

Rearrangement inequalities for functionals with monotone integrands

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Dedicated to Albert Baernstein, II on the occasion of his 65th birthday.

Abstract

The inequalities of Hardy-Littlewood and Riesz say that certain integrals involving products of two or three functions increase under symmetric decreasing rearrangement. It is known that these inequalities extend to integrands of the form $F(u_1, \dots, u_m)$ where F is supermodular; in particular, they hold when F has nonnegative mixed second derivatives $\partial_i \partial_j F$ for all $i \neq j$. This paper concerns the regularity assumptions on F and the equality cases. It is shown here that extended Hardy-Littlewood and Riesz inequalities are valid for supermodular integrands that are just Borel measurable. Under some nondegeneracy conditions, all equality cases are equivalent to radially decreasing functions under transformations that leave the functionals invariant (i.e., measure-preserving maps for the Hardy-Littlewood inequality, translations for the Riesz inequality). The proofs rely on monotone changes of variables in the spirit of Sklar's theorem.

1 Introduction

The systematic study of rearrangements begins with the final chapter of “Inequalities” by Hardy, Littlewood, and Pólya [1]. Two inequalities are discussed there at length, the *Hardy-Littlewood inequality* (Theorems 368-370 and 378 of [1])

$$\int_{\mathbb{R}} u(x)v(x) dx \leq \int_{\mathbb{R}} u^*(x)v^*(x) dx, \quad (1.1)$$

and the *Riesz rearrangement inequality* ([2, 3], Theorem 370 of [1])

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u(x)v(x')w(x-x') dx dx' \leq \int_{\mathbb{R}} \int_{\mathbb{R}} u^*(x)v^*(x')w^*(x-x') dx dx'. \quad (1.2)$$

Here, u , v , and w are nonnegative measurable functions that vanish at infinity, and u^* , v^* , and w^* are their symmetric decreasing rearrangements.

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The Hardy-Littlewood inequality is a very basic inequality that holds, with suitably defined rearrangements, on arbitrary measure spaces [4]. Its main implication is that rearrangement decreases L^2 -distances [5]. In contrast, the Riesz rearrangement inequality is specific to \mathbb{Z} and to \mathbb{R}^n , where it is closely related with the Brunn-Minkowski inequality of convex geometry. The generalization of Eq. (1.2) from \mathbb{R} to \mathbb{R}^n is due to Sobolev [6], and the inequality is also known as the Riesz-Sobolev inequality. For many applications, the third function in Eq. (1.2) is already radially decreasing, i.e., $w(x-x') = K(|x-x'|)$ with some nonnegative nonincreasing function K , such as the heat kernel or the Coulomb kernel (Theorems 371-373 and 380 of [1]). This special case of the inequality also holds on the standard spheres and hyperbolic spaces [7, 8], and it still contains the isoperimetric inequality as a limit.

It is a natural question whether these inequalities carry over to more general integral functionals. Under what conditions on F do the *extended Hardy-Littlewood inequality*

$$\int F(u_1(x), \dots, u_m(x)) dx \leq \int F(u_1^*(x), \dots, u_m^*(x)) dx \quad (1.3)$$

and the *extended Riesz inequality*

$$\begin{aligned} & \int \cdots \int F(u_1(x_1), \dots, u_m(x_m)) \prod_{i < j} K_{ij}(d(x_i, x_j)) dx_1 \dots dx_m \\ & \leq \int \cdots \int F(u_1^*(x_1), \dots, u_m^*(x_m)) \prod_{i < j} K_{ij}(d(x_i, x_j)) dx_1 \dots dx_m \end{aligned} \quad (1.4)$$

hold for all choices of u_1, \dots, u_m ? In Eq. (1.4) the K_{ij} are given nonnegative nonincreasing functions on \mathbb{R}_+ , and $d(x, y)$ denotes the distance between x and y . Eq. (1.3) can be recovered from Eq. (1.2) by choosing K_{ij} as a Dirac sequence and passing to the limit. Note that Eq. (1.4) contains only the case of Eq. (1.2) where the third function is a symmetric decreasing kernel. The full generalization of Riesz' inequality to products of more than three functions was found by Brascamp-Lieb-Luttinger [9]; again, one may ask to what class of integrands the Brascamp-Lieb-Luttinger inequality naturally extends.

The main condition on F is a second-order monotonicity property identified by Lorentz [10],

$$F(\mathbf{y} + h\mathbf{e}_i + k\mathbf{e}_j) + F(\mathbf{y}) \geq F(\mathbf{y} + h\mathbf{e}_i) + F(\mathbf{y} + k\mathbf{e}_j) \quad (i \neq j, h, k > 0), \quad (1.5)$$

where $\mathbf{y} = (y_1, \dots, y_m)$, and \mathbf{e}_i denotes the i -th standard basis vector in \mathbb{R}^m . Functions satisfying Eq. (1.5) are called *supermodular* or *2-increasing* in Economics. A smooth function is supermodular, if all its mixed second partial derivatives are nonnegative. Eqs. (1.3) and (1.4) were proved for continuous supermodular integrands depending on $m = 2$ functions by Crowe-Zweibel-Rosenbloom [11] and Almgren-Lieb (Theorem 2.2 of [12]). For $m > 2$, Eq. (1.3) is due to Brock [13], and Eq. (1.4) is a recent result of Draghici [14]. The purpose of this paper is to dispense with the continuity assumptions on F in the theorems of Brock and Draghici, and to characterize the equality cases in some relevant situations. This continues prior work of the second author [15-18].

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2 Statement of the results

Let \mathbb{X} denote either the Euclidean space \mathbb{R}^n , the sphere \mathbb{S}^n , or the hyperbolic space \mathbb{H}^n , equipped with the standard distance function $d(\cdot, \cdot)$ and the uniform volume measure λ . Choose a distinguished point $x^* \in \mathbb{X}$ to serve as the origin or the north pole. Consider a nonnegative measurable function u on \mathbb{X} . When $\mathbb{X} = \mathbb{R}^n$ or \mathbb{H}^n , we require u to *vanish at infinity* in the sense that all its positive level sets $\{x \in \mathbb{X} : u(x) > t\}$ have finite measure; when $\mathbb{X} = \mathbb{S}^n$ this requirement is void. By definition, the *symmetric decreasing rearrangement* u^* of u is the unique upper semicontinuous, nonincreasing function of $d(x, x^*)$ that is equimeasurable with u . Explicitly, if

$$\rho(t) = \lambda(\{x \in \mathbb{X} : u(x) > t\})$$

is the distribution function of u , and B_r denotes the open ball of radius r centered at x^* , then

$$u^*(x) := \sup\{t \geq 0 : \rho(t) \geq \lambda(B_{d(x, x^*)})\}.$$

Theorem 1 (Extended Hardy-Littlewood inequality.) *Eq. (1.3) holds for any nonnegative measurable functions u_1, \dots, u_m that vanish at infinity on $\mathbb{X} = \mathbb{R}^n, \mathbb{S}^n$, or \mathbb{H}^n , provided that the integrand F is a supermodular Borel measurable function on the closed positive cone \mathbb{R}_+^m with $F(\mathbf{0}) = 0$, and that its negative part satisfies*

$$\int_{\mathbb{X}} F_-(u_i(x) \mathbf{e}_i) dx < \infty \quad (2.1)$$

for $i = 1, \dots, m$.

Suppose Eq. (1.3) holds with equality, and the integrals are finite. If F satisfies Eq. (1.5) with strict inequality for some $i \neq j$, all $\mathbf{y} \in \mathbb{R}_+^m$ and all $h, k > 0$, then

$$(u_i(x) - u_i(x'))(u_j(x) - u_j(x')) \geq 0$$

for almost all $x, x' \in \mathbb{X}$; in particular, if $u_i = u_i^$ is strictly radially decreasing, then $u_j = u_j^*$.*

The Borel measurability of F and the integrability assumption in Eq. (2.1) ensure that the integrals are well-defined, though they may take the value $+\infty$.

