Advanced Nonlinear Studies **4** (2004), 469–501

On the Variational Approach to the Stability of Standing Waves for the Nonlinear Schrödinger Equation

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Received 12 July 2004

Abstract

We consider the orbital stability of standing waves of the nonlinear Schrdinger equation

$$i\frac{\partial\Phi}{\partial t}(t,\,x)+\Delta\Phi(t,\,x)+g(x,\,|\Phi|^2)\Phi(t,\,x)=0$$

by the approach that was laid down by Cazenave and Lions in 1992. Our work covers several situations that do not seem to be included in previous treatments, namely,
(i) $g(x,s)-g(x,0)\to 0$ as $|x|\to\infty$ for all $s\geq 0$. This includes linear problems.
(ii) $g(x,s)$ is a periodic function of $x\in\mathbb{R}^N$ for all $s\geq 0$.
(iii) $g(x,s)$ is asymptotically periodic in the sense that $g(x,s)-g^{\infty}(x,s)\to 0$ as $|x|\to\infty$ for some function g^{∞} that is periodic with respect to $x\in\mathbb{R}^N$ for all $s\geq 0$.
Furthermore, we focus attention on the form of the set that is shown to be stable and may be bigger than what is usually known as the orbit of the standing wave.

*The author acknowledges the support the Fonds National Suisse pour la Recherche Scientifique.

1991 Mathematics Subject Classification. 35Q55, 37K45.
Key words. orbital stability, standing waves, nonlinear Schrödinger equation.

1 Introduction

This paper concerns the orbital stability of standing waves for nonlinear Schrödinger equations of the form

$$\left.\begin{array}{c} i\frac{\partial \Phi}{\partial t}(t,\,x)+\Delta\Phi(t,\,x)+g(x,\,|\Phi|^2)\Phi(t,\,x)=0 \\ \Phi(0,\,x)=\Phi_0(x) \end{array}\right\} \tag{1.1}$$

where $t\in\mathbb{R}$, $x\in\mathbb{R}^N$ and $\Phi_0\in H^1(\mathbb{R}^N,\,\mathbb{C})$.

Global existence and uniqueness of solutions of (1.1) when $g(x,\,s^2)=|s|^{p-1}$ for $1<p<1+\frac{4}{N}$ has been studied by many people, notably Cazenave. We are particularly interested in cases where g is not independent of x, and we cite a result from Cazenave [5], see Proposition 2.4 below, giving the information we need. The inhomogeneous version of (1.1) arises in several different contexts, notably, the study of spatial solitons in nonlinear waveguides (see [9],[21]) and the theory of Bose-Einstein condensates (see [3]), where it is sometimes called the Gross-Pitaevskii equation. Cases where $g(x,s)$ converges to a function $g^\infty(x,s)$ that depends periodically on x are of particular interest in these applications.

An important issue concerning (1.1) is the orbital stability of standing waves for (1.1), see [15]. First, let us recall that a standing wave of (1.1) is a solution of (1.1) having the form $z(x)e^{-i\lambda t}$ where $\lambda\in\mathbb{R}$. For solutions of this type with $z\in H^1(\mathbb{R}^N,\,\mathbb{C})$, (1.1) is equivalent to

$$\Delta z(x)+g(x,\,|z(x)|^2)z(x)+\lambda z(x)=0, \tag{1.2}$$

which can be written as a 2×2 real elliptic system for (u,v) where $z=u+iv$, namely,

$$\begin{cases}\Delta u+g(x,\,u^2+v^2)u+\lambda u=0\\ \Delta v+g(x,\,u^2+v^2)v+\lambda v=0.\end{cases} \tag{1.3}$$

Seeking solutions with $v\equiv 0$ leads to the scalar equation

$$\Delta u+g(x,\,u^2)u+\lambda u=0 \tag{1.4}$$

which has been intensively studied and constitutes in itself an important chapter of nonlinear analysis going back to the early work of Nehari and Berger, developed by Strauss, Berestyski and Lions and continued by a multitude of other contributions.

Broadly speaking, there are two approaches to determining the stability of standing waves of (1.1), see Chapter 3 of [24] for example. One method uses what is sometimes known as the Vakhitov-Kolokolov criterion (see [15], [22]) and it has been given a rigorous mathematical basis in [10], [18], [23], [8]. Under appropriate conditions, it reduces the stability question to that of checking that $\frac{d}{d\lambda}\int u_\lambda^2 dx<0$ for certain solutions u_λ of (1.4). For non-autonomous equations, finding explicit conditions on the nonlinearity g that imply this monotonicity seems to be rather difficult, [17]. The other method exploits the Hamiltonian structure of (1.1) through the characterization of the standing wave as a constrained minimum, [6]. According to Section 2 of [1], this approach, which is the one we follow here, can be traced back to Boussinesq [2]. It establishes stability under explicit assumptions on g but, except in some special cases, the conclusion that is obtained is weaker that what is usually referred to as orbital stability because the set which plays the role of the orbit is bigger. We explore this issue further below.

Before formulating a notion of stability of standing waves for (1.1), it is convenient to introduce some notation and definitions.

