3.4).

Proof. Observe that for $k \in \{1, ..., n\}$, if $\lambda_i |(a_i - y_{l_i})_k| > \frac{M_1}{\sqrt{n}}$, we have

$$\int_{\mathbb{R}^n} \frac{\big| x_k + \lambda_i (a_i - y_{l_i})_k \big|^{\beta_i - 1} x_k}{(1 + |x|^2)^n} dx = O\Big((\lambda_i |(a_i - y_{l_i})_k|)^{\beta_i - 2} \Big), \tag{3.14}$$

taking M_1 large enough. If not, we have

$$\int_{\mathbb{R}^n} \frac{\big| x_k + \lambda_i (a_i - y_{l_i})_k \big|^{\beta_i - 1} |x_k|}{(1 + |x|^2)^n} dx = O(1).$$

Using the fact that k_i defined in (3.4) satisfies $\lambda_i |(a_i - y_{l_i})_{k_i}| > \frac{M_1}{\sqrt{n}}$, if $i \in L_2$, Lemma 3.4 follows from Proposition A.1

Lemma 3.5 *For* $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon)$*, such that* $a_i \in B(y_{l_i}, \rho)$*,* $y_{l_i} \in \mathcal{K}$*,* $\forall\ i = 1, ..., p$*, we have*

$$\begin{aligned} < \partial J(u), X_i(u) > &\leq O\Big(\sum_{j \neq i} \frac{1}{\lambda_i} \Big| \frac{\partial \varepsilon_{ij}}{\partial a_i} \Big| \Big) + O\Big[\Big(\frac{1}{\lambda_i^{\beta_i}} \Big), \textit{if } i \in L_1 \Big] \\ &+ \Big[-c \Big(\frac{1}{\lambda_i^{\beta_i}} + \frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 1}}{\lambda_i} \Big), \textit{if } i \in L_2 \Big] + o\Big(\sum_{j=1}^{p} \frac{1}{\lambda_j^{\beta_j}} \Big), \end{aligned}$$

where k_i *is defined in* (3.4).

Proof. Using proposition A.2, we have

$$\begin{aligned} < \partial J(u), X_i(u) > &\leq -c \frac{1}{\lambda_i^{\beta_i}} \Big(\int_{\mathbb{R}^n} b_{k_i} \frac{|x_k + \lambda_i (a_i - y_{l_i})_{k_i}|^{\beta_i}}{(1 + \lambda_i |(a_i - y_{l_i})_{k_i}|)^{(\beta_i - 1)/2}} \frac{x_{k_i}}{(1 + |x|^2)^{n+1}} dx \Big)^2 \\ &+ O\Big(\sum_{j \neq i} \frac{1}{\lambda_i} \Big| \frac{\partial \varepsilon_{ij}}{\partial a_i} \Big| \Big) + o\Big(\sum_{j=1}^{p} \frac{1}{\lambda_j^{\beta_j}} \Big). \end{aligned} \tag{3.15}$$

Using (3.3) and the fact that $\lambda_i|(a_i-y_{l_i})_{k_i}| > \frac{M_1}{\sqrt{n}}$, if $i\in L_2$, lemma 3.5 follows.

In order to construct the required pseudo-gradient, we have to divide the set $V_1(p,\varepsilon)$ in four different regions, to construct an appropriate pseudo-gradient in each region and then glue up through convex combinations. Let

$V_1^1(p,\varepsilon) := \Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon),\ y_{l_i}\neq y_{l_j},\ \forall i\neq j,\ -\sum_{k=1}^{n} b_k(y_{l_i})>0, \text{and } \lambda_i|a_i - y_{l_i}|<\delta,\ \forall i=1,...,p\Big\}.$

$V_1^2(p,\varepsilon) := \Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon),\ y_{l_i}\neq y_{l_j},\ \forall i\neq j,\ \lambda_i|a_i-y_{l_i}|<\delta,\ \forall i = 1,..,p$ and there exist $\exists\ i_1,...,i_q$
such that $-\sum_{k=1}^{n} b_k(y_{l_{i_j}})<0,\ \forall j=1,...,q\Big\}.$

$V_1^3(p,\varepsilon) := \Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon),\ y_{l_i}\neq y_{l_j},\ \forall i\neq j,$ and there exist j (at least), $s,t,\lambda_j|a_j - y_{l_j}|\geq \frac{\delta}{2}\Big\}.$

$V_1^4(p,\varepsilon) := \Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon),$ such that there exist $i\neq j$ satisfying $y_{l_i}=y_{l_j}\Big\}.$

<u>Pseudo-gradient in $V_1^1(p,\varepsilon)$</u>. Let $u=\sum_{i=1}^{p}\alpha_i\delta_i\in V_1^1(p,\varepsilon)$. For any $i\neq j$, we have $|a_i-a_j|>\rho$, therefore

$$\varepsilon_{ij}=O\Big(\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}}\Big)=o\Big(\frac{1}{\lambda_i^{\beta_i}}\Big)+o\Big(\frac{1}{\lambda_j^{\beta_j}}\Big), \tag{3.16}$$

since $\beta_i,\beta_j<n-2\sigma$. Let $W_1^1(u)=\sum_{i=1}^{p}Z_i(u)$, using the fact that $\frac{|\nabla K(a_i)|}{\lambda_i}$ is small with respect to $\frac{1}{\lambda_i^{\beta}}$, we obtain from Proposition A.1

$$\Big\langle \partial J(u), W_1^1(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

<u>Pseudo-gradient in $V_1^2(p,\varepsilon)$</u>. Let $u=\sum_{i=1}^{p}\alpha_i\delta_i\in V_1^2(p,\varepsilon)$. Without loss of generality, we can assume that $1,...,q$ are the indices which satisfy $-\sum_{k=1}^{n}b_k(y_{l_i})<0,\ \forall i=1,...,q$. Let

$$I=\Big\{i=1,...,p\ \text{ s.t }\ \lambda_i^{\beta_i}\leq\frac{1}{10}\min_{1\leq j\leq q}\lambda_j^{\beta_j}\Big\}.$$

In this region we define $W_1^2(u) = \sum_{i=1}^{q}(-Z_i)(u) + \sum_{i\in I} Z_i(u)$. Using similar calculation than [7], we obtain

$$\Big\langle \partial J(u), W_1^2(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Pseudo-gradient in $V_1^3(p,\varepsilon)$. Let $u = \sum_{i=1}^{p}\alpha_i\delta_i \in V_1^3(p,\varepsilon)$. Without loss of generality, we can assume that $\lambda_1^{\beta_1} = \min\{\lambda_j^{\beta_j}$ s.t $\lambda_j|a_j - y_{l_j}| \geq \delta\}$. Let $J := \Big\{i, 1 \leq i \leq p$ s.t $\lambda_i^{\beta_i} \geq \frac{1}{2}\lambda_1^{\beta_1}\Big\}$. Observe that if $i \notin J$ we have $\lambda_i|a_i - y_{l_i}| \geq \delta$. We write u as follows $u = \sum_{i\in J^C}\alpha_i\delta_i + \sum_{i\in J}\alpha_i\delta_i = u_1 + u_2$. Observe that u_1 has to satisfy one of two above cases that is $u_1 \in V_1^1(\sharp J^C,\varepsilon)$ or $u_1 \in V_1^2(\sharp J^C,\varepsilon)$. Let $\widetilde{W}$ be a pseudo-gradient on $V_1^3(p,\varepsilon)$ defined by $\widetilde{W}(u) = W_1^1(u_1)$, if $u_1 \in V_1^1(\sharp J^C,\varepsilon)$ or $\widetilde{W}(u) = W_1^2(u_1)$, if $u_1 \in V_1^2(\sharp J^C,\varepsilon)$. Let in this region $W_1^3(u) = \widetilde{W}(u) + X_1(u) + \sum_{i\in J\cap L_2} X_i(u) - M_1Z_1(u)$. By Propositions A.1 and A.2, we have