The left hand side of Eq. (1.3) is invariant under volume-preserving diffeomorphisms of \mathbb{X} . More generally, if (Ω, μ) and (Ω', μ') are measure spaces and $\tau : \Omega \rightarrow \Omega'$ pushes μ forward to μ' in the sense that $\mu'(A) = \mu(\tau^{-1}(A))$ for all μ' -measurable subsets $A \subset \Omega'$, then

$$\int_{\Omega} F(u_1(\omega), \dots, u_m(\omega)) d\mu(\omega) = \int_{\Omega'} F(u_1 \circ \tau(\omega'), \dots, u_m \circ \tau(\omega')) d\mu'(\omega').$$

The right hand side of Eq. (1.3) can also be expressed in an invariant form. Define the *nonincreasing rearrangement* $u^\#$ of u as the unique nonincreasing upper semicontinuous function on \mathbb{R}_+ which is equimeasurable with u ,

$$u^\#(\xi) := \sup\{t \geq 0 : \rho(t) \geq \xi\}.$$

By construction, $(u \circ \tau)^\# = u^\#$ for any map $\tau : \Omega \rightarrow \Omega'$ that pushes μ forward to μ' . On $\mathbb{X} = \mathbb{R}^n, \mathbb{S}^n$ and \mathbb{H}^n , the nonincreasing rearrangement is related with the symmetric decreasing rearrangement by $u^*(x) = u^\#(\lambda(B_{d(x,x^*)}))$. Theorem 1 implies that

$$\int_{\Omega} F(u_1(\omega), \dots, u_m(\omega)) d\mu(\omega) \leq \int_0^{\mu(\Omega)} F(u_1^\#(\xi), \dots, u_m^\#(\xi)) d\xi \quad (2.2)$$

for all nonnegative measurable functions u_1, \dots, u_m on Ω that vanish at infinity.

When μ is a probability measure, Eq. (2.2) says that the expected value of $F(Y_1, \dots, Y_m)$ is maximized among all random variables Y_1, \dots, Y_m with given marginal distributions by the perfectly correlated random variables $Y_1^\#, \dots, Y_m^\#$. The joint distribution of the maximizer is uniquely determined, if Y_i is continuously distributed for some i and Eq. (1.5) is strict for all $j \neq i$. In this formulation, the invariance under measure-preserving transformations is evident, since the expected value depends only on the joint distribution of Y_1, \dots, Y_m . The assumption that F is supermodular signifies that each of the random variables enhances the contribution of the others.

Theorem 2 (Extended Riesz inequality.) *Eq. (1.4) holds for all nonnegative measurable functions u_1, \dots, u_m on $\mathbb{X} = \mathbb{R}^n, \mathbb{S}^n$, or \mathbb{H}^n that vanish at infinity, provided that F is a supermodular Borel measurable function on \mathbb{R}_+^m with $F(\mathbf{0}) = 0$, each K_{ij} is nonincreasing and nonnegative, and the negative part of F satisfies*

$$\int_{\mathbb{X}} \dots \int_{\mathbb{X}} F_-(u_\ell(x_\ell) \mathbf{e}_\ell) \prod_{i < j} K_{ij}(d(x_i, x_j)) dx_1 \dots dx_m < \infty \quad (2.3)$$

for $i = 1, \dots, m$.

Suppose Eq. (1.4) holds with equality. Assume additionally that the integrals are finite, and that $K_{ij}(t) > 0$ for all $i < j$ and all $t < \text{diam } \mathbb{X}$. Let Γ_0 be the graph on the vertex set $\{1, \dots, m\}$ which has an edge between i and j whenever K_{ij} is a strictly decreasing function, and let $i \neq j$ be from the same component of Γ_0 . If Eq. (1.5) is strict for all $\mathbf{y} \in \mathbb{R}_+^m$ and $h, k > 0$, and u_i and u_j are non-constant, then $u_i = u_i^* \circ \tau$ and $u_j = u_j^* \circ \tau$ for some translation τ on \mathbb{X} .

3 Related work

There are several proofs of the extended Hardy-Littlewood inequality in the literature. For continuous integrands, Lorentz showed by discretization and elementary manipulations of the u_i that Eq. (2.2) holds for all measurable functions u_1, \dots, u_m on $\Omega = (0, 1)$ if and only if F is supermodular [10]. By the invariance under measure-preserving transformations, this implies Eq. (1.3), as well as Eq. (2.2) for arbitrary finite measure spaces Ω . However, Lorentz' paper has had little impact on subsequent developments.

More than thirty years later, Crowe-Zweibel-Rosenbloom proved Eq. (1.3) for $m = 2$ on $\mathbb{X} = \mathbb{R}^n$ [11]. They expressed a given continuous supermodular function F on \mathbb{R}_+^2 that vanishes on the boundary as the distribution function of a Borel measure μ_F ,

$$F(y_1, y_2) = \mu_F([0, y_1) \times [0, y_2)) .$$

With Fubini's theorem, this provides a *layer-cake representation* (see Theorem 1.13 in [19])

$$\int F(u_1(x), u_2(x)) dx = \int_{\mathbb{R}_+^2} \left\{ \int \mathbf{1}_{u_1(x) > y_1} \mathbf{1}_{u_2(x) > y_2} dx \right\} d\mu_F(y_1, y_2), \quad (3.1)$$

which reduces Eq. (1.3) to the case where F is a product of characteristic functions. Another reduction to products was proposed by Tahraoui [20]. The regularity and boundary conditions on F were relaxed by Hajaiej-Stuart, who assumed it to be supermodular, of Carathéodory type (i.e., Borel measurable in the first, continuous in the second variable), and to satisfy some growth and integrability restrictions [15]. Equality statements for their results were obtained by Hajaiej [16, 17]. Using a slightly different layer-cake decomposition, Van Schaftingen-Willem recently established this inequality, under additional assumptions on F , for any equimeasurable rearrangement that preserves inclusions [21].

The drawback of the layer-cake representation is that for $m > 2$ it requires an m -th order monotonicity condition on the integrand, which amounts for smooth F to the nonnegativity of all (non-repeating) mixed partial derivatives [18]. Brock proved Eq. (1.3) under the much weaker assumption that F is continuous and supermodular [13].

Carlier viewed maximizing the left hand side of Eq. (2.2) for a given right hand side as an optimal transportation problem where the distribution functions of u_1, \dots, u_m define mass distributions μ_i on \mathbb{R} , the joint distribution defines a transportation plan, and the functional represents the cost after multiplying by a minus sign [22]. He showed that the functional achieves its maximum (i.e., the cost is minimized) when the joint distribution is concentrated on a curve in \mathbb{R}^m that is nondecreasing in all coordinate directions, and obtained Eq. (2.2) as a corollary. His proof takes advantage of the dual problem of minimizing

$$\sum_{i=1}^m \int_{\mathbb{R}} f_i(y) d\mu_i(y)$$

over f_1, \dots, f_m , subject to the constraint that $\sum f_i(y_i) \geq F(y_1, \dots, y_m)$ for all y_1, \dots, y_m .

Theorem 1 can be applied to some integrands that depend explicitly on the radial variable [10,22,15-18]. If G is a function on $\mathbb{R}_+ \times \mathbb{R}_+^m$ such that $F(y_0, \dots, y_m) := G(y_0^{-1}, y_1, \dots, y_m)$ satisfies the assumptions of Theorem 1, then

$$\int_{\mathbb{R}^n} G(|x|, u_1(x), \dots, u_m(x)) dx \leq \int_{\mathbb{R}^n} G(|x|, u_1^*(x), \dots, u_m^*(x)) dx. \quad (3.2)$$

This inequality was studied by Hajaiej-Stuart in connection with the following problem in nonlinear optics [15, 18]. The profiles of stable electromagnetic waves traveling along a planar waveguide are given by the ground states of the energy functional

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}} |u'|^2 dx - \int_{\mathbb{R}} G(|x|, u) dx$$

under the constraint $\|u\|_2 = c$. Here, x is the position relative to the optical axis, G is determined by the index of refraction, and $c > 0$ is a parameter related to the wave speed [23]. If the index of refraction of the optical media decreases with $|x|$, then $F(r, y) = G(r^{-1}, y)$ satisfies the assumptions of Theorem 1. Then the first integral shrinks under symmetric decreasing rearrangement by the Pólya-Szegő inequality, the second integral grows by Eq. (3.2),

and the L^2 -constraint is conserved. Thus, one may rearrange any minimizing sequence to obtain a minimizing sequence of symmetric decreasing functions. This is a crucial step in the construction of ground states — if G violates the monotonicity conditions, then a ground state need not exist [24]. Hajaiej-Stuart worried about restrictive regularity assumptions, because G may jump at interfaces between layers of different media.