Let
$$H = \{z = (u, v) \in H^1 \times H^1\} \text{ where } H^1 \equiv H^1(\mathbb{R}^N, \mathbb{R}).$$

We shall sometimes identify $z = (u, v)$ with $u + iv$ and H with $H^1(\mathbb{R}^N, \mathbb{C})$. For $z = (u, v) \in H$,

$$\|z\|_H^2 = \|z\|_2^2 + \|\nabla z\|_2^2 \text{ where } \|z\|_2^2 = |u|_2^2 + |v|_2^2 \text{ and } \|\nabla z\|_2^2 = |\nabla u|_2^2 + |\nabla v|_2^2.$$

Here and elsewhere $|\cdot|_p$ denotes the usual norm on $L^p(\mathbb{R}^N, \mathbb{R}) = L^p$.

We associate with (1.1) the energy functionals, $\tilde{E} : H \to \mathbb{R}$ and $E : H^1 \to \mathbb{R}$, defined by

$$\begin{aligned}
\tilde{E}(z) &= \frac{1}{2}\|\nabla z\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} G(x, |z|)dx && (1.5)\\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 - G(x, \sqrt{u^2 + v^2})dx,\\
E(u) &= \tilde{E}(u, 0) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - G(x, |u|)dx && (1.6)
\end{aligned}$$

where

$$G(x, s) = \int_0^{s^2} g(x, t)\, dt. \quad (1.7)$$

For $c > 0$, we set

$$\begin{aligned}
\tilde{S}_c &= \{z \in H : \|z\|_2^2 = c^2\}, & S_c &= \{u \in H^1 : |u|_2^2 = c^2\},\\
M_c &= \inf\{\tilde{E}(z) : z \in \tilde{S}_c\}, & m_c &= \inf\{E(u) : u \in S_c\},\\
Z_c &= \{z \in \tilde{S}_c : \tilde{E}(z) = M_c\}, & W_c &= \{u \in S_c \cap C^1(\mathbb{R}^N) : E(u) = m_c \text{ and } u > 0\}.
\end{aligned}$$

From now on $c > 0$ is fixed.

Following the terminology of [6],[5], we say that Z_c is stable if:
$Z_c \neq \emptyset$ and, $\forall v \in Z_c$ and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for any $\Phi_0 \in H$ such that $\|\Phi_0 - v\|_H < \delta$ it follows that $\inf_{z \in Z_c} \|\Phi(t, \cdot) - z\|_H < \varepsilon$ for all $t \in \mathbb{R}$, where $\Phi(t, \cdot)$ is the solution of (1.1) corresponding to the initial condition Φ_0. (This definition implicitly requires that (1.1) has a unique global solution for every initial value

$\Phi_0 \in H$. Under our hypotheses, this follows from results of Cazenave which we recall in Proposition 2.4 below.)

If $w \in Z_c$, there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that (1.2) is satisfied and hence $e^{-i\lambda t}w$ is a solution of (1.1) with initial condition $\Phi_0 = w$. Now $e^{-i\lambda t}w$ can be regarded as a periodic solution of (1.1) and its orbit is $\Theta(w) = \{e^{-i\lambda t}w : t \in \mathbb{R}\} \subset Z_c$.The usual definition for orbital stability in H of the periodic solution $e^{-i\lambda t}w$ is that:

$\forall \varepsilon > 0, \exists \delta > 0$ such that for any $\Phi_0 \in H$ such that $\|\Phi_0 - w\|_H < \delta$ it follows that $\inf_{v\in\Theta(w)} \|\Phi(t,\cdot) - v\|_H < \varepsilon$ for all $t \in \mathbb{R}$, where $\Phi(t,\cdot)$ is the solution of (1.1) corresponding to the initial condition Φ_0.

We observe that the two definitions of stability coincide only when $Z_c = \Theta(w)$ for some $c > 0$. In general, Z_c is larger than the orbit of a particular periodic solution and a discussion of the stability of Z_c should be accompanied by a thorough description of this set.

In the special case $g(x, s^2) = |s|^{p-1}$ where $1 < p < 1 + \frac{4}{N}$, Cazenave and Lions [6],[5] established the stability of Z_c for all $c > 0$ and they also proved that

$$Z_c = \{e^{i\theta}\psi(\cdot + y) : \theta \in \mathbb{R} \text{ and } y \in \mathbb{R}^N\} \tag{1.8}$$

where ψ is the unique Schwarz symmetric solution of (1.4) such that

$$W_c = \{\psi(\cdot + y) : y \in \mathbb{R}^N\}. \tag{1.9}$$

The proof of (1.8)(1.9) depends heavily on both the autonomous and homogeneous properties of the nonlinearity. It is shown in Remark 8.3.3 of [5] that the inclusion of the translations by y in (1.8) is essential for the stability. They also stated that their approach still applies to some more general autonomous nonlinearities [remark II.3, 3]. See also [13],[7] where this issue is again raised for quasilinear autonomous equations.