$$\Big\langle \partial J(u), W_1^3(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Pseudo-gradient in $V_1^4(p,\varepsilon)$. We study now the case of u=$\sum_{i=1}^{p}\alpha_i\delta_i \in V_1^4(p,\varepsilon)$. Let, $B_k = \{j,\ 1 \leq j \leq p$ s.t $a_j \in B(y_{l_k},\rho)\}$. In this case, there is at least one B_k which contains at least two indices. Without loss of generality, we can assume that $1,...,q$ are the indices such that the set B_k, $1 \leq k \leq q$ contains at least two indices. We will decrease the λ_i's for $i \in B_k$ with different speed. For this purpose, let

$$\begin{array}{rcl} \chi : \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ t & \longmapsto & \begin{cases} 0 & \text{if } |t| \leq \widetilde{\gamma} \\ 1 & \text{if } |t| \geq 1. \end{cases} \end{array}$$

Here $\widetilde{\gamma}$ is a small constant. For $j \in B_k$, set $\overline{\chi}(\lambda_j) = \sum_{i\neq j,\ i\in B_k}\chi\left(\frac{\lambda_j}{\lambda_i}\right)$. Let, $I_1 = \Big\{i,\ 1 \leq i \leq p,\ \lambda_i|a_i - y_{l_i}| \geq \delta\Big\}$.

We distinguish two cases:

case 1. $I_1 \neq \emptyset$, let in this case

$$J = \Big\{j, 1 \leq j \leq p, s.t,\ \lambda_j^{\beta_j} \geq \frac{1}{2}\min_{i\in I_1}\lambda_i^{\beta_i}\Big\}.$$

Observe that, if $a_i \in B(\, y_{l_i}, \rho\,)$, we have $|\nabla K(a_i)| \sim \sum_{k=1}^n |b_k||(a_i - y_{l_i})_k|^{\beta_i - 1}$. So, if $i \in L_1$ we have $\dfrac{|\nabla K(a_i)|}{\lambda_i} \leq \dfrac{c}{\lambda_i^{\beta_i}}$, and if $i \in L_2$ we have $\dfrac{|\nabla K(a_i)|}{\lambda_i} \leq c\,\dfrac{|(a_i - y_{l_i})_k|^{\beta_i - 1}}{\lambda_i}$. Thus by lemma 3.5 we obtain

$$\begin{aligned}\Big\langle \partial J(u), \sum_{i\in I_1} X_i(u)\Big\rangle &\leq -c_\delta\Big(\sum_{i\in J}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in J}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\in I_1\cap L_2}\frac{|(a_i - y_{l_i})|^{\beta_i-1}}{\lambda_i}\Big)\\ &+ O\Big(\sum_{i\neq j,\, i\in I_1}\Big|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big|\Big) + o\Big(\sum_{i=1}^p\frac{1}{\lambda_i^{\beta_i}}\Big).\end{aligned}$$

Let $\widetilde{C} = \Big\{(i,j) \text{ s.t } \gamma \leq \frac{\lambda_i}{\lambda_j} \leq \frac{1}{\gamma}\Big\}$, where γ is a small positive constant. Observe that

$$\Big|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big| = o(\varepsilon_{ij}),\ \forall\, i \neq j\, \in \widetilde{C}.$$

This with (3.3) yields

$$\begin{aligned}\Big\langle \partial J(u), \sum_{i\in I_1} X_i(u)\Big\rangle &\leq -c_\delta\Big(\sum_{i\in J}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in J}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\in I_1\cap L_2}\frac{|(a_i - y_{l_i})|^{\beta_i-1}}{\lambda_i}\Big)\\ &+ o\Big(\sum_{k=1}^q\sum_{i\neq j\in B_k\cap\widetilde{C},\, i\in I_1}\varepsilon_{ij}\Big) + O\Big(\sum_{k=1}^q\sum_{i\neq j\in B_k,(i,j)\notin\widetilde{C},\, i\in I_1}\varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^p\frac{1}{\lambda_i^{\beta_i}}\Big).\end{aligned} \tag{3.17}$$

For any $k = 1, ..., q$, let $\lambda_{i_k} = \min\{\lambda_i, i \in B_k\}$. Define

$$\overline{Z} = -\sum_{k=1}^q\sum_{j\in B_k,(i_k,j)\notin\widetilde{C}}\overline{\chi}(\lambda_j)Z_j - \gamma_1\sum_{k=1}^q\sum_{j\in B_k,(i_k,j)\in\widetilde{C}}\overline{\chi}(\lambda_j)Z_j,$$

where γ_1 is a small positive constant. Using Lemma 3.4, we find that

$$\begin{aligned}\Big\langle \partial J(u), \overline{Z}(u)\Big\rangle &\leq c\sum_{k=1}^q\sum_{i\neq j, j\in B_k,(j,i_k)\notin\widetilde{C}}\overline{\chi}\,(\lambda_j)\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} + c\gamma_1\sum_{k=1}^q\sum_{j\in B_k,(j,i_k)\in\widetilde{C}, i\neq j}\overline{\chi}\,(\lambda_j)\\ &\times\ \lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} + O\Big(\sum_{k=1}^q\sum_{j\in B_k,(j,i_k)\notin\widetilde{C}}\Big(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j - y_{l_j})|^{\beta_j-2}}{\lambda_j^2}, \text{if } j\in L_2\Big)\Big)\\ &+\ \gamma_1 O\Big(\sum_{k=1}^q\sum_{j\in B_k,(j,i_k)\in\widetilde{C}}\Big(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j - y_{l_j})|^{\beta_j-2}}{\lambda_j^2}, \text{if } j\in L_2\Big)\Big).\end{aligned}$$

Observe that by using a direct calculation, we have

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \leq -c\,\varepsilon_{ij},\ \text{if } \lambda_i \geq \lambda_j \text{ or } \lambda_i \sim \lambda_j \text{ or } |a_i - a_j| \geq \delta_0 > 0. \tag{3.18}$$

Let $j \in B_k$, $1 \le k \le q$ and let i, $1 \le i \le p$ such that $i \neq j$. If $i \notin B_k$ or $i \in B_k$, with $(i,j) \in \widetilde{C}$, then we have by (3.18)

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \le -c\, \varepsilon_{ij} \text{ and } \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \le -c\, \varepsilon_{ij}.$$

In the case where $i \in B_k$ with $(i,j) \notin \widetilde{C}$, (assuming that $\lambda_i << \lambda_j$), we have $\overline{\chi}(\lambda_j) - \overline{\chi}(\lambda_i) \ge 1$. Thus,

$$\overline{\chi}\,(\lambda_j)\, \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \overline{\chi}\,(\lambda_i)\, \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \le \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \le -c\, \varepsilon_{ij}.$$