The Riesz inequality in Eq. (1.4) is non-trivial even when F is just a product of two functions. Ahlfors introduced two-point rearrangements to treat this case on $\mathbb{X} = \mathbb{S}^1$ [25], Baernstein-Taylor proved the corresponding result on \mathbb{S}^n [7], and Beckner noted that the proof remains valid on \mathbb{H}^n and \mathbb{R}^n [8]. When F is a product of $m > 2$ functions, Eq. (1.4) has applications to spectral invariants of heat kernels via the Trotter product formula [26]. This case was settled by Friedberg-Luttinger [27], Burchard-Schmuckenschläger [28], and by Morpurgo, who proved Eq. (1.4) more generally for integrands of the form

$$F(y_1, \dots, y_m) = \Phi\left(\sum_{i=1}^m y_i\right) \quad (3.3)$$

with Φ convex (Theorem 3.13 of [29]). In the above situations, equality cases have been determined [30, 31, 28, 29]. Almgren-Lieb used the technique of Crowe-Zweibel-Rosenbloom to prove Eq. (1.4) for $m = 2$ [12]. The special case where $F(u, v) = \Phi(|u - v|)$ for some convex function Φ was identified by Baernstein as a *master inequality* from which many classical geometric inequalities can be derived quickly [32]. Eq. (1.4) for continuous supermodular integrands with $m > 2$ is due to Draghici [14].

4 Outline of the arguments

In their proofs of Eqs. (1.3) and (1.4), Brock and Draghici showed that the left hand sides increase under two-point rearrangements if F is any supermodular Borel integrand [13, 14]. Then they approximated the symmetric decreasing rearrangement with sequences of repeated two-point rearrangements. Baernstein-Taylor had established that such sequences can be made to converge to the symmetric decreasing rearrangement in a space of continuous functions [7], and Brock-Solynin had proved this convergence in L^p -spaces [33]. To pass to the desired limits, Brock and Draghici assumed that F is continuous and satisfies some boundary and growth conditions.

No new proofs of these inequalities will be given here. Rather, we reduce general supermodular integrands to the known cases of integrands that are also bounded and continuous. This reduction needs more care than the usual density arguments, because pointwise *a.e.* convergence of a sequence of integrands F_k does not guarantee pointwise *a.e.* convergence of the compositions $F_k(u_1, \dots, u_m)$. Approximation within a class of functions with specified positivity or monotonicity properties can be subtle; for instance, nonnegative functions of m variables cannot always be approximated by positive linear combinations of products of nonnegative functions of the individual variables (contrary to Theorem 2.1 and Lemma 4.1 of [20]).

In Section 5, we prove a variant of Sklar's theorem [34] which factorizes a given supermodular function on \mathbb{R}_+^m as the composition of a Lipschitz continuous supermodular function on \mathbb{R}_+^m with m monotone functions on \mathbb{R}_+ , and a cutoff lemma that replaces a given supermodular function by a bounded supermodular function. Section 6 is dedicated to the two-point

versions of Theorems 1 and 2. Here, we review the proofs of the two-point rearrangement inequalities of Lorentz [10], Brock [13], and Draghici [14] and find their equality cases. The main theorems are proved in Section 7 by combining the results from Sections 5 and 6. Adapting Beckner's argument from [31], we note that the inequalities in Eq. (1.3) and Eq. (1.4) are strict unless u_1, \dots, u_m produce equality in *all* of the corresponding two-point inequalities, and then apply the results from Section 6. In the final Section 8, we briefly discuss extensions for the Brascamp-Lieb-Luttinger and related inequalities.

5 Monotone functions

In this section, we provide two technical results about functions with higher-order monotonicity properties. We begin with an auxiliary lemma for functions of a single variable.

Lemma 5.1 (Monotone change of variable.) *Let ϕ be a nondecreasing real-valued function defined on an interval I . Then, for every function f on I satisfying*

$$|f(z) - f(y)| \leq C(\phi(z) - \phi(y)) \quad (5.1)$$

for all points $y < z \in I$ with some constant C , there exists a Lipschitz continuous function $\tilde{f} : \mathbb{R} \rightarrow [\inf f, \sup f]$ such that $f = \tilde{f} \circ \phi$. Furthermore, if f is nondecreasing, then \tilde{f} is nondecreasing.

PROOF. If $t = \phi(y)$ we set $\tilde{f}(t) := f(y)$. For $s < t$ with $s = \phi(y)$, $t = \phi(z)$, Eq. (5.1) implies that

$$|\tilde{f}(t) - \tilde{f}(s)| = |f(z) - f(y)| \leq C(\phi(z) - \phi(y)) = C(t - s). \quad (5.2)$$

Since \tilde{f} is uniformly continuous on the image of ϕ , it has a unique continuous extension to the closure of the image. The complement consists of a countable number of open disjoint bounded intervals, each representing a jump of ϕ , and possibly one or two unbounded intervals. On each of the bounded intervals, we interpolate \tilde{f} linearly between the values that have already been assigned at the endpoints. If ϕ is bounded either above or below, we extrapolate \tilde{f} to $t > \sup \phi$ and $t < \inf \phi$ by constants.

By construction, $f = \tilde{f} \circ \phi$ and $\tilde{f}(\mathbb{R}) = [\inf f, \sup f]$. The continuous extension and the linear interpolation preserve the modulus of continuity of \tilde{f} , and hence, by Eq. (5.2),

$$|\tilde{f}(t) - \tilde{f}(s)| \leq C|t - s| \quad (5.3)$$

for all $s, t \in \mathbb{R}$. If f is nondecreasing, then \tilde{f} is nondecreasing on the image of ϕ by definition, and on the complement by continuous extension and linear interpolation. ■

Lemma 5.1 is related to the elementary fact that a continuous random variable can be made uniform by a monotone change of variables. More generally, if ϕ is nondecreasing and right continuous and $\psi(t) = \inf\{y : \phi(y) \geq t\}$ is its generalized inverse, then the cumulative distribution functions of two random variables that are related by $Y = \psi(\tilde{Y})$ satisfy

$$F(y) = P(Y \leq y) = P(\tilde{Y} \leq \phi(y)) = \tilde{F} \circ \phi(y).$$

Choosing $\phi = F$ results in a uniform distribution for \tilde{Y} .

The corresponding result for $m \geq 2$ random variables is known as *Sklar's theorem* [34]. The theorem asserts that a collection of random variables Y_1, \dots, Y_m with a given joint distribution function F can be replaced by random variables $\tilde{Y}_1, \dots, \tilde{Y}_m$ whose marginals \tilde{Y}_i are uniformly distributed on $[0, 1]$, and whose joint distribution function \tilde{F} is continuous. The next lemma contains Sklar's theorem for supermodular functions. Since the lemma follows from the arguments outlined in [35] rather than from the statement of the theorem, we include its proof for the convenience of the reader.