In this paper, we are chiefly concerned with non-autonomous nonlinearities. In this context, we establish the stability of Z_c and we show that

$$Z_c = \{e^{i\theta}u : \theta \in \mathbb{R} \text{ and } u \in W_c\}. \tag{1.10}$$

For general non-autonomous non-linearities it does not seem possible to give a complete description of the set W_c. However, in some cases it can reduce to a singleton, in which case we obtain orbital stability in the usual sense as mentioned above. For an example of this in the case $N = 1$, see [17]. To facilitate the discussion of several types of non-autonomous non-linearity without unnecessary repetition, we formulate four properties (see $(P0)$ and $(P3)$ below) and show that they imply stability of Z_c (Theorem 3.1) and also some information about its form (Theorem 4.1). In our context, Z_c may be stable for some values of c but not for others. Our assumption $(g0)$ on g ensures that $(P0)$ and $(P2)$ are satisfied as is shown in Proposition 2.3. Some results of Cazenave [5] (see Proposition 2.4 in Section 2) show that $(P1)$ follows from our assumptions $(g0)$ and $(g1)$ about the nonlinearity g.

Theorem 3.1 shows that if $(g0)$ and $(g1)$ are satisfied, then Z_c is stable provided that $(P3)$ holds.In the last part of the paper we exhibit various hypotheses on the nonlinearity $g(x,s)$ under which the property $(P3)$ can be verified for appropriate values of c. We deal with the following situations which do not seem to have been treated before.

(i) $g(x,s)-g(x,0)\to 0$ as $|x|\to\infty$ for all $s\geq 0$. This includes linear problems where $g(x,s)$ is independent of s. See Proposition 5.1.
(ii) $g(x,s)$ is a periodic function of $x\in\mathbb{R}^N$ for all $s\geq 0$. See Proposition 5.2.
(iii) $g(x,s)$ is asymptotically periodic in the sense that $g(x,s)-g^\infty(x,s)\to 0$ as $|x|\to\infty$ for some function g^∞ that is periodic with respect to $x\in\mathbb{R}^N$ for all $s\geq 0$. See Proposition 5.3.

The properties mentioned above are as follows:

$(P0)$ $\tilde{E}\in C^1(H,\mathbb{R})$ and all minimizing sequences for M_c are bounded in H.

$(P1)$ For any $\Phi_0\in H$, (1.1) admits a unique global solution $\Phi\in C(\mathbb{R},H)$ satisfying

$$\|\Phi(t)\|_2=\|\Phi_0\|_2 \text{ and } \widetilde{E}(\Phi(t))=\widetilde{E}(\Phi_0) \text{ for all } t\in\mathbb{R}.$$

The next two properties refer to a fixed value of $c>0$.

$(P2)$ For any sequence $\{c_n\}$ converging to c, $\liminf m_{c_n}\geq m_c$.

$(P3)$ Any sequence $\{u_n\}\subset H^1$ such that $|u_n|_2\to c$ and $E(u_n)\to m_c$ is relatively compact in H^1.

We end this introduction by outlining the arguments we use to establish the stability of Z_c starting from $(P0)$ to $(P3)$, rigorous proofs being given in Sections 3and 4 after some technical preliminaries have been put in place in Section 2.

Suppose that Z_c is not stable. Then either $Z_c=\emptyset$ or there exist $v\in Z_c,\varepsilon_0>0$ and a sequence $\{\Phi_0^n\}\subset H$ such that

$$\|\Phi_0^n-v\|_H\to 0 \text{ as } n\to\infty, \text{ but } \inf_{z\in Z_c}\|\Phi^n(t_n,\cdot)-z\|_H\geq\varepsilon_0 \tag{1.11}$$

for some sequence $\{t_n\}\subset\mathbb{R}$, where Φ^n is the solution of (1.1) corresponding to the initial condition Φ_0^n. Let $z_n=\Phi^n(t_n,\cdot)$.
Since $v\in\widetilde{S_c}$ and $\widetilde{E}(v)=M_c$, it follows from the continuity of $\|\cdot\|_2$ and $\widetilde{E}$ on H, that $\|\Phi_0^n\|_2\to c$ and $\widetilde{E}(\Phi_0^n)\to M_c$. Using $(P1)$, this implies that $\|z_n\|_2=\|\Phi_0^n\|_2\to c$ and $\widetilde{E}(z_n)=\widetilde{E}(\Phi_0^n)\to M_c$. If $\{z_n\}$ contains a subsequence converging in H to an element w, we find that $\|w\|_2=c$ and $\widetilde{E}(w)=M_c$ showing that $w\in Z_c$ and

$$\inf_{z\in Z_c}\|\Phi^n(t_n,\cdot)-z\|_H\leq\|z_n-w\|_H\,,$$

contradicting (1.11). Hence, to establish the stability of Z_c it is enough to show that

$$\left.\begin{array}{c} Z_c\neq\emptyset \text{ and that any sequence } \{z_n\}\subset H \text{ such that} \\ \|z_n\|_2\longrightarrow c \text{ and } \tilde{E}(z_n)\longrightarrow M_c \text{ is relatively compact in } H. \end{array}\right\} \tag{1.12}$$