We therefore have

$$\begin{aligned} \Big\langle \partial J(u), \overline{Z}(u) \Big\rangle \;&\le\; -c\Big(\sum_{k=1}^{q} \sum_{i\neq j, j\in B_k, (j,i_k)\notin \widetilde{C}} \varepsilon_{ij} + \gamma_1 \sum_{k=1}^{q} \sum_{i\neq j, j \in B_k, (j,i_k)\in \widetilde{C}} \varepsilon_{ij} \Big) \\ &+\; O\Big(\sum_{k=1}^{q} \sum_{j\in B_k, (j,i_k)\notin \widetilde{C}} \Big(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j - y_{l_j})|^{\beta_j - 2}}{\lambda_j^2}, \text{if } j \in L_2 \Big)\Big) \\ &+\; \gamma_1 O\Big(\sum_{k=1}^{q} \sum_{j\in B_k, (j,i_k)\in \widetilde{C}} \Big(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j - y_{l_j})|^{\beta_j - 2}}{\lambda_j^2}, \text{if } j \in L_2 \Big)\Big). \end{aligned} \tag{3.19}$$

Observe that if $j \in B_k$ with $(j, i_k) \in \widetilde{C}$, we have j or $i_k \in I_1$. Thus for M_1 large enough, and γ_1 very small, we obtain from (3.17) and (3.19)

$$\begin{aligned} \Big\langle \partial J(u), \sum_{i\in I_1} X_i + M_1 \overline{Z}(u) \Big\rangle \;&\le\; -c\Big(\sum_{i\in J} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in J} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k=1}^{q} \sum_{i\neq j, j\in B_k} \varepsilon_{ij} \Big) \\ &+\; O\Big(\sum_{k=1}^{q} \sum_{j\in B_k, (i_k,j)\notin \widetilde{C}} \frac{1}{\lambda_j^{\beta_j}} \Big), \end{aligned} \tag{3.20}$$

since $$\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 2}}{\lambda_i^2} = o\Big(\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 1}}{\lambda_i} \Big), \text{ for any } i \in L_2, \text{ as } M_1 \text{ large enough.} \tag{3.21}$$

Now, let in this region

$$W_1^4 := M_1 \Big(\sum_{i\in I_1} X_i + M_1 \overline{Z} \Big) + \sum_{i\notin J} (-\sum_{k=1}^{n} b_k) Z_i.$$

We obtain from the above estimates

$$\Big\langle \partial J(u), W_1^4(u) \Big\rangle \le -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij} \Big).$$

case 2. $I_1 = \emptyset$, we order the λ_i's in an increasing order, for sake of simplicity, we can assume that $\lambda_1 \leq ... \leq \lambda_p$. Let

$$I_2 = \{1\} \cup \{i, 1 \leq i \leq p \; s.t \; \lambda_i \sim \lambda_1\}.$$

We write u as follows

$$u = \sum_{i \in I_2} \alpha_i \delta_i + \sum_{i \notin I_2} \alpha_i \delta_i := u_1 + u_2.$$

Observe that, $\forall i \neq j \in I_2$ such that $i \neq j$ we have $|a_i - a_j| \geq \delta$. Indeed, if $|a_i - a_j| < \delta$, so $i, j \in B_k$, we get $|a_i - a_j| \leq |a_i - y_{l_i}| + |a_j - y_{l_i}| \leq \dfrac{2\delta}{\lambda_i}$, since $I_1 = \emptyset$ and $\lambda_i \sim \lambda_j \; \forall \, i, j \in I_2$. This implies that

$$\left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{n-2\sigma}{2}} \leq c_1,$$

and hence $\varepsilon_{ij} \geq c$ which is a contradiction. Thus $u_1 \in V_1^j(\sharp I_2, \varepsilon)$, $j = 1$ or 2 or 3. Apply the associated pseudo-gradient denoted by $\overline{W}$, we obtain

$$\Big\langle \partial J(u), \overline{W}(u) \Big\rangle \leq -c \Big(\sum_{i \in I_2} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in I_2} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j, \, i,j \in I_2} \varepsilon_{ij} \Big) + O\Big(\sum_{i \in I_2, j \notin I_2} \varepsilon_{ij} \Big).$$

Let, $$J_2 = \{i, 1 \leq i \leq p, \; \lambda_i^{\beta_i} \geq \min_{j \in I_2} \lambda_j^{\beta_j} \}.$$

We can add to the above estimates all indices i such that $i \in J_2$. So using the estimate (3.16) we obtain

$$\Big\langle \partial J(u), \overline{W}(u) \Big\rangle - c \Big(\sum_{i \in J_2} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in J_2} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j, \, i,j \in I_2} \varepsilon_{ij} \Big)$$

$$+ o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}} \Big) + O\Big(\sum_{i \in I_2, j \notin I_2, i,j \in B_k} \varepsilon_{ij} \Big).$$

Let $M_1 > 0$ large enough, the above estimate and (3.19) yields

$$\begin{aligned} \Big\langle \partial J(u), M_1 \overline{Z}(u) + \overline{W}(u) \Big\rangle \; &\leq \; -c \Big(\sum_{i \in J_2} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in J_2} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k=1}^{q} \sum_{i \neq j \in B_k} \varepsilon_{ij} \\ &\quad + \sum_{i \neq j, \, i,j \in I_2} \varepsilon_{ij} \Big) + O\Big(\sum_{k=1}^{q} \sum_{i \in B_k, (i_k, i) \notin \widetilde{C}} \frac{1}{\lambda_i^{\beta_i}} \Big). \end{aligned} \quad (3.22)$$

From another part, by (iii) of proposition 3.3 and (3.16), we have

$$\Big\langle \partial J(u), \sum_{i \notin J_2} (-\sum_{k=1}^{n} b_k) Z_i(u) \Big\rangle \;\leq\; -c \Big(\sum_{i \notin J_2} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \notin J_2} \frac{|\nabla K(a_i)|}{\lambda_i} \Big) \quad (3.23)$$

$$+O\Big(\sum_{k=1}^{q}\sum_{i\neq j\in B_k,\, i\notin J_2}\varepsilon_{ij}\Big)+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big).$$

Define
$$W_1^4(u)=M_1\Big(M_1\overline{Z}(u)+\overline{W}(u)\Big)+\sum_{i\notin J_2}(-\sum_{k=1}^{n}b_k)Z_i(u).$$

Using (3.23), we get

$$\Big\langle\partial J(u),W_1^4(u)\Big\rangle\leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big),$$

since $\frac{1}{\lambda_i^{\beta_i}}=o\Big(\frac{1}{\lambda_{i_k}^{\beta_{i_k}}}\Big)$ $\forall\, i\in B_k$ such that $(i,i_k)\notin\widetilde{C}$.

The vector field W_1 in $V_1(p,\varepsilon)$ will be a convex combination of $W_1^j, j=1,...,4$. From the definitions of $W_1^j, j=1,...,4$ the only case where the maximum of the λ_i's increase is when $a_i\in B(y_{l_i},\rho),\ y_{l_i}\in\mathcal{K}^+,\ \forall\ i=1,...,p$, with $y_{l_i}\neq y_{l_j}, \forall\ i\neq j$. This conclude the proof of proposition 3.2.