We first introduce some notation. Let F be a real-valued function on the closed positive cone \mathbb{R}_+^m . For $i = 1, \dots, m$ and $h \geq 0$, consider the finite difference operators

$$\Delta_i F(\mathbf{y}; h) := F(\mathbf{y} + h\mathbf{e}_i) - F(\mathbf{y}).$$

The operators commute, and higher order difference operators are defined recursively by

$$\Delta_{i_1 \dots i_\ell} F(\mathbf{y}; h_1, \dots, h_\ell) := \Delta_{i_1 \dots i_{\ell-1}} \Delta_{i_\ell} F(\mathbf{y}; h_\ell; h_1, \dots, h_{\ell-1}).$$

If F is ℓ times continuously differentiable, then

$$\Delta_{i_1 \dots i_\ell} F(\mathbf{y}; h_1, \dots, h_\ell) = \int_0^{h_1} \dots \int_0^{h_\ell} \partial_{i_1} \dots \partial_{i_\ell} F\left(\mathbf{y} + \sum_{i=1}^{\ell} t_i \mathbf{e}_i\right) dt_1 \dots dt_\ell.$$

A function F is nondecreasing in each variable if $\Delta_i F \geq 0$ for $i = 1, \dots, m$; it is supermodular, if $\Delta_{ij} F \geq 0$ for all $i \neq j$. The joint distribution function of m random variables satisfies $\Delta_{i_1} \dots \Delta_{i_\ell} F \geq 0$ for any choice of distinct indices i_1, \dots, i_ℓ .

Lemma 5.2 (Sklar's theorem.) *Assume that F is bounded, supermodular, and nondecreasing in each variable on \mathbb{R}_+^m . Then there exist bounded nondecreasing functions ϕ_1, \dots, ϕ_m on \mathbb{R}_+ with $\phi_i(0) = 0$ and a Lipschitz continuous function \tilde{F} on \mathbb{R}_+^m such that*

$$F(y_1, \dots, y_m) = \tilde{F}(\phi_1(y_1), \dots, \phi_m(y_m)).$$

Furthermore, \tilde{F} is bounded, nondecreasing in each variable, and supermodular. If, in addition, $\Delta_{i_1 \dots i_\ell} F \geq 0$ on $\mathbb{R}_+^m \times \mathbb{R}_+^\ell$ for some distinct indices i_1, \dots, i_ℓ , then $\Delta_{i_1 \dots i_\ell} \tilde{F} \geq 0$.

PROOF. Set

$$\phi_i(y) = \lim_{y_j \rightarrow \infty, j \neq i} \left\{ F(y_1, \dots, y_m) \Big|_{y_i=y} - F(y_1, \dots, y_m) \Big|_{y_i=0} \right\}.$$

These functions are nonnegative and bounded by $\sup F - \inf F$. Since F is nondecreasing in each variable, they are nonnegative, and since F is supermodular, they are nondecreasing and satisfy

$$F(\mathbf{y} + h\mathbf{e}_i) - F(\mathbf{y}) \leq \phi_i(y_i + h) - \phi_i(y_i) \tag{5.4}$$

for all $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}_+^m$ and all $h > 0$.

We construct \tilde{F} by changing one variable at a time. For the first variable, we write $\mathbf{y} = (y, \hat{\mathbf{y}})$ where $y \in \mathbb{R}_+$ and $\hat{\mathbf{y}} \in \mathbb{R}_+^{m-1}$. By Eq. (5.4), for each $\hat{\mathbf{y}} \in \mathbb{R}_+^{m-1}$, the function

$f(y) = F(y, \hat{y})$ satisfies Eq. (5.1) with $C = 1$ and $\phi = \phi_1$. By Lemma 5.1, there exists a function F_1 satisfying

$$F(y, \hat{y}) = F_1(\phi_1(y), \hat{y})$$

for all $(y, \hat{y}) \in \mathbb{R}_+^m$. Furthermore, F_1 is Lipschitz continuous in the first variable,

$$|F_1(t, \hat{y}) - F_1(s, \hat{y})| \leq |t - s|.$$

We claim that F_1 satisfies Eq. (5.4) for all $j > 1$ with the same function ϕ_j as F . To see this, note that for each $h > 0$ and every \hat{y} ,

$$f(y) = \Delta_j F(y, \hat{y}; h)$$

satisfies the assumptions of Lemma 5.1 with $C = 2$ and $\phi = \phi_1$. A moment's consideration shows that

$$\tilde{f}(t) = \Delta_j F_1(t, \hat{y}; h)$$

and the claim follows since $\sup \tilde{f} = \sup f \leq \phi_j(y_j + h) - \phi_j(y_j)$ by Lemma 5.1.

Next we verify that F_1 inherits the monotonicity properties of F . Suppose that $\Delta_{i_1 \dots i_\ell} F \geq 0$ for some set of $\ell \geq 1$ distinct indices i_1, \dots, i_ℓ . If $1 \notin \{i_1, \dots, i_\ell\}$, we apply Lemma 5.1 to $f(y) = \Delta_{i_1 \dots i_\ell} F(y, \hat{y}; h_1, \dots, h_\ell)$, which satisfies Eq. (5.1) with $C = 2^\ell$ and $\phi = \phi_1$ for all $\hat{y} \in \mathbb{R}^{m-1}$ and all $h_1, \dots, h_\ell \geq 0$. It follows that $\tilde{f}(t) = \Delta_{i_1 \dots i_\ell} F_1(t, \hat{y}; h_1, \dots, h_\ell) \geq 0$. On the other hand, if $i_1 = 1$, we apply Lemma 5.1 to $f(y) = \Delta_{i_2, \dots, i_\ell} F(y, \hat{y}; h_2, \dots, h_\ell)$. Since $f(y)$ is nondecreasing by assumption, $\tilde{f}(t) = \Delta_{i_2, \dots, i_\ell} F_1(t, \hat{y}; h_2, \dots, h_\ell)$ is again nondecreasing, and we conclude that $\Delta_{i_1 \dots i_\ell} F_1 \geq 0$ also in this case.

Iterating the change of variables for $i = 2, \dots, m$ gives functions F_i satisfying

$$F_{i-1}(t_1, \dots, t_{i-1}, y_i, \dots, y_m) = F_i(t_1, \dots, t_{i-1}, \phi_i(y_i), y_{i+1}, \dots, y_m),$$

as well as

$$0 \leq \Delta_j F_i(t_1, \dots, t_i, y_{i+1}, \dots, y_m; h) \leq \begin{cases} h, & j \leq i \\ \phi_j(y_j + h) - \phi_j(y_j), & j > i. \end{cases} \quad (5.5)$$

The construction is completed by setting $\tilde{F} = F_m$. It follows from Eq. (5.5) that \tilde{F} satisfies the Lipschitz condition $|\tilde{F}(\mathbf{z}) - \tilde{F}(\mathbf{y})| \leq \sum |z_i - y_i| \leq \sqrt{m} |\mathbf{z} - \mathbf{y}|$ for all $\mathbf{y}, \mathbf{z} \in \mathbb{R}_+^m$. ■

The distribution function of a Borel measure on \mathbb{R}_+^m can be conveniently approximated from below by restricting the measure to a large cube $[0, L]^m$. The next lemma constructs the corresponding approximation for functions with weaker monotonicity properties.

Lemma 5.3 (Cutoff.) *Given a real-valued function F in \mathbb{R}_+^m , set*

$$F^L(y_1, \dots, y_m) := F(\min\{y_1, L\}, \dots, \min\{y_m, L\}).$$

If F is nondecreasing in each variable, then $F^L \leq F$. If $\Delta_{i_1 \dots i_\ell} F \geq 0$ on $\mathbb{R}_+^m \times \mathbb{R}_+^\ell$ for some distinct indices i_1, \dots, i_ℓ , then $\Delta_{i_1 \dots i_\ell} F^L \geq 0$. In particular, if F is supermodular, so is F^L . If F has the property that $\Delta_{i_1 \dots i_\ell} F \geq 0$ on $\mathbb{R}_+^m \times \mathbb{R}_+^\ell$ for every set of distinct indices i_1, \dots, i_ℓ , then $F - F^L$ also has this property.