Thus we see that the main point is to show how the properties $(P2)$ and $(P3)$ can be used to prove (1.12). This is the gist of the variational approach to the stability of standing waves for the (1.1), going back to [6]. The principal ingredient of our method for doing this is a comparison between $\tilde{E}(z)$ and $E(|z|)$ which turns out to be very fruitful. More precisely, for $z=(u,v)\in H$, we first observe that $\varphi=|z|=(u^2+v^2)^{\frac{1}{2}}\in H^1$ (see Proposition 2.2) and

$$\begin{aligned}\widetilde{E}(z)-E(|z|) &= \frac{1}{2}\left\{\|\nabla z\|_2^2-|\nabla\varphi|_2^2\right\}\\ &= \int_{\{u^2+v^2>0\}}\sum_{j=1}^{N}\frac{(u\partial_j v-v\partial_j u)^2}{u^2+v^2}dx\geq 0.\end{aligned}\tag{1.13}$$

Now consider a sequence $\{z_n\}\subset H$ such that

$$\|z_n\|_2^2\longrightarrow c^2\quad\text{and}\quad\tilde{E}(z_n)\longrightarrow M_c.$$

For $\varphi_n=|z_n|$ we obtain

$$|\varphi_n|_2^2\longrightarrow c^2\text{ and }E(\varphi_n)\leq\widetilde{E}(z_n)$$

and, using $(P2)$, $\liminf E(\varphi_n)\geq m_c$ showing that $M_c\leq m_c\leq\liminf E(\varphi_n)\leq\lim_{n\to\infty}\tilde{E}(z_n)=M_c$.
Thus

$$M_c=m_c\quad\text{and}\quad\lim_{n\to\infty}E(\varphi_n)=m_c.$$

By $(P3)$, there exists an element $\varphi\in H^1$ such that (up to a subsequence) $\varphi_n\longrightarrow\varphi$ in H^1. On the other hand, by $(P0)$, we know that there exists $z\in H$ such that $z_n\rightharpoonup z$ weakly in H. We then show that $\varphi=|z|$ and finally that $z_n\to z$ in H, proving (1.12). This leads to Theorem 3.1.

Moreover $(P0)$ to $(P3)$ also enable us to characterize Z_c. In fact, if $z\in Z_c$ we show that $\varphi=|z|\in W_c$ and (1.13) yields

$$\int_{\mathbb{R}^N}\sum_{j=1}^{N}\frac{(u\partial_j v-v\partial_j u)^2}{u^2+v^2}\,dx=0,$$

which is the crucial step in establishing (1.10) in Theorem 4.1

Notation In what follows, a positive constant whose value is unimportant for the purposes of the discussion will usually be denoted by C or K. The value of the constant denoted by C or K may change from line to line.

2 Preliminaries

Proposition 2.1 *Let $F\in C^1(\mathbb{R}^2,\mathbb{R})$ be such that $F(0)=0$ and $sup|\nabla F|<\infty$. Then, for every u and $v\in H^1$, $F(u,v)\in H^1$ and*

$$\partial_i\{F(u,v)\}=\partial_1F(u,v)\partial_iu+\partial_2F(u,v)\partial_iv.\ \textit{for}\ 1\leq i\leq N.$$

Proof. Since $F(0)=0$ we have that

$$|F(s)| \leq A|s| \text{ for all } s \in \mathbb{R}^2$$

where $A = \sup|\nabla F|$. This shows that $F(u,\, v) \in L^2$ for all $u, v \in H^1$.

Next, fixing $1 \leq i \leq N$, we remark that $\partial_1 F(u,\, v)\partial_i u + \partial_2 F(u,\, v)\partial_i v \in L^2$ since $\sup|\nabla F| < \infty$. Furthermore, for any $u, v \in H^1$, there exist sequences $\{u_n\}$ and $\{v_n\} \in C_0^\infty$ satisfying:

- $u_n \to u, v_n \to v$ in H^1
- $u_n \to u, \partial_i u_n \to \partial_i u, v_n \to v$ and $\partial_i v_n \to \partial_i v$ a.e. on $\mathbb{R}^N$
- $|u_n|, |\partial_i u_n|, |v_n|, |\partial_i v_n| \leq k$ a.e. on $\mathbb{R}^N$ where $k \in L^2$.

For any $\xi \in C_0^\infty$,

$$\int_{\mathbb{R}^N} F(u_n, v_n)\partial_i \xi \, dx = -\int_{\mathbb{R}^N} \{\partial_1 F(u_n,\, v_n)\partial_i u_n + \partial_2 F(u_n, v_n)\partial_i v_n\} \, \xi \, dx$$

and by the properties of $\{u_n\}$ and $\{v_n\}$ and the fact that $\sup|\nabla F| < \infty$, we obtain

$$\int_{\mathbb{R}^N} F(u, v)\partial_i \xi \, dx = -\int_{\mathbb{R}^N} \{\partial_1 F(u,\, v)\partial_i u + \partial_2 F(u,\, v)\partial_i v\} \, \xi \, dx$$

using the dominated convergence theorem. □

Remark In the case $N = 1$, the hypothesis $\sup|\nabla F| < \infty$ becomes superfluous, since we can easily prove the previous proposition using the continuous embedding of H^1 in L^∞.