Proof of proposition 3.3. We divide the set $V_2(p,\varepsilon)$ into five sets.

$$V_2^1(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon),\ y_{l_i}\neq y_{l_j}\ \forall i\neq j,\ -\sum_{k=1}^{n}b_k(y_{l_i})>0,\ \lambda_i|a_i-y_{l_i}|<\delta,\ \forall i=1,...,p \text{ and } \rho(y_{l_1},...,y_{l_p})>0\Big\}.$$

$$V_2^2(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon),\ y_{l_i}\neq y_{l_j}\ \forall i\neq j,\ -\sum_{k=1}^{n}b_k(y_{l_i})>0,\ \lambda_i|a_i-y_{l_i}|<\delta,\ \forall i=1,...,p \text{ and } \rho(y_{l_1},...,y_{l_p})<0\Big\}.$$

$$V_2^3(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon),\ y_{l_i}\neq y_{l_j}\ \forall i\neq j,\ \lambda_i|a_i-y_{l_i}|<\delta,\ \forall i=1,...,p, \text{and there exist } j \text{ (at least) such that } -\sum_{k=1}^{n}b_k(y_{l_j})<0\Big\}.$$

$$V_2^4(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon),\ y_{l_i}\neq y_{l_j}\ \forall i\neq j, \text{and there exist } j \text{ (at least) such that } \lambda_j|a_j-y_{l_j}|\geq\frac{\delta}{2}\Big\}.$$

$$V_2^5(p,\varepsilon) = \Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon), \text{ such that there exist } i\neq j \text{ satisfying } y_{l_i}=y_{l_j}\Big\}.$$

We break up the proof into five steps. We construct an appropriate pseudo-gradient in each region and then glue up via convex combinations. Let Z_1 and Z_2 be two vector fields. A convex combination of Z_1 and Z_2 is given by $\theta Z_1+(1-\theta)Z_2$ where θ is cut-off function.

Step 1: First, we consider the case of $u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2^1(p,\varepsilon)$, we have for any $i\neq j$, $|a_i-a_j|>\rho$ and therefore,

$$\begin{aligned}\varepsilon_{ij} &= \Big(\frac{2}{(1-\cos d(a_i,a_j))\lambda_i\lambda_j}\Big)^{\frac{n-2\sigma}{2}}(1+o(1))\\ &= 2^{\frac{n-2\sigma}{2}}\frac{G(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}}(1+o(1)).\end{aligned}$$

Where $G(a_i,a_j)$ is defined in (1.5). Thus,

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}=-\frac{n-2\sigma}{2}2^{\frac{n-2\sigma}{2}}\frac{G(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}}(1+o(1)).$$

Using proposition A.1 with $\beta=n-2\sigma$ and the fact that $\alpha_i^{\frac{4\sigma}{n-2\sigma}}K(a_i)J(u)^{\frac{n}{n-2\sigma}}=1+o(1)\ \forall i=1,...,p.$, we derive that

$$\begin{aligned}\Big\langle\partial J(u),\alpha_i\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\Big\rangle &= \frac{n-2\sigma}{2}\,J(u)^{1-\frac{n}{2}}\Bigg[\frac{n-2\sigma}{n}\,\widetilde{c}_1\,\frac{\sum_{i=1}^{p}b_k}{K(a_i)^{\frac{n}{2\sigma}}}\,\frac{1}{\lambda_i^{n-2\sigma}}\\ &+ c_1\,2^{\frac{n-2\sigma}{2}}\sum_{i\neq j}\frac{G(y_{l_i},y_{l_j})}{\Big(K(a_i)K(a_j)\Big)^{\frac{n-2\sigma}{4\sigma}}}\frac{1}{\Big(\lambda_i\lambda_j\Big)^{\frac{n-2\sigma}{2}}}\Bigg]\\ &+ o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2\sigma}}+\sum_{i\neq j}\varepsilon_{ij}\Big).\end{aligned}$$

Where $\widetilde{c}_1=c_0^{\frac{2n}{n-2\sigma}}\displaystyle\int_{\mathbb{R}^n}\frac{|(x_1)|^{n-2\sigma}}{(1+|x|^2)^n}dx$. Hence, using the fact that $\big|a_i-y_{l_i}\big|<\delta$, δ very small, we get,

$$\begin{aligned}\Big\langle\partial J(u),\sum_{i=1}^{p}\alpha_iZ_i\Big\rangle &\le -c\ {}^t\Lambda\ M(y_{l_1},...,y_{l_p})\Lambda+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2\sigma}}+\sum_{i\neq j}\varepsilon_{ij}\Big)\\ &\le -c\,\rho(y_{l_1},...,y_{l_p})\,|\Lambda|^2+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2\sigma}}+\sum_{i\neq j}\varepsilon_{ij}\Big),\end{aligned}$$

where $\Lambda = \ {}^t\Big(\frac{1}{\lambda_1^{\frac{n-2\sigma}{2}}}, \dots, \frac{1}{\lambda_p^{\frac{n-2\sigma}{2}}}\Big)$. Here $M(y_{l_1}, ..., y_{l_p})$ is defined in (1.4) and $\rho(y_{l_1}, ..., y_{l_p})$ is the least eigenvalue of $M(y_{l_1}, ..., y_{l_p})$. Using the fact that $\forall i \neq j$, we have $\varepsilon_{ij} \leq \frac{c}{(\lambda_i \lambda_j)^{\frac{n-2\sigma}{2}}}$, since $|a_i - a_j| \geq \delta$, we then obtain

$$\Big\langle \partial J(u), \sum_{i=1}^{p} \alpha_i Z_i \Big\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \Big).$$

In addition, $\forall i = 1, ..., p$, we have $\lambda_i |a_i| < \delta \implies \frac{|\nabla K(a_i)|}{\lambda_i} \sim \frac{|(a_i)_k|^{\beta-1}}{\lambda_i} \leq \frac{c}{\lambda_i^{\beta}}$. Thus, we derive for $W_2^1 := \sum_{i=1}^{p} \alpha_i Z_i$

$$\Big\langle \partial J(u), W_2^1 \Big\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \Big).$$

<u>Step 2</u>: Secondly, we study the case of $u = \sum_{i=1}^{p} \alpha_i \delta_{a_i \lambda_i} \in V_2^2(p, \varepsilon)$. Let, $e = (e_i)_{i=1,...,p}$ an eigenvector associated to $\rho(y_{l_1}, ..., y_{l_p})$ such that $|e| = 1$ with $e_i > 0$ $\forall i = 1, ..., p$. Let $\gamma > 0$ such that for any $x \in B(e, \gamma) = \{y \in S^{p-1} \text{ s.t } |y - e| \leq \gamma\}$, we have

$${}^t x M(y_{l_1}, ..., y_{l_p}) x \leq \frac{1}{2} \rho(y_{l_1}, ..., y_{l_p}).$$

Two cases may occur.