PROOF. As in the proof of Lemma 5.2, we modify the variables one at a time. The function $F^{1,L}(y, \hat{y}) := F(\min\{y, L\}, \hat{y})$ has the same monotonicity properties as F because $\min\{y, L\}$ is nondecreasing in y .

If $\Delta_{i_1 \dots i_\ell} F \geq 0$ for all collections of distinct indices i_1, \dots, i_ℓ , we write

$$F(y, \hat{y}) - F^{1,L}(y, \hat{y}) = \Delta_1 F(y, \hat{y}; [y - L]_+),$$

and it follows that $\Delta_{i_1 \dots i_\ell} (F - F^{1,L}) \geq 0$ whenever $1 \notin \{i_1, \dots, i_\ell\}$. For $i_1 = 1$, we write

$$\Delta_1 (F(y, \hat{y}; h) - F^{1,L}(y, \hat{y}; h)) = \Delta_1 F(\max\{y, L\}, \hat{y}; [h - [L - y]_+]_+),$$

and conclude that $\Delta_{i_1 \dots i_\ell} (F - F^{1,L}) \geq 0$ also in this case.

Repeating the construction for the variables y_2, \dots, y_m gives the claims. \blacksquare

6 Two-point rearrangements

Let \mathbb{X} be \mathbb{R}^n , \mathbb{S}^n , or \mathbb{H}^n . A *reflection* on \mathbb{X} is an isometry characterized by the properties that (i) $\sigma^2 x = x$ for all $x \in \mathbb{X}$; (ii) the fixed point set H_0 of σ separates M into two half-spaces H_+ and H_- that are interchanged by σ ; and (iii) $d(x, x') < d(x, \sigma x')$ for all $x, x' \in H_+$. We call H_+ and H_- the *positive* and *negative half-spaces* associated with σ . By convention, we always choose H_+ to contain the distinguished point x^* of \mathbb{X} in its closure. The *two-point rearrangement*, or *polarization* of a real-valued function u with respect to a reflection σ is defined by

$$u^\sigma(x) = \begin{cases} \max\{u(x), u(\sigma x)\}, & x \in H_+ \cup H_0 \\ \min\{u(x), u(\sigma x)\}, & x \in H_- . \end{cases}$$

This definition makes sense, and the two-point versions of Eqs. (1.3) and (1.4) hold for any space with a reflection symmetry.

On $\mathbb{X} = \mathbb{R}^n$, \mathbb{S}^n , and \mathbb{H}^n , any pair of points is connected by a unique reflection. The space of reflections forms an n -dimensional submanifold of the $n(n+1)/2$ -dimensional space of isometries, and thus has a natural uniform metric. If u is measurable, both the composition $u \circ \sigma$ and the rearrangement u^σ depend continuously on σ in the sense that $\sigma_k \rightarrow \sigma$ implies that $u \circ \sigma_k \rightarrow u \circ \sigma$ and $u^{\sigma_k} \rightarrow u^\sigma$ in measure.

Two-point rearrangements are particularly well-suited for identifying symmetric decreasing functions, because

$$u = u^* \iff u = u^\sigma \text{ for all } \sigma . \quad (6.1)$$

Functions that are radially decreasing about some point are characterized by

$$u = u^* \circ \tau \text{ for some translation } \tau \iff \text{for all } \sigma, \text{ either } u = u^\sigma \text{ or } u = u^\sigma \circ \sigma \quad (6.2)$$

(see Lemma 2.8 of [28]).

Integral inequalities for two-point rearrangements typically reduce to elementary combinatorial inequalities for the integrands. The following lemma supplies the elementary inequality for the Hardy-Littlewood and Riesz functionals.

Lemma 6.1 (Lorentz two-point inequality.) *A real-valued function F on \mathbb{R}_+^m is supermodular, if and only if for any pair of points $\mathbf{z}, \mathbf{w} \in \mathbb{R}_+^m$.*

$$\begin{aligned} & F(z_1, \dots, z_m) + F(w_1, \dots, w_m) \\ & \leq F(\min\{z_1, w_1\}, \dots, \min\{z_m, w_m\}) + F(\max\{z_1, w_1\}, \dots, \max\{z_m, w_m\}). \end{aligned} \quad (6.3)$$

If $\Delta_{ij}F > 0$ for some $i \neq j$ then Eq. (6.3) is strict unless $(z_i - w_i)(z_j - w_j) \geq 0$.

PROOF. Given $\mathbf{z}, \mathbf{w} \in \mathbb{R}_+^m$, define $\mathbf{y}, \mathbf{h} \in \mathbb{R}_+^m$ by $y_i = \min\{z_i, w_i\}$ and $h_i = |z_i - w_i|$ for $i = 1, \dots, m$. If $I \subset \{1, \dots, m\}$, we use the notation $\mathbf{h}_I = \sum_{i \in I} h_i \mathbf{e}_i$. Subtracting the left hand side of Eq. (6.3) from the right hand side results in the equivalent statement

$$\Delta_{IJ}F(\mathbf{y}; \mathbf{h}_I, \mathbf{h}_J) := F(\mathbf{y} + \mathbf{h}_{I \cup J}) - F(\mathbf{y} + \mathbf{h}_I) - F(\mathbf{y} + \mathbf{h}_J) + F(\mathbf{y}) \geq 0, \quad (6.4)$$

where $I = \{i : z_i < w_i\}$, and $J = \{i : z_i > w_i\}$. If either I or J is empty, Eq. (6.4) is trivially satisfied. If I and J each have exactly one element, Eq. (6.4) is equivalent to Eq. (1.5). If one of the sets, say I , has several elements, then decomposing it into disjoint subsets as $I = I' \cup I''$ gives

$$\Delta_{IJ}F(\mathbf{y}; \mathbf{h}_I, \mathbf{h}_J) = \Delta_{I'J}F(\mathbf{y} + \mathbf{h}_{I''}, \mathbf{h}_{I'}, \mathbf{h}_J) + \Delta_{I''J}F(\mathbf{y}, \mathbf{h}_{I'}, \mathbf{h}_J),$$

and Eq. (6.4) follows by recursion. The same recursion implies that if $\Delta_{ij}F > 0$ and $z_i - w_i$ and $z_j - w_j$ have opposite signs, then the inequality in Eq. (6.4) is strict whenever I contains i , J contains j , and $h_i, h_j > 0$. ■

Brock proved that the left hand side of Eq. (1.3) increases under two-point rearrangement [13]:

Lemma 6.2 (Hardy-Littlewood two-point inequality.) *Let F be a supermodular Borel measurable function on \mathbb{R}_+^m , and let u_1, \dots, u_m be nonnegative measurable functions on \mathbb{X} satisfying the integrability condition in Eq. (2.1). Then, for any reflection σ on \mathbb{X} ,*

$$\int_{\mathbb{X}} F(u_1(x), \dots, u_m(x)) dx \leq \int_{\mathbb{X}} F(u_1^\sigma(x), \dots, u_m^\sigma(x)) dx. \quad (6.5)$$

Assume furthermore that $\Delta_{ij}F > 0$ on $\mathbb{R}_+^m \times (0, \infty)^2$ for some $i \neq j$. If Eq. (6.5) holds with equality, and the integrals are finite, then

$$(u_i(x) - u_i(\sigma x))(u_j(x) - u_j(\sigma x)) \geq 0 \quad a.e..$$

In particular, if $u_i = u_i^*$ is strictly radially decreasing and $\sigma(x^*) \neq x^*$, then $u_j = u_j^\sigma$.

PROOF. (The inequality [13].) The left hand side of Eq. (6.5) can be written as an integral over the positive half-space,

$$\mathcal{I}(u_1, \dots, u_m) := \int_{H_+} F(u_1(x), \dots, u_m(x)) + F(u_1(\sigma x), \dots, u_m(\sigma x)) dx.$$

By Lemma 6.1, with $z_i = u_i(x)$ and $w_i = u_i(\sigma x)$, the integrand satisfies

$$\begin{aligned} & F(u_1(x), \dots, u_m(x)) + F(u_1(\sigma x), \dots, u_m(\sigma x)) \\ & \leq F(u_1^\sigma(x), \dots, u_m^\sigma(x)) + F(u_1^\sigma(\sigma x), \dots, u_m^\sigma(\sigma x)) \end{aligned} \quad (6.6)$$

for all $x \in H_+$. Integrating over H_+ yields Eq. (6.5).