Proposition 2.2 *Let $u,\, v \in H^1$. Then $\varphi = (u^2+v^2)^{\frac{1}{2}} \in H^1$ and for every $1 \leq i \leq N$,*

$$\partial_i \varphi = \begin{cases} \frac{u\partial_i u + v\partial_i v}{(u^2+v^2)^{\frac{1}{2}}} & \textit{if } u^2+v^2 \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

In particular,

$$|\nabla\varphi|_2^2 \leq 2(|\nabla u|_2^2 + |\nabla v|_2^2) \textit{ for all } u,\, v \in H^1.$$

Proof. For $\varepsilon > 0$, we define $\psi^\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ by

$$\psi^\varepsilon(s_1, s_2) = (|(s_1, s_2)|^2 + \varepsilon^2)^{1/2} - \varepsilon \text{ pour } (s_1, s_2) \in \mathbb{R}^2.$$

Clearly ψ^ε satisfies all the hypotheses of the previous proposition. For $u,\, v \in H^1$and $1 \leq i \leq N$, we have

$$\int_{\mathbb{R}^N} \{(u^2+v^2+\varepsilon^2)^{\frac{1}{2}} - \varepsilon\}\partial_i \xi \, dx = -\int_{\mathbb{R}^N} \frac{u\partial_i u + v\partial_i v}{(u^2+v^2+\varepsilon^2)^{\frac{1}{2}}}\, \xi \, dx$$

for any $\xi \in C_0^\infty$.

Since $0 \le (u^2+v^2+\varepsilon^2)^{\frac{1}{2}} - \varepsilon \le (u^2+v^2)^{\frac{1}{2}}$ and $|\frac{u\partial_i u+v\partial_i v}{(u^2+v^2+\varepsilon^2)^{\frac{1}{2}}}| \le |\partial_i u| + |\partial_i v|$, we obtain

$$\int_{\mathbb{R}^N} \{u^2(x)+v^2(x)\}^{\frac{1}{2}} \partial_i \xi(x)\, dx = -\int_{\mathbb{R}^N} \lim_{\varepsilon\to 0^+} \frac{u(x)\partial_i u(x) + v(x)\partial_i v(x)}{(u^2(x)+v^2(x)+\varepsilon^2)^{\frac{1}{2}}}\, \xi(x)\, dx$$

by using the dominated convergence theorem. □

Henceforth we suppose that the nonlinearity in (1.1) is of the following type:

$(g0)$ $g : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ is a Carathéodory function and there exist constants $K > 0$ and $0 < l < \frac{4}{N}$ such that

$$|g(x,\, s)| \le K(1 + s^{\frac{l}{2}})$$

for almost every $x \in \mathbb{R}^N$ and every $s \in [0,\infty)$.

Proposition 2.3 *Under the hypothesis $(g0)$ the functionals $\widetilde{E}$ and E have the following properties:*

(i) $\tilde{E} \in C^1(H,\mathbb{R})$ and $E \in C^1(H^1,\mathbb{R})$. Furthermore there exists a constant $C > 0$ such that

$$\|E'(u)\|_{H^{-1}} \le C\{\|u\|_{H^1} + \|u\|_{H^1}^{1+\frac{4}{N}}\} \text{ for all } u \in H^1.$$

(ii) There exists a constant $D > 0$ such that

$$\tilde{E}(z) \ge \frac{1}{4}\, \|\nabla z\|_2^2 - D\{c^2 + c^\gamma\} \text{ for all } z \in \tilde{S}_c \text{ and all } c > 0,$$

where $\gamma = \frac{2(2l+4-Nl)}{(4-Nl)} > 2$.

(iii) For all $c > 0$, $m_c \ge M_c \ge -D\{c^2 + c^\gamma\} > -\infty$.

(iv) All minimizing sequences for M_c are bounded in H and all minimizing sequences for m_c are bounded in H^1.

(v) $c \mapsto m_c$ is a continuous function on $(0,\infty)$.

In particular, the assumption $(g0)$ ensures that the properties $(P0)$ and $(P2)$ hold.

Remark If $l = \frac{4}{N}$ in $(g0)$, the conclusions of the above proposition (with minor modifications to (ii)) remain true for small values of c. See [11] for details.
If $l > \frac{4}{N}$, Z_c is unstable for all $c > 0$.

Proof. Let $\psi : \mathbb{R} \to \mathbb{R}$ be the function

$$\psi(s) = 1 \text{ for } |s| \le 1; \psi(s) = -|s| + 2 \text{ for } 1 \le |s| \le 2; \psi(s) = 0 \text{ for } |s| \ge 2$$

and then set

$$g_1(x, s^2) = \psi(|s|)g(x, s^2) \text{ and } g_2(x, s^2) = (1 - \psi(|s|))g(x, s^2).$$

Then

$$\left|g_1(x, s^2)s\right| \le (1 + 2^l)K |s| \text{ and } \left|g_2(x, s^2)s\right| \le 2K |s|^{\frac{4}{N}+1} \tag{2.14}$$

for all $s \in \mathbb{R}$. Let

$$\Gamma_i(u) = g_i(\cdot, u^2)u$$

and let

$$p = \begin{cases} \frac{2N}{N+2} & \text{for } N \ge 3 \\ \frac{4}{3} & \text{for } N \le 2 \end{cases} \quad \text{and } q = (1 + \frac{4}{N})p.$$