<u>case 1:</u> $\frac{\Lambda}{|\Lambda|} \in B(e, \gamma)$, where $\Lambda = \ {}^t\Big(\frac{1}{\lambda_1^{\frac{n-2\sigma}{2}}}, \dots, \frac{1}{\lambda_p^{\frac{n-2\sigma}{2}}}\Big)$. In this case, we define $W_2^2 = -\sum_{i=1}^{p} \alpha_i Z_i$. As in step 1, we find that,

$$\Big\langle \partial J(u), W_2^2(u) \Big\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \Big).$$

<u>case 2:</u> $\frac{\Lambda}{|\Lambda|} \notin B(e, \gamma)$. In this case, we define

$$W_2^2 = -\frac{2}{n-2\sigma} |\Lambda| \sum_{i=1}^{p} \alpha_i \lambda_i^{\frac{n}{2}} \Big[\frac{|\Lambda| \, e_i - \Lambda_i}{|\Lambda|} - \frac{\Lambda_i < |\Lambda| \, e - \Lambda, \Lambda >}{|\Lambda|^3} \Big] \frac{\partial \delta_{a_i \lambda_i}}{\partial \lambda_i}.$$

Using proposition A.1, we find that

$$\Big\langle \partial J(u), W_2^2(u)\Big\rangle = -c|\Lambda|^2\frac{\partial}{\partial t}\Big({}^t\Lambda(t)M\ \Lambda(t)\Big)_{/t=0} + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-4}}\Big) + o\Big(\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Where $M = M(y_{l_1},...,y_{l_p})$ and $\Lambda(t) = \dfrac{(1-t)\Lambda + t|\Lambda|e}{\Big|(1-t)\Lambda + t|\Lambda|e\Big|}\Lambda$. Observe that,

$${}^t\Lambda(t)M\Lambda(t) = \rho + \frac{(1-t)^2}{\Big|(1-t)\Lambda + t|\Lambda|e\Big|}\Big({}^t\Lambda M\Lambda - \rho|\Lambda|^2\Big).$$

Thus we obtain, $\dfrac{\partial}{\partial t}\Big({}^t\Lambda(t)M\ \Lambda(t)\Big) < -c$ and therefore we get,

$$\Big\langle \partial J(u), W_2^2(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Step 3: Now, we deal with the case of $u = \sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i} \in V_2^3(p,\varepsilon)$.

Without loss of generality, we can assume that 1,...,q are the indices which satisfy $-\sum_{k=1}^{n}b_k(y_{l_i}) < 0\ \forall i = 1,...,q$. Let,

$$\widetilde{W}_2^1 = \sum_{i=1}^{q} -\alpha_i Z_i.$$

By proposition A.1 and (3.18), we obtain

$$\Big\langle \partial J(u), \widetilde{W}_2^1(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{q}\frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\ 1\leq i\leq q}\varepsilon_{ij}\Big).$$

Set

$$I = \Big\{ i, 1\leq i\leq p \ \text{ s.t } \ \lambda_i \leq \frac{1}{10}\min_{1\leq j\leq q}\lambda_j\Big\}.$$

It is easy to see that, we can add to the above estimates all indices i such that $i\notin I$. Thus

$$\Big\langle \partial J(u), \widetilde{W}_2^1(u)\Big\rangle \leq -c\Big(\sum_{i\notin I}\frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\ i\notin I}\varepsilon_{ij}\Big).$$

If $I\neq\emptyset$, in this case, we write u as follows

$$u = \sum_{i\in I}\alpha_i\delta_{a_i\lambda_i} + \sum_{i\notin I}\alpha_i\delta_{a_i\lambda_i} = u_1 + u_2.$$

Observe that u_1 has to satisfy one of two cases above that is $u_1 \in V_2^1(\sharp I, \varepsilon)$ or $u_1 \in V_2^2(\sharp I, \varepsilon)$. Thus we can apply the associated vector field which we will denote $\widetilde{W}_2^2$. We then have

$$\Big\langle \partial J(u), \widetilde{W}_2^2(u) \Big\rangle \leq -c\Big(\sum_{i\in I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\, i\in I} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + O\Big(\sum_{i\neq j,\, i\notin I} \varepsilon_{ij}\Big).$$

Let in this subset $W_2^3 = \widetilde{W}_2^1 + m_1 \widetilde{W}_2^2$, m_1 be a small positive constant. We get,

$$\Big\langle \partial J(u), W_2^3(u) \Big\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\Big).$$

Step 4: We consider her the case of $u = \sum_{i=1}^{p} \alpha_i \delta_{a_i\lambda_i} \in V_2^4(p, \varepsilon)$.

We order the λ_i's in an increasing order, for sake of simplicity, we can assume that $\lambda_1 \leq ... \leq \lambda_p$. Let $\lambda_{i_1} = \inf\{\lambda_j \text{ s.t } \lambda_j |a_j| \geq \delta\}$. For $m_1 > 0$ small enough, we need to prove the following claim

$$\Big\langle \partial J(u), (X_{i_1} - m_1 Z_{i_1})(u) \Big\rangle \leq -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j} + \sum_{i=1}^{p} \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\Big).$$

Indeed, for $i \neq j$, we have $|a_i - a_j| > \rho$, thus in proposition A.2 the term $\Big|\dfrac{1}{\lambda_i} \dfrac{\partial \varepsilon_{ij}}{\partial (a_i)_k}\Big|$ is very small with respect ε_{ij}, hence,

$$\begin{aligned} \Big\langle \partial J(u), X_{i_1}(u) \Big\rangle \ &\leq\ -\frac{c}{\lambda_{i_1}^{n-2\sigma}} \Big(\int_{\mathbb{R}^n} b_{k_{i_1}} \frac{|x_{k_{i_1}} + \lambda_{i_1} (a_{i_1})_{k_{i_1}}|^\beta}{\Big(1 + \lambda_{i_1} |(a_{i_1})_{k_{i_1}}|\Big)^{\frac{\beta-1}{2}}} \frac{x_{k_{i_1}}}{\Big(1+|x|^2\Big)^{n+1}} dx \Big)^2 \\ &+\ o\Big(\frac{1}{\lambda_{i_1}^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j}\Big). \end{aligned}$$

If $i_1 \in L_1$ in this case $\delta \leq \lambda_{i_1} |a_{i_1}| \leq M_1$, using elementary calculation, we have

$$\Big(\int_{\mathbb{R}^n} b_{k_i} \frac{|x_{k_i} + \lambda_i (a_1)_{k_i}|^\beta}{(1 + \lambda_i |(a_1)_{k_i}|)^{\frac{\beta-1}{2}}} \frac{x_{k_i}}{(1+|x|^2)^n} dx \Big)^2 \geq c > 0. \tag{3.24}$$

Using (3.24) , we get

$$\Big\langle \partial J(u), X_{i_1}(u) \Big\rangle \leq -c \frac{1}{\lambda_{i_1}^{n-2\sigma}} + o\Big(\sum_{j\neq i_1} \varepsilon_{i_1 j}\Big) \leq -c \sum_{i=i_1}^{p} \frac{1}{\lambda_i^{\beta}} + o\Big(\sum_{j\neq i_1} \varepsilon_{i_1 j}\Big). \tag{3.25}$$

From another part, we have by proposition A.1 and (3.18),

$$\Big\langle \partial J(u), Z_{i_1}(u) \Big\rangle \leq -c \sum_{j\neq i_1} \varepsilon_{i_1 j} + O\Big(\frac{1}{\lambda_{i_1}^{n-2\sigma}}\Big). \tag{3.26}$$