(Equality statement.) Assume that $\mathcal{I}(u_1, \dots, u_m) = \mathcal{I}(u_1^\sigma, \dots, u_m^\sigma)$ is finite. Then Eq. (6.6) must hold with equality almost everywhere on H_+ . If $\Delta_{ij}F > 0$ on $\mathbb{R}_+^m \times (0, \infty)^2$, then Lemma 6.1 implies that $u_i(x) - u_i(\sigma x)$ and $u_j(x) - u_j(\sigma x)$ cannot have opposite signs except on a set of zero measure. If moreover $u_i = u_i^*$ is strictly radially decreasing and $\sigma x^* \neq x^*$, then $u_i(x) > u_i(\sigma x)$ for a.e. $x \in H_+$, and Lemma 6.1 implies that $u_j(x) \geq u_j(\sigma x)$ for a.e. $x \in H_+$. ■

Brock completed the proof of Eq. (1.3) by approximating the symmetric decreasing rearrangement with a sequence of two-point rearrangements à la Baernstein-Taylor [7]. We sketch his argument in the simplest case where F is a continuous supermodular function that vanishes on the boundary of the positive cone \mathbb{R}_+^m , and u_1, \dots, u_m are bounded and compactly supported.

By Theorem 6.1 of [33] there exists a sequence of reflections $\{\sigma_k\}_{k \geq 1}$ such that

$$u_i^{\sigma_1, \dots, \sigma_k} \rightarrow u_i^* \quad \text{in measure } (k \rightarrow \infty) \quad (6.7)$$

for $i = 1, \dots, m$. By Lemma 6.2, the functional increases monotonically along such a sequence. If B is a ball centered at x^* that contains the supports of u_1, \dots, u_m , then the rearranged functions $u_i^{\sigma_1, \dots, \sigma_k}$ are also supported on B , and dominated convergence yields

$$\mathcal{I}(u_1, \dots, u_m) \leq \mathcal{I}(u_1^{\sigma_1, \dots, \sigma_k}, \dots, u_m^{\sigma_1, \dots, \sigma_k}) \rightarrow \mathcal{I}(u_1^*, \dots, u_m^*) \quad (k \rightarrow \infty). \quad (6.8)$$

The corresponding results for Eq. (1.4) are due to Draghici [14]. The two-point inequality is not an immediate consequence of Lemma 6.1, but requires an additional combinatorial argument. This argument was used previously by Morpurgo [29], and a simpler version appears in [28].

Lemma 6.3 (Riesz two-point inequality.) *Assume that F is a supermodular Borel measurable function on \mathbb{R}_+^m . For each pair of indices $1 \leq i < j \leq m$, let K_{ij} be a nonincreasing function on \mathbb{R}_+ , and let u_1, \dots, u_m be nonnegative measurable functions on \mathbb{X} satisfying the integrability condition in Eq. (2.3). Then, for any reflection σ ,*

$$\begin{aligned} & \int_{\mathbb{X}} \dots \int_{\mathbb{X}} F(u_1(x_1), \dots, u_m(x_m)) \prod_{i < j} K_{ij}(d(x_i, x_j)) dx_1 \dots dx_m \\ & \leq \int_{\mathbb{X}} \dots \int_{\mathbb{X}} F(u_1^\sigma(x_1), \dots, u_m^\sigma(x_m)) \prod_{i < j} K_{ij}(d(x_i, x_j)) dx_1 \dots dx_m. \end{aligned} \quad (6.9)$$

Assume additionally that that $K_{ij}(t) > 0$ for all $i < j$ and all $t < \text{diam } \mathbb{X}$. Let Γ_0 be the graph on $\{1, \dots, m\}$ with an edge between i and j whenever K_{ij} is strictly decreasing. If $\Delta_{ij}F > 0$ for some $i \neq j$ lying in the same connected component of Γ_0 , and that u_i and u_j are not symmetric under σ . If the integrals in Eq. (6.9) have the same finite value, then either $u_i = u_i^\sigma$ and $u_j = u_j^\sigma$, or $u_i = u_i^\sigma \circ \sigma$ and $u_j = u_j^\sigma \circ \sigma$.

PROOF. (*The inequality [14].*) The left hand side of Eq. (6.9) can be written as an m -fold integral over the positive half-space

$$\begin{aligned} \mathcal{I}(u_1, \dots, u_m) &:= \int_{H_+} \dots \int_{H_+} \sum_{\varepsilon_i \in \{0,1\}, i=1, \dots, m} \left\{ F(u_1(\sigma^{\varepsilon_1} x_1), \dots, u_m(\sigma^{\varepsilon_m} x_m)) \times \right. \\ &\quad \left. \times \prod_{i < j} K_{ij}(d(\sigma^{\varepsilon_i} x_i, \sigma^{\varepsilon_j} x_j)) \right\} dx_1 \dots dx_m. \end{aligned} \quad (6.10)$$

Fix $x_1, \dots, x_m \in H_+$. For each $i < j$, set $a_{ij} = K_{ij}(d(x_i, \sigma x_j))$ and $b_{ij} = K_{ij}(d(x_i, x_j)) - K_{ij}(d(x_i, \sigma x_j))$, so that

$$K_{ij}(d(\sigma^{\varepsilon_i} x_i, \sigma^{\varepsilon_j} x_j)) = a_{ij} + b_{ij} \mathbf{1}_{\varepsilon_i = \varepsilon_j}.$$

The product term in Eq. (6.10) expands to

$$\prod_{i < j} K_{ij}(d(\sigma^{\varepsilon_i} x_i, \sigma^{\varepsilon_j} x_j)) = \sum_{\Gamma} \left(\prod_{ij \notin E} a_{ij} \right) \left(\prod_{ij \in E} b_{ij} \mathbf{1}_{\varepsilon_i = \varepsilon_j} \right) =: C_{\Gamma} \mathbf{1}_{\varepsilon_i = \varepsilon_j, ij \in E},$$

where Γ runs over all proper graphs on the vertex set $V = \{1, \dots, m\}$, and E is the set of edges of Γ . Inserting the expansion into Eq. (6.10) and exchanging the order of summation, we see that each graph contributes a nonnegative term

$$C_{\Gamma} \sum_{\varepsilon_i \in \{0,1\}, i \in V} F(u_1(\sigma^{\varepsilon_1} x_1), \dots, u_m(\sigma^{\varepsilon_m} x_m)) \mathbf{1}_{\varepsilon_i = \varepsilon_j, ij \in E} \quad (6.11)$$

to the integral in Eq. (6.10). If Γ is connected, then

$$\begin{aligned} &\sum_{\varepsilon_i \in \{0,1\}, i \in V} F(u_1(\sigma^{\varepsilon_1} x_1), \dots, u_m(\sigma^{\varepsilon_m} x_m)) \mathbf{1}_{\varepsilon_i = \varepsilon_j, ij \in E} \\ &= F(u_1(x_1), \dots, u_m(x_m)) + F(u_1(\sigma x_1), \dots, u_m(\sigma x_m)) \\ &\leq F(u_1^{\sigma}(x_1), \dots, u_m^{\sigma}(x_m)) + F(u_1^{\sigma}(\sigma x_1), \dots, u_m^{\sigma}(\sigma x_m)) \\ &= \sum_{\varepsilon_i \in \{0,1\}, i \in V} F(u_1^{\sigma}(\sigma^{\varepsilon_1} x_1), \dots, u_m^{\sigma}(\sigma^{\varepsilon_m} x_m)) \mathbf{1}_{\varepsilon_i = \varepsilon_j, ij \in E}, \end{aligned} \quad (6.12)$$

where the second step follows from Lemma 6.1 with $z_i = u_i(x_i)$ and $w_i = u_i(\sigma x_i)$.