The estimates (2.14) imply that $\Gamma_1 \in C(L^2, L^2)$ and $\Gamma_2 \in C(L^q, L^p)$ and that there exists a constant $C > 0$ such that

$$\begin{aligned} |\Gamma_1(u)|_2 &\le C |u|_2 \text{ for all } u \in L^2 \\ |\Gamma_2(u)|_p &\le C |u|_q^{1+\frac{4}{N}} \text{ for all } u \in L^q. \end{aligned}$$

(See Chapter 1.16 of [14], for example.) Note that H^1 is continuously embedded in L^q since $q \in [2, \frac{2N}{N-2}]$ for $N \ge 3$ and $q \in [2, \infty)$ for $N \le 2$. Also L^p is continuously embedded in H^{-1} since $p' \in [2, \frac{2N}{N-2}]$ for $N \ge 3$ and $p' \in [2, \infty)$ for $N \le 2$. Hence

$$\Gamma \equiv \Gamma_1 + \Gamma_2 \in C(H^1, H^{-1})$$

and there exists a constant $C > 0$ such that

$$\|\Gamma(u)\|_{H^{-1}} \le C\{\|u\|_{H^1} + \|u\|_{H^1}^{1+\frac{4}{N}}\} \text{ for all } u \in H^1. \tag{2.15}$$

Since

$$G(x, s) = \int_0^{s^2} g(x, t)dt \text{ and } |G(x, s)| \le K(s^2 + \frac{2}{l+2} |s|^{l+2}),$$

where $l + 2 < \frac{2N}{N-2}$ for $N \ge 3$, it follows that

$$\begin{aligned} \left|\int_{\mathbb{R}^N} G(x, u)dx\right| &\le K(|u|_2^2 + \frac{2}{l+2} |u|_{l+2}^{l+2}) \\ &\le C\{\|u\|_{H^1}^2 + \|u\|_{H^1}^{l+2}\} \text{ for all } u \in H^1. \end{aligned} \tag{2.16}$$

Standard arguments (see Chapter 1.17 of [14], for example) show that $E \in C^1(H^1, \mathbb{R})$ with

$$E'(u)v = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v - \Gamma(u)vdx \text{ for all } u, v \in H^1$$

and

$$\|E'(u)\|_{H^{-1}} \leq C\{\|u\|_{H^1} + \|u\|_{H^1}^{1+\frac{4}{N}}\} \text{ for all } u \in H^1$$

by (2.15). Similarly $\widetilde{E} \in C^1(H, \mathbb{R})$.

(ii) For $z = (u, v) \in H$ and $\varphi = |z| \in S_c$, (2.16) and the Gagliardo-Nirenberg inequality with $\alpha = \frac{Nl}{2(l+2)}$ show that

$$\begin{aligned}
\tilde{E}(z) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \, dx - G(x, \varphi) dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \, dx - K(|\varphi|_2^2 + \frac{2}{l+2} |\varphi|_{l+2}^{l+2}) \\
&\geq \frac{1}{2}(|\nabla u|_2^2 + |\nabla v|_2^2) - K\{c^2 + \frac{2B}{l+2} c^{(1-\alpha)(l+2)} |\nabla \varphi|_2^{\alpha(l+2)}\} \\
&\geq \frac{1}{2}(|\nabla u|_2^2 + |\nabla v|_2^2) - K\{c^2 + K_1 c^{(1-\alpha)(l+2)}(|\nabla u|_2^2 + |\nabla v|_2^2)^{\alpha(l+2)/2}\},
\end{aligned}$$

by Proposition 2.2. Using Young's inequality we now find that

$$\tilde{E}(z) \geq \frac{1}{4}(|\nabla u|_2^2 + |\nabla v|_2^2) - D\{c^2 + c^\gamma\} = \frac{1}{4}\|\nabla z\|_2^2 - D\{c^2 + c^\gamma\}$$

where $\gamma = \frac{2(2l+4-Nl)}{4-Nl}$.

Parts (iii) and (iv) follow immediately from (ii). For part (v), consider $c > 0$ and a sequence $\{c_n\} \subset (0, \infty)$ such that $c_n \to c$. For all n, there exists $u_n \in S_{c_n}$ such that $m_{c_n} \leq E(u_n) \leq m_{c_n} + \frac{1}{n}$. By (ii), there exists $K > 0$ such that $\|u_n\|_{H^1} \leq K$ for all n. Setting $w_n = \frac{c}{c_n} u_n$, we have that

$$w_n \in S_c \text{ and } \|u_n - w_n\|_{H^1} \leq \left|1 - \frac{c}{c_n}\right| \|u_n\|_{H^1} \leq K \left|1 - \frac{c}{c_n}\right| \text{ for all } n.$$