Using (3.25) and (3.26) our claim follows in this case.
If $i_1 \in L_2$, using (3.3), we find

$$\begin{aligned}\Big\langle \partial J(u), X_{i_1}(u) \Big\rangle &\leq -c\Big(\frac{1}{\lambda_{i_1}^{n-2\sigma}} + \frac{|(a_{i_1})_{k_{i_1}}|^{\beta-1}}{\lambda_{i_1}}\Big) + o\Big(\sum_{j\neq i_1} \varepsilon_{i_1 j}\Big) \\ &\leq -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \frac{|(a_{i_1})_{k_{i_1}}|^{\beta-1}}{\lambda_{i_1}}\Big) + o\Big(\sum_{j\neq i_1} \varepsilon_{i_1 j}\Big)\end{aligned}$$

and by proposition A.1 and (3.3), we have

$$\Big\langle \partial J(u), -Z_{i_1}(u) \Big\rangle \leq -c \sum_{j\neq i_1} \varepsilon_{i_1 j} + O\Big(\frac{|(a_{i_1})_{k_{i_1}}|^{\beta-2}}{\lambda_{i_1}^2}\Big).$$

Now using (3.21), we obtain

$$\begin{aligned}\Big\langle \partial J(u), (X_{i_1} - m_1 Z_{i_1})(u) \Big\rangle &\leq -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j} + \frac{|(a_{i_1})_k|^{\beta-1}}{\lambda_{i_1}}\Big) \\ &\leq -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j} + \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\Big),\end{aligned}$$

since $|\nabla K(a_{i_1})| \sim |(a_{i_1})_{k_i}|^{\beta-1}$ hence our claim is valid.
Now let,

$$I = \Big\{ i, 1 \leq i \leq p \quad \text{s.t} \;\; \lambda_i < \frac{1}{10}\lambda_{i_1} \Big\},$$

it is easy to see that

$$\Big\langle \partial J(u), (X_{i_1} - m_1 Z_{i_1})(u) \Big\rangle \leq -c\Big(\sum_{i\notin I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\neq i,\, i\notin I} \varepsilon_{ij} + \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\Big).$$

Furthermore, using (3.3), we have

$$\Big\langle \partial J(u), \Big(X_{i_1} - m_1 Z_{i_1} + \sum_{i\notin I,\, i\in L_2} X_i\Big)(u) \Big\rangle \leq -c\Big(\sum_{i\notin I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\notin I} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j,\, i\notin I} \varepsilon_{ij}\Big)$$

since for $i \notin I$ and $i \in L_1$ we have $\dfrac{|\nabla K(a_i)|}{\lambda_i} \leq \dfrac{c}{\lambda_i^{\beta}}$.
We need to add the remainder terms (if $I \neq \emptyset$). Let $u_1 = \sum_{i\in I} \alpha_i \delta_{a_i \lambda_i}$, $\forall i \in I$ we have

$\lambda_i|a_i| < \delta$, thus $u_1 \in V_2^j(\sharp I, \varepsilon)$, $j = 1$ or 2 or 3, we can apply then the associated vector field which we will denote $\widetilde{W}_2^4$. We then have

$$\Big\langle \partial J(u), \widetilde{W}_2^4 \Big\rangle \leq -c\Big(\sum_{i\in I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\, i,j\in I} \varepsilon_{ij} + \sum_{i\in I} \frac{|\nabla K(a_i)|}{\lambda_i} \Big) + O\Big(\sum_{i\in I,\, j\notin I} \varepsilon_{ij} \Big).$$

Let $W_2^4 = X_{i_1} - m_1 Z_{i_1} + \sum_{i\notin I,\, i\in L_2} X_i + m_2 \widetilde{W}_2^4$, m_2 is positive small enough, we get

$$\Big\langle \partial J(u), W_2^4(u) \Big\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij} \Big).$$

Step 5: We study now the case of $u = \sum_{i=1}^{p} \alpha_i \delta_{a_i\lambda_i} \in V_2^5(p, \varepsilon)$.

Let,

$$B_k = \{ j,\ 1 \leq j \leq p \ \text{ s.t } \ a_j \in B(y_{l_k}, \rho) \}.$$

In this case, there is at least one B_k which contains at least two indices. Without loss of generality, we can assume that $1, ..., q$ are the indices such that the set B_k, $1 \leq k \leq q$ contains at least two indices. We will decrease the λ_i's for $i \in B_k$ with different speed. For this purpose, let

$$\begin{array}{rcl} \chi : \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ t & \longmapsto & \begin{cases} 0 & \text{if } |t| \leq \gamma' \\ & \\ 1 & \text{if } |t| \geq 1. \end{cases} \end{array}$$

Where γ' is a small constant.

For $j \in B_k$, set $\overline{\chi}\,(\lambda_j) = \sum_{i\neq j,\, i\in B_k} \chi\left(\frac{\lambda_j}{\lambda_i}\right)$. Define

$$\widetilde{W}_2^5 = -\sum_{k=1}^{q} \sum_{j\in B_k} \alpha_j\, \overline{\chi}\,(\lambda_j)\, Z_j.$$

Using proposition A.1 and (3.3), we obtain

$$\begin{aligned} \Big\langle \partial J(u), \widetilde{W}_2^5(u) \Big\rangle \ \leq\ & c \sum_{k=1}^{q} \Big[\sum_{i\neq j,\, j\in B_k} \overline{\chi}\,(\lambda_j) \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \sum_{j\in B_k,\, j\in L_1} \overline{\chi}\,(\lambda_j) O\Big(\frac{1}{\lambda_j^{n-2\sigma}} \Big) \\ & + \sum_{j\in B_k,\, j\in L_2} \overline{\chi}\,(\lambda_j) O\Big(\frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2} \Big) \Big]. \end{aligned}$$

For $j \in B_k$, with $k \leq q$, if $\overline{\chi}\ (\lambda_j) \neq 0$, then there exists $i \in B_k$ such that $\dfrac{1}{\lambda_j^{n-2\sigma}} = o(\varepsilon_{ij})$ (for ρ small enough). Furthermore, for $j \in B_k$, if $i \notin B_k$ (or $i \in B_k$ with $\lambda_i \sim \lambda_j$), then we have by (3.18),

$$\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c\, \varepsilon_{ij} \text{ and } \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq -c\, \varepsilon_{ij}.$$

In the case where $i \in B_k$ with (assuming $\lambda_i << \lambda_j$), we have $\overline{\chi}(\lambda_j) - \overline{\chi}(\lambda_i) \geq 1$. Thus

$$\overline{\chi}\ (\lambda_j)\ \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \overline{\chi}\ (\lambda_i)\ \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c\, \varepsilon_{ij}.$$

Thus we obtain

$$\begin{aligned} \Big\langle \partial J(u), \widetilde{W}_2^5(u) \Big\rangle \ &\leq\ -c \sum_{k=1}^{q} \sum_{j \in B_k} \overline{\chi}\ (\lambda_j) \Big(\sum_{i \neq j} \varepsilon_{ij} + \frac{1}{\lambda_j^{n-2\sigma}} \Big) \\ &+\ \sum_{k=1}^{q} \sum_{j \in B_k, j \in L_2} \overline{\chi}\ (\lambda_j) O\Big(\frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2} \Big). \end{aligned} \tag{3.27}$$