If Γ is not connected, choose a connected component Γ' and let Γ'' be its complement. Let E', E'', V' , and V'' be the corresponding edge and vertex sets. The sum in Eq. (6.11) can be decomposed as

$$\sum_{\varepsilon_i \in \{0,1\}, i \in V''} \left\{ \sum_{\varepsilon_i \in \{0,1\}, i \in V'} F(u_1(\sigma^{\varepsilon_1} x_1), \dots, u_m(\sigma^{\varepsilon_m} x_m)) \mathbf{1}_{\varepsilon_i = \varepsilon_j, ij \in E'} \right\} \mathbf{1}_{\varepsilon_i = \varepsilon_j, ij \in E''}.$$

The key observation is that Eq. (6.12) applies to the term in braces for fixed $\varepsilon_i, i \in V''$; in other words, the contribution of Γ can only increase if u_i is replaced by u_i^{σ} for all $i \in V'$. An

induction over the connected components of Γ shows that

$$\begin{aligned} \sum_{\varepsilon_i \in \{0,1\}, i \in V} F(u_1(\sigma^{\varepsilon_1} x_1), \dots, u_m(\sigma^{\varepsilon_m} x_m)) \mathbf{1}_{\varepsilon_i = \varepsilon_j, ij \in E} \\ \leq \sum_{\varepsilon_i \in \{0,1\}, i \in V} F(u_1^\sigma(\sigma^{\varepsilon_1} x_1), \dots, u_m^\sigma(\sigma^{\varepsilon_m} x_m)) \mathbf{1}_{\varepsilon_i = \varepsilon_j, ij \in E} \end{aligned}$$

for any graph $\Gamma = (E, V)$. Adding the contributions of all graphs shows that the integrand in Eq. (6.10) increases pointwise under two-point rearrangement, and Eq. (6.9) follows.

(Equality statement.) Let Γ_0 be the graph defined in the statement of the lemma, and let E_0 be its edge set. By assumption,

$$C_{\Gamma_0} = \left(\prod_{ij \notin E_0} K_{ij}(d(x_i, x_j)) - K_{ij}(d(\sigma x_i, x_j)) \right) \left(\prod_{ij \in E_0} K_{ij}(d(\sigma x_i, x_j)) \right) > 0$$

for a.e. $x_1, \dots, x_m \in H_+$. If $\Delta_{ij} F > 0$, then Lemma 6.1 implies that Eq. (6.12) is strict unless

$$(u_i(x_i) - u_i(\sigma x_i))(u_j(x_j) - u_j(\sigma x_j)) \geq 0, \quad a.e. x_i, x_j \in H_+.$$

If u_i and u_j are not symmetric under σ , the product is not identically zero. Since x_i and x_j can vary independently, this means that $u_i(x) - u_i(\sigma x)$ and $u_j(x) - u_j(\sigma x)$ cannot change sign on H_+ . We conclude that equality in Eq. (6.9) implies that either $u_i = u_i^\sigma$ and $u_j = u_j^\sigma$, or $u_i = u_i^\sigma \circ \sigma$ and $u_j = u_j^\sigma \circ \sigma$. \blacksquare

Draghici also used Baernstein-Taylor approximation to obtain Eq. (1.4) from Eq. (6.9). If F is bounded and continuous and K_{ij} is bounded for $1 \leq i < j \leq m$, then for bounded functions u_1, \dots, u_m that are supported in a common ball B the inequality follows from Lemma 6.3 by approximating the symmetric decreasing rearrangement with a sequence of two-point rearrangements, see Eq. (6.7). Dominated convergence applies as in Eq. (6.8), since the integrations extend only over the bounded set B^m .

7 Proof of the main results

PROOF OF THEOREM 1. (The inequality for Borel integrands.) Let F be a supermodular Borel function with $F(\mathbf{0}) = 0$, and let u_1, \dots, u_m be nonnegative measurable functions that vanish at infinity, as in the statement of the theorem. Denote by

$$\mathcal{I}(u_1, \dots, u_m) := \int_{\mathbb{X}} F(u_1(x), \dots, u_m(x)) dx$$

the left hand side of Eq. (1.3). Replacing $F(\mathbf{y})$ by $F(\mathbf{y}) - \sum_{i=1}^m F(y_i \mathbf{e}_i)$ and using that $F(u_i(\cdot) \mathbf{e}_i)$ and $F(u_i^*(\cdot) \mathbf{e}_i)$ contribute equally to the two sides of Eq. (1.3), we may assume F to be nondecreasing in each variable.

Fix $L > 0$, and replace u_i by the bounded function

$$u_i^L(x) := \min \{u_i(x), L\} \mathbf{1}_{\{|x| < L\}}$$

for $i = 1, \dots, m$. Then

$$F(u_1^L, \dots, u_m^L) = F^L(u_1^L, \dots, u_m^L), \quad (7.1)$$

where F^L is the function defined in Lemma 5.3. By construction, F^L is bounded, and by Lemma 5.3 it is nondecreasing and supermodular. By Lemma 5.2, there exist nondecreasing functions ϕ_i with $\phi_i(0) = 0$ and a continuous supermodular function \tilde{F}^L on \mathbb{R}_+^m such that

$$F^L(y_1, \dots, y_m) = \tilde{F}^L(\phi_1(y_1), \dots, \phi_m(y_m)). \quad (7.2)$$

Since ϕ_i is nondecreasing and vanishes at zero, u_i^L is compactly supported, and $(u_i^L)^* \leq (u_i^*)^L$ pointwise by construction, we have

$$(\phi_i \circ u_i^L)^* = \phi_i \circ (u_i^L)^* \leq \phi_i \circ (u_i^*)^L \quad (7.3)$$

for $i = 1, \dots, m$. By Theorem 1 of [13],

$$\int_{\mathbb{X}} \tilde{F}^L(\phi_1 \circ u_1^L(x), \dots, \phi_m \circ u_m^L(x)) dx \leq \int_{\mathbb{X}} \tilde{F}^L((\phi_1 \circ u_1^L)^*(x), \dots, (\phi_m \circ u_m^L)^*(x)) dx.$$

With Eqs. (7.1)-(7.3), this becomes

$$\mathcal{I}(u_1^L, \dots, u_m^L) \leq \mathcal{I}((u_1^*)^L, \dots, (u_m^*)^L).$$

Since $u_i^L(x) = u_i(x)$ for $L \geq \max\{u_i(x), |x|\}$, we see that $F(u_i^L(x), \dots, u_m^L(x))$ converges pointwise to $F(u_1(x), \dots, u_m(x))$, and Eq. (1.3) follows by monotone convergence.

(Equality statement.) Combining Eq. (6.5) with Eq. (1.3) and using that u_i^σ is equimeasurable with u_i , we see that

$$\mathcal{I}(u_1, \dots, u_m) \leq \mathcal{I}(u_1^\sigma, \dots, u_m^\sigma) \leq \mathcal{I}(u_1^*, \dots, u_m^*).$$

Hence equality in Eq. (1.3) implies equality in Eq. (6.5) for every choice of the reflection σ . Given two points x, x' in \mathbb{X} , choose σ such that $\sigma(x) = x'$. If $\Delta_{ij}F > 0$ for some $i \neq j$, then $u_i(x) - u_i(x')$ and $u_j(x) - u_j(x')$ cannot have opposite signs by Lemma 6.2. If $u_i = u_i^*$ is strictly radially decreasing, then it follows that $u_j^\sigma = u_j$ for every reflection σ that does not fix x^* . By Eq. (6.1), $u_j = u_j^*$ as claimed. \blacksquare

PROOF OF THEOREM 2. (The inequality for Borel integrands.) The proof of Eq. (1.4) proceeds along the same lines as the proof of Eq. (1.3). Let

$$\mathcal{I}(u_1, \dots, u_m) := \int_{\mathbb{X}} \cdots \int_{\mathbb{X}} F(u_1(x_1), \dots, u_m(x_m)) \prod_{i < j} K_{ij}(d(x_i, x_j)) dx_1 \dots dx_m$$

be the left hand side of Eq. (1.4). As before, we may assume that F is nondecreasing in each variable. We replace F with \tilde{F}^L , u_i with $\phi_i \circ u_i^L$, K_{ij} with $K_{ij}^L = \min\{K_{ij}, L\}$, and set

$$\mathcal{I}^L(u_1, \dots, u_m) := \int_{\mathbb{X}} \cdots \int_{\mathbb{X}} F^L(u_1(x_1), \dots, u_m(x_m)) \prod_{i < j} K_{ij}^L(d(x_i, x_j)) dx_1 \dots dx_m.$$

Applying Theorem 2.2 of [14], we obtain with the help of Eqs. (7.1)-(7.3)

$$\mathcal{I}^L(u_1^L, \dots, u_m^L) \leq \mathcal{I}^L((u_1^*)^L, \dots, (u_m^*)^L) .$$

Eq. (1.4) follows by taking $L \rightarrow \infty$ and using monotone convergence.