In particular, there exists n_1 such that $\|u_n - w_n\|_{H^1} \leq K + 1$ for all $n \geq n_1$. By part (i), there exists a constant $L(K) > 0$ such that $\|E'(u)\|_{H^{-1}} \leq L(K)$ for all $u \in H^1$ such that $\|u\|_{H^1} \leq 2K + 1$. Now, for all $n \geq n_1$,

$$\begin{aligned}
|E(w_n) - E(u_n)| &= \left|\int_0^1 \frac{d}{dt} E(t w_n + (1-t) u_n) dt\right| \\
&\leq \sup_{\|u\|_{H^1} \leq 2K+1} \|E'(u)\|_{H^{-1}} \|u_n - w_n\|_{H^1} \\
&\leq L(K) K \left|1 - \frac{c}{c_n}\right|.
\end{aligned}$$

Hence we have that

$$\begin{aligned}
m_{c_n} &\geq E(u_n) - \frac{1}{n} \geq E(w_n) - L(K)K\left|1 - \frac{c}{c_n}\right| - \frac{1}{n} \\
&\geq m_c - L(K)K\left|1 - \frac{c}{c_n}\right| - \frac{1}{n}
\end{aligned}$$

and so

$$\liminf_{n\to\infty} m_{c_n} \geq m_c \text{ since } c_n \to c. \tag{2.17}$$

On the other hand, there exists a sequence $\{u_n\} \in S_c$ such that $E(u_n) \to m_c$ and, by (ii), there exists $K > 0$ such that $\|u_n\|_{H^1} \leq K$. Let $w_n = \frac{c_n}{c} u_n$. As above, we have that

$$w_n \in S_{c_n} \text{ and } \|u_n - w_n\|_{H^1} \leq K\left|1 - \frac{c_n}{c}\right| \text{ and } |E(w_n) - E(u_n)| \leq L(K)K\left|1 - \frac{c_n}{c}\right|,$$

so that

$$m_{c_n} \leq E(w_n) \leq E(u_n) + L(K)K\left|1 - \frac{c_n}{c}\right|$$

showing that we have

$$\limsup_{n\to\infty} m_{c_n} \leq \lim_{n\to\infty} E(u_n) = m_c. \tag{2.18}$$

Recalling (2.17), we now have that $\lim_{n\to\infty} m_{c_n} = m_c$. □

Proposition 2.4 *(Cazenave [5]) In addition to* $(g0)$ *suppose that the function* g *satisfies the following condition:*

$(g1)$ *If* $N \geq 2$, *there exist constants* $C > 0$ *and* $\alpha \in [0, \frac{4}{N-2})$ *for* $N \geq 3$, $\alpha \in [0, \infty)$ *for* $N = 2$ *such that*
$\left|g(x, s^2)s - g(x, t^2)t\right| \leq C\{1 + |s|^\alpha + |t|^\alpha\}\,|s - t|$ *for all* $s, t \in \mathbb{R}$.
If $N = 1$, *for every* $R > 0$ *there exists a constant* $L(R) > 0$ *such that*
$\left|g(x, s^2)s - g(x, t^2)t\right| \leq L(R)\,|s - t|$ *for all* $s, t \in \mathbb{R}$ *such that* $|s| + |t| \leq R$.

Then for every $\Phi_0 \in H$, *the initial value problem (1.1) has a unique solution*

$$\Phi \in C(\mathbb{R}, H) \cap C^1(\mathbb{R}, H^{-1}).$$

Furthermore,

$$\sup_{t\in\mathbb{R}} \|\Phi(t)\|_H < \infty,$$

$$\|\Phi(t)\|_2 = \|\Phi_0\|_2 \text{ and } \widetilde{E}(\Phi(t)) = \widetilde{E}(\Phi_0) \text{ for all } t \in \mathbb{R}.$$

In particular, the assumptions $(g0)$ *and* $(g1)$ *ensure that the conditions* $(P0), (P1)$ *and* $(P2)$ *hold.*

Proof. The assumptions $(g0)$ and $(g1)$ imply that the hypotheses Theorem 4.3.1 of [5] are satisfied, see also Remark 4.3.2 of [5]. Furthermore, for $z \in H$ and $\varphi = |z|$, (2.16) and the Gagliardo-Nirenberg inequality yield

$$\begin{aligned}\left|\int_{\mathbb{R}^N} G(x, |z|)dx\right| &\leq K\{|\varphi|_2^2 + \frac{2}{l+2}\,|\varphi|_{l+2}^{l+2}\} \\ &\leq K_1\{|\varphi|_2^2 + |\varphi|_2^{(1-\alpha)(l+2)}\,|\nabla\varphi|_2^{\alpha(l+2)}\}\end{aligned}$$

with $\alpha = \frac{Nl}{2(l+2)}$ where $\alpha(l+2) < 2$ by $(g0)$. Young's inequality now shows that condition (6.1.3) of [5] is satisfied and global existence follows from Corollary 6.1.2 in [5]. □

3 Orbital Stability

Theorem 3.1 *Suppose that g satisfies the conditions $(g0)$ and $(g1)$, and consider some value of $c > 0$ for which the property $(P3)$ holds.*

(i) Then $m_c = M_c, Z_c \neq \emptyset$ and Z_c is stable.

(ii) If $z \in Z_c$, then $|z| \in W_c$.