We need to add the indices j, $j \in {}^{C}\Big(\bigcup_{K=1}^{q} B_k \Big) \bigcup \Big\{ j \in B_k \text{ s.t } \overline{\chi}\ (\lambda_j) = 0 \Big\}$. Let,

$$\lambda_{i_0} = \inf\{\lambda_i,\ i = 1, ..., p\}.$$

We distinguish two cases.
<u>case 1:</u> there exists j such that $\overline{\chi}\ (\lambda_j) \neq 0$ and $\lambda_{i_0} \sim \lambda_j$, $\Big(\gamma' \leq \dfrac{\lambda_{i_0}}{\lambda_j} \leq 1 \Big)$, then we can appear on the above estimate $-\dfrac{1}{\lambda_{i_0}^{n-2\sigma}}$ and therefore $-\displaystyle\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}}$ and $-\displaystyle\sum_{k \neq r} \varepsilon_{kr}$. Thus we obtain

$$\Big\langle \partial J(u), \widetilde{W}_2^5(u) \Big\rangle \leq -c \Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \Big) + O\Big(\sum_{k=1}^{q} \sum_{j \in B_k,\, j \in L_2} \frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2} \Big).$$

Now let,

$$W_2^5 = \widetilde{W}_2^5 + m_1 \sum_{i=1}^{p} X_i,$$

using the above estimates with proposition A.2 and (3.21), we obtain

$$\Big\langle \partial J(u), W_2^5(u) \Big\rangle \leq -c \Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \Big).$$

case 2: For each $j \in B_k$, $1 \leq k \leq q$ we have

$$\lambda_{i_0} << \lambda_j \ \Big(i.e. \ \frac{\lambda_{i_0}}{\lambda_j} < \gamma'\Big) \text{ or if } \lambda_{i_0} \sim \lambda_j \text{ we have } \overline{\chi}\,(\lambda_j) = 0.$$

In this case we define

$$D = \Big[\Big\{i, \ \overline{\chi}\,(\lambda_i) = 0\Big\} \bigcup \ {}^C\Big(\bigcup_{k=1}^q B_k\Big)\Big] \bigcap \Big\{i, \ \frac{\lambda_i}{\lambda_{i_0}} < \frac{1}{\gamma'}\Big\}.$$

It is easy to see that $i_0 \in D$ and if $i \neq j \in \Big\{i, \ \overline{\chi}\,(\lambda_i) = 0\Big\} \bigcup \ {}^C\Big(\bigcup_{k=1}^q B_k\Big)$ we have $a_i \in B(y_{l_i}, \rho)$ and $a_j \in B(y_{l_j}, \rho)$ with $y_{l_i} \neq y_{l_j}$. Let,

$$u_1 = \sum_{i \in D} \alpha_i \delta_{a_i \lambda_i},$$

u_1 has to satisfy one of the four subsets above, that is $u_1 \in V_2^j(\sharp I, \varepsilon)$ for $j = 1, 2, 3$ or 4. Thus we can apply the associated vector field which we will denote Y and we have the estimate

$$\Big\langle \partial J(u), Y(u) \Big\rangle \leq -c\Big(\sum_{i \in D} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \in D} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j, \, i,j \in D} \varepsilon_{ij}\Big) + O\Big(\sum_{i \in D, \, j \notin D} \varepsilon_{ij}\Big).$$

Observe in the above majoration we have the term $-\frac{1}{\lambda_{i_0}^{n-2\sigma}}$, thus we can make appear $-\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}}$. Now concerning the term $-\sum_{i \neq j} \varepsilon_{ij}$, if $i \in D$ and $j \in {}^C D$, observe that,

$${}^C D = \Big\{i, \ \frac{\lambda_i}{\lambda_{i_0}} > \frac{1}{\gamma'}\Big\} \cup \Big[\Big\{i, \ \overline{\chi}\,(\lambda_i) \neq 0\Big\} \cap \Big(\cup_{k=1}^q B_k\Big)\Big],$$

we have two situations: either $j \in \Big[\Big\{i, \ \overline{\chi}\,(\lambda_i) \neq 0\Big\} \bigcap \Big(\cup_{k=1}^q B_k\Big)\Big]$, then we have $-\varepsilon_{ij}$ in the estimates (3.27) or $j \in \Big\{i, \ \frac{\lambda_i}{\lambda_{i_0}} > \frac{1}{\gamma'}\Big\}$, we can prove in this cases that $|a_i - a_j| \geq \rho$. Thus

$$\varepsilon_{ij} \leq \frac{c}{(\lambda_i \lambda_j)^{\frac{n-2\sigma}{2}}} < \frac{c\gamma'^{\frac{n-2\sigma}{2}}}{(\lambda_{i_0} \lambda_i)^{\frac{n-2\sigma}{2}}} = o(\varepsilon_{i_0 i}) \qquad \text{(for } \gamma' \text{ small enough).}$$

Thus we derive,

$$\begin{aligned} \Big\langle \partial J(u), (\widetilde{W}_2^5 + m_1 Y)(u) \Big\rangle \ &\leq \ -c\Big(\sum_{i \in D} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij}\Big) \\ &+ \ \sum_{K=1}^{q} \ \sum_{j \in B_k, \, j \in L_2} \overline{\chi}(\lambda_j) O\Big(\frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2}\Big), \end{aligned}$$

and hence, by (3.21), we have

$$\Big\langle \partial J(u), \Big(\widetilde{W}_2^5 + m_1 Y + m_2 \sum_{i=1,\, i\in L_2} X_i \Big)(u) \Big\rangle \leq -c \Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \Big),$$

for m_1 and m_2 two small positive constants. In this case we denote

$$W_2^5 := \widetilde{W}_2^5 + m_1 Y + m_2 \sum_{i=1,\, i\in L_2} X_i.$$

The vector field W_2 in $V_2(p,\varepsilon)$ will be a convex combination of W_2^j, $j = 1, ..., 5$. This conclude the proof of proposition 3.3.

Corollary 3.6 *Let $p \geq 1$. The critical points at infinity of J in $V(p,\varepsilon)$ correspond to*

$$(y_{l_1}, ..., y_{l_p})_\infty := \sum_{i=1}^{p} \frac{1}{K(y_{l_i})^{\frac{n-2\sigma}{2}}} \delta_{(y_{l_i},\infty)},$$

where $(y_{l_1}, ..., y_{l_p}) \in \mathcal{P}^\infty$. Moreover, such a critical point at infinity has an index equal to $i(y_{l_1}, ..., y_{l_p})_\infty = p - 1 + \sum_{i=1}^{p} n - \widetilde{i}(y)$.