(*Equality statement.*) Consider the set S_i of all reflections σ of \mathbb{X} that fix u_i . If u_i is non-constant, then S_i is a closed proper subset of the space of all reflections on \mathbb{X} . This subset is nowhere dense, since any open set of reflections generates the entire isometry group of \mathbb{X} . If Eq. (1.4) holds with equality, then the two-point rearrangement inequality in Eq. (6.9) holds with equality for every reflection σ . For $\sigma \notin S_i$, Lemma 6.3 implies that either $u_j = u_j^\sigma$ or $u_j = u_j^\sigma \circ \sigma$. Since S_i is nowhere dense, it follows from the continuous dependence of u^σ on σ that u_j agrees with either u_j^σ or $u_j^\sigma \circ \sigma$ also for $\sigma \in S_i$. By Eq. (6.2), there exists a translation τ such that $u_j = u_j^* \circ \tau$. Lemma 6.3 implies furthermore that u_i agrees with u_i^σ when $u_j = u_j^\sigma$, and with $u_i^\sigma \circ \sigma$ when $u_j = u_j^\sigma \circ \sigma$. We conclude that $u_i = u_i^* \circ \tau$. ■

8 Concluding remarks

In the proof of Eq. (1.4) and its two-point version in Eq. (6.9), the kernels $K_{ij}(d(x_i, x_j))$ played a very different role from the functions u_1, \dots, u_m that enter into the integrand. However, the Riesz functional on the left hand side of Eq. (1.2) depends equally on u, v , and w . We will use the connection of Riesz' inequality with the Brunn-Minkowski inequality to construct examples where the two-point rearrangement fails for Eq. (1.2).

The Brunn-Minkowski inequality says that the measures of two subsets $A, B \subset \mathbb{R}^n$ are related to the measure of their Minkowski sum $A + B = \{a + b : a \in A, b \in B\}$ by

$$\lambda(A)^{1/n} + \lambda(B)^{1/n} \leq \lambda(A + B)^{1/n} .$$

Recognizing the two sides of the inequality as proportional to the radii of the balls $A^* + B^*$ and $(A + B)^*$, we rewrite it as the rearrangement inequality

$$\lambda(A^* + B^*) \leq \lambda(A + B) . \quad (8.1)$$

Eq. (8.1) follows rather directly from Riesz' inequality in Eq. (1.2), because the support of the convolution of two nonnegative functions is essentially the Minkowski sum of their supports. Conversely, the Brunn-Minkowski inequality enters into the proof of the Brascamp-Lieb-Luttinger inequality [9], of which Eq. (1.2) is a special case.

Equality in the Brunn-Minkowski inequality implies that A and B differ only by sets of measure zero from two independently scaled and translated copies of a convex body [36]. Let $A = B$ be an ellipsoid in \mathbb{R}^n with $n > 1$ that is centered at a point $c \neq 0$, so that Eq. (8.1) holds with equality. If σ is the reflection at a hyperplane through c that is not a hyperplane of symmetry for A and B , then A^σ and B^σ are non-convex, and therefore

$$\lambda(A^\sigma + B^\sigma) > \lambda(A^* + B^*) = \lambda(A + B) .$$

Choosing u, v , and w as the characteristic functions of $A, A + B$, and B provides an example where the Riesz functional strictly decreases under two-point rearrangement. For an example in one dimension, consider the symmetric decreasing functions

$$u(x) = \mathbf{1}_{|x-2| < \varepsilon} , \quad v(x) = w(x) = \mathbf{1}_{|x-1| < \varepsilon} ,$$

and let σ be the reflection at $x = 1$. Then

$$u^\sigma(x) = \mathbf{1}_{|x| < \varepsilon}, \quad v^\sigma(x) = w^\sigma(x) = \mathbf{1}_{|x-1| < \varepsilon},$$

and if $0 < \varepsilon \leq \frac{1}{2}$, Riesz' inequality fails for σ ,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u(x)v(x')w(x-x') dx dx' > 0 = \int_{\mathbb{R}} \int_{\mathbb{R}} u^\sigma(x)v^\sigma(x)w^\sigma(x-x') dx dx'.$$

While the two-point rearrangement is not useful for Eq. (1.2), the layer-cake representation of Crowe-Zweibel-Rosenbloom shows that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(u(x), v(x'), w(x-x')) dx dx' \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(u^*(x), v^*(x'), w^*(x-x')) dx dx' \quad (8.2)$$

for any integrand that can be written as the joint distribution function of a Borel measure μ_F on \mathbb{R}_+^3 ,

$$F(y_1, y_2, y_3) = \mu_F([0, y_1] \times [0, y_2] \times [0, y_3]).$$

Such integrands are left continuous, vanish at the origin, and satisfy $\Delta_{i_1, \dots, i_\ell} F \geq 0$ for every choice of $\ell \leq 3$ distinct indices. Lemma 5.2 allows to accommodate integrands in Eq. (8.2) that are only Borel measurable. The main condition is that $\Delta_{123} F \geq 0$; the second-order monotonicity conditions can be replaced by integrability assumptions on the negative part F_- similar to Eq. (2.3). To ensure that the functional is finite at least when u, v, w are bounded and compactly supported, F should vanish on the coordinate axes. For example, Eq. (8.2) holds for

$$F(u, v, w) = \frac{uvw}{(1+u)(1+v)(1+w)} - (uv + uw + vw)$$

since $\Delta_{123} F > 0$, even though $\Delta_{ij} F < 0$ for all $i \neq j$.

For Borel integrands satisfying $\Delta_{123} F > 0$, equality in Eq. (8.2) implies that every triple of level sets of u, v, w produces equality in Eq. (1.2). These equality cases were described in [38]. In particular, if two of the three functions u, v, w are known to have continuous distribution functions and the value of the functional is finite, then equality implies that u, v, w are equivalent to u^*, v^*, w^* under the symmetries of the functional (see Theorem 2 of [38]).

By the same line of reasoning, the Brascamp-Lieb-Luttinger inequality [9] implies that

$$\mathcal{I}(u_1, \dots, u_m) := \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} F\left(u_1\left(\sum_{j=1}^k a_{1j}x_j\right), \dots, u_m\left(\sum_{j=1}^k a_{mj}x_j\right)\right) dx_1 \dots dx_k$$

increases under symmetric decreasing rearrangement, if $\Delta_{i_1, \dots, i_\ell} F \geq 0$ for all choices of distinct indices i_1, \dots, i_ℓ with $\ell \leq m$. Interesting examples are integrands of the form in Eq. (3.3), where Φ is completely monotone in the sense that all its distributional derivatives are nonnegative. If $\Delta_{i_1 \dots i_\ell} F > 0$ for all choices of i_1, \dots, i_ℓ , then the last statement of Lemma 5.3 can be used to show that the extended Brascamp-Lieb-Luttinger inequality has the same equality cases as the original inequality. The characterization of these equality cases remains an open problem.

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