Proof. Let $\{z_n\} = \{(u_n, v_n)\} \subset H$ be a sequence such that $\|z_n\|_2^2 \longrightarrow c^2$ and $\tilde{E}(z_n) \longrightarrow M_c$. Our first objective is to prove that $\{z_n\}$ has a subsequence converging in H.

By Proposition 2.3(ii), $\{z_n\}$ is bounded in H and hence, passing to a subsequence, there exists $z = (u, v) \in H$ such that

$$u_n \rightharpoonup u, v_n \rightharpoonup v \text{ weakly in } H^1 \text{ and } \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + |\nabla v_n|^2 \, dx \text{ exists.} \tag{3.19}$$

Setting $\varphi_n = |z_n|$, Proposition 2.2 assures us that $\{\varphi_n\} \subset H^1$ and that, for all $n \in \mathbb{N}$ and $1 \leq i \leq N$,

$$\partial_i \varphi_n(x) = \begin{cases} \frac{u_n(x)\partial_i u_n(x) + v_n(x)\partial_i v_n(x)}{(u_n^2(x)+v_n^2(x))^{\frac{1}{2}}} & \text{if } u_n^2 + v_n^2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \tilde{E}(z_n) - E(\varphi_n) &= \frac{1}{2}\{\|\nabla z_n\|_2^2 - |\nabla \varphi_n|_2^2\} \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + |\nabla v_n|^2 - |\nabla (u_n^2 + v_n^2)^{\frac{1}{2}}|^2 \, dx \\ &= \frac{1}{2}\int_{\{u_n^2+v_n^2>0\}} \sum_{i=1}^{N} \frac{(u_n \partial_i v_n - v_n \partial_i u_n)^2}{u_n^2 + v_n^2} \, dx \geq 0 \end{aligned} \tag{3.20}$$

proving that $M_c = \lim_{n\to\infty} \tilde{E}(z_n) \geq \limsup E(\varphi_n)$.
On the other hand,

$$\|z_n\|_2^2 = |\varphi_n|_2^2 = c_n^2 \longrightarrow c^2 \tag{3.21}$$

and so, by Proposition 2.3(v), we have that $\liminf E(\varphi_n) \geq \liminf m_{c_n} \geq m_c \geq M_c$. Thus

$$\lim_{n\to\infty} E(\varphi_n) = \lim_{n\to\infty} \tilde{E}(z_n) = m_c = M_c. \tag{3.22}$$

Furthermore (3.20) and (3.22) imply that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + |\nabla v_n|^2 - |\nabla (u_n^2 + v_n^2)^{\frac{1}{2}}|^2 \, dx = 0$$

and it follows from (3.19) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^2 + |\nabla v_n|^2 dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla (u_n^2+v_n^2)^{\frac{1}{2}}|^2\, dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla \varphi_n|^2\, dx. \tag{3.23}$$

Combining (3.21) and (3.22) and using ($P3$), we may assume henceforth (after passing to a subsequence) that there exists $\varphi \in H^1$ such that $\varphi_n \longrightarrow \varphi$ in H^1. Certainly $\varphi \in S_c$, and using Proposition 2.3(i), we also have that $E(\varphi) = m_c$. Thus φ is weak solution of (1.4) and, using elliptic regularity theory and maximum principle in standard way, we deduce that $\varphi \in C^1(\mathbb{R}^N)$ and $\varphi > 0$. Hence

$$\varphi \in W_c \subset Z_c \text{ since } m_c = M_c. \tag{3.24}$$

On the other, hand we claim that the element $z = (u, v)$ given by (3.19) is such that $(u^2+v^2)^{\frac{1}{2}} = \varphi$. Indeed it follows from (3.19) that, for any $R > 0$,

$$u_n \to u \text{ and } v_n \to v \text{ in } L^2(B(0,R)).$$

Furthermore

$$\begin{aligned}[(u_n^2+v_n^2)^{1/2} - (u^2+v^2)^{1/2}]^2 &= u_n^2+v_n^2+u^2+v^2 - 2(u_n^2+v_n^2)^{1/2}(u^2+v^2)^{1/2}\\ &\le u_n^2+v_n^2+u^2+v^2 - 2\{|u_n|\,|u| + |v_n|\,|v|\}\\ &= |u_n-u|^2 + |v_n-v|^2\end{aligned}$$

so we see that $(u_n^2+v_n^2)^{1/2} \to (u^2+v^2)^{1/2}$ in $L^2(B(0,R))$ for all $R > 0$. But $(u_n^2+v_n^2)^{1/2} = \varphi_n \to \varphi$ in L^2 so we must have $(u^2+v^2)^{1/2} = \varphi$ a.e. on $\mathbb{R}^N$.

Now $\|z_n\|_2 = |\varphi_n|_2 \to |\varphi|_2 = \|z\|_2$. To establish (1.12), we still have to show that $\lim_{n\to\infty} \|\nabla z_n\|_2^2 = \|\nabla z\|_2^2$. From (3.23) we have that

$$\lim_{n\to\infty} \|\nabla z_n\|_2^2 = \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla \varphi_n|^2\, dx$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla \varphi_n|^2\, dx = \int_{\mathbb