4 Proof of Theorem 1.1

Using corollary 3.6, the only critical points at infinity associated to problem (1.1) correspond to $w_\infty = (y_{i_1}, ..., y_{i_p}) \in \mathcal{P}^\infty$. We prove Theorem 1.1 by contradiction. Therefore, we assume that equation (1.1) has no solution. For any $w_\infty \in \mathcal{P}^\infty$, let $c(w)_\infty$ denote the associated critical value at infinity. Here we choose to consider a simplified situation where for any $w_\infty \neq w'_\infty$, $c(w)_\infty \neq c(w')_\infty$ and thus order the $c(w)_\infty$'s, $w_\infty \in \mathcal{P}^\infty$ as

$$c(w_1)_\infty < ... < c(w_{k_0})_\infty.$$

For any $\overline{c} \in \mathbb{R}$, let $J_{\overline{c}} = \{u \in \Sigma^+,\ J(u) \leq \overline{c}\}$. By using a deformation lemma (see [6]), we know that if $c(w_{k-1})_\infty < a < c(w_k)_\infty < b < c(w_{k+1})_\infty$, then

$$J_b \simeq J_a \cup W_u^\infty(w_k)_\infty, \tag{4.1}$$

Here $W_u^\infty(w_k)_\infty$ denote the unstable manifolds at infinity of $(w_k)_\infty$(see [4]) and $\simeq$ denotes retracts by deformation.
We apply the Euler-Poincaré characteristic of both sides of (4.1), we find that

$$\chi(J_b) = \chi(J_a) + (-1)^{i(w_k)_\infty}, \tag{4.2}$$

where $i(w_k)_\infty$ denotes the index of the critical point at infinity $(w_k)_\infty$. Let

$$b_1 < c(w_1)_\infty = \min_{u\in\Sigma^+} J(u) < b_2 < c(w_2)_\infty < ... < b_{k_0} < c(w_{k_0})_\infty < b_{k_0+1}.$$

Since we have assumed that (1.1) has no solution, $J_{b_{k_0+1}}$ is a retard by deformation of Σ^+. Therefore $\chi(J_{b_{k_0+1}}) = 1$, since Σ^+ is a contractible set. Now using (4.2), we derive after recalling that $\chi(J_{b_1}) = \chi(\emptyset) = 0$,

$$1 = \sum_{j=1}^{k_0} (-1)^{i(w_j)_\infty}. \tag{4.3}$$

So, if (4.3) is violated, then (1.1) has a solution. This complete the proof of Theorem 1.1.

A Appendix A

This appendix is devoted to some useful expansions of the gradient of J near a potential critical points at infinity consisting of p masses. Those propositions are proved under some technical estimates of the different integral quantities, extracted from [3] (with some change). In order to simplify the notations, in the remainder we write δ_i instead of $\delta_{(a_i,\lambda_i)}$.

Proposition A.1 *Assume that K satisfies $(f)_\beta$, $1 < \beta < n$. For any $U = \sum_{j=1}^{p} \alpha_j\delta_j$ in $V(p,\varepsilon)$, the following expansion hold*

$$(i)\Big\langle \partial J(U), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\Big\rangle = -2c_2 J(u) \sum_{i\neq j} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\Big(\sum_{i\neq j}\varepsilon_{ij}\Big) + o\Big(\frac{1}{\lambda_i}\Big),$$

where $c_2 = c_0^{\frac{2n}{n-2\sigma}} \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^{\frac{n+2\sigma}{2}}}$.

(ii) If $a_i \in B(y_{j_i},\rho)$, $y_{j_i} \in \mathcal{K}$ and ρ is a positive constant small enough , we have

$$\begin{aligned}
&\Big\langle \partial J(U), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\Big\rangle \\
&= 2J(u)\Big[-c_2 \sum_{j\neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2\sigma}{2n} c_0^{\frac{2n}{n-2\sigma}} \beta \frac{\alpha_i}{K(a_i)} \frac{1}{\lambda_i^\beta} \sum_{k=1}^{n} b_k \\
&\times \int_{\mathbb{R}^n} sign\Big(x_k + \lambda_i(a_i - y_{j_i})_k\Big)\Big|x_k + \lambda_i(a_i-y_{j_i})_k\Big|^{\beta-1} \frac{x_k}{(1+|x|^2)^n} dx \\
&+ o\Big(\sum_{j\neq i}\varepsilon_{ij} + \sum_{j=1}^{p}\frac{1}{\lambda_j^\beta}\Big)\Big].
\end{aligned} \tag{A.1}$$

(iii) Furthermore, if $\lambda_i|a_i - y_{j_i}| < \delta$, for δ very small, we then have

$$\begin{aligned}\Big\langle \partial J(U), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\Big\rangle &= 2J(u)\Big[\frac{n-2\sigma}{2n}\,\beta\, c_3\,\frac{\alpha_i}{K(a_i)}\,\frac{\sum_{k=1}^n b_k}{\lambda_i^\beta} - c_2\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\\ &+ o\Big(\sum_{j\neq i}\varepsilon_{ij} + \sum_{j=1}^p \frac{1}{\lambda_j^\beta}\Big)\Big], \qquad (A.2)\end{aligned}$$

where $c_3 = c_0^{\frac{2n}{n-2\sigma}}\displaystyle\int_{S^n}\frac{|x_1|^\beta}{(1+|x|^2)^n}dx.$

Proposition A.2 *Under condition $(f)_\beta$, $1 < \beta < n$, for each $U = \sum_{j=1}^p \alpha_j\delta_j \in V(p,\varepsilon)$, we have*

$$\begin{aligned}(i)\Big\langle \partial J(U), \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial a_i}\Big\rangle &= -c_5 J(u)^2\alpha_i^{\frac{n+2\sigma}{n-2\sigma}}\frac{\nabla K(a_i)}{\lambda_i} + O\Big(\sum_{i\neq j}\frac{1}{\lambda_i}\big|\frac{\partial\varepsilon_{ij}}{\partial a_i}\big|\Big)\\ &+ o\Big(\sum_{i\neq j}\varepsilon_{ij} + \frac{1}{\lambda_i}\Big),\end{aligned}$$

where $c_5 = \displaystyle\int_{\mathbb{R}^n}\frac{dy}{\big(1+|y|^2\big)^n}.$

(ii) if $a_i \in B(y_{j_i},\rho)$, $y_{j_i}\in\mathcal{K}$, we have

$$\begin{aligned}&\Big\langle \partial J(U), \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_k}\Big\rangle =\\ &- 2(n-2\sigma)c_0^{\frac{2n}{n-2\sigma}}\alpha_i^{\frac{n+2\sigma}{n-2\sigma}}J(u)^2\frac{1}{\lambda_i^\beta}\int_{\mathbb{R}^n} b_k\big|x_k + \lambda_i(a_i - y_{j_i})_k\big|^\beta\frac{x_k}{\big(1+|x|^2\big)^{n+1}}dy\\ &+ o\Big(\sum_{i\neq j}\varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^p\frac{1}{\lambda_i^\beta}\Big) + O\Big(\sum_{i\neq j}\frac{1}{\lambda_i}\big|\frac{\partial\varepsilon_{ij}}{\partial a_i}\big|\Big),\end{aligned}$$

where $k = 1,...,n$ and $(a_i)_k$ is the k^{th} component if a_i in some geodesic normal coordinates system.

Proposition A.3 *Let $n \geq 2$. Suppose that K satisfies $(f)_\beta$, with $1 < \beta < n$. There exists $c > 0$ such that the following holds*

$$\|\overline{v}\| \leq c\sum_{i=1}^p\Big[\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log\lambda_i)^{\frac{n+2\sigma}{2n}}}{\lambda_i^{\frac{n+2\sigma}{2}}}\Big] + c\begin{cases}\displaystyle\sum_{k\neq r}\varepsilon_{k\,r}^{\frac{n+2\sigma}{2(n-2\sigma)}}\Big(\log\varepsilon_{kr}^{-1}\Big)^{\frac{n+2\sigma}{2n}}, & \text{if } n\geq 3\\ \displaystyle\sum_{k\neq r}\varepsilon_{k\,r}\Big(\log\varepsilon_{kr}^{-1}\Big)^{\frac{n-2\sigma}{n}}, & \text{if } n<3.\end{cases}$$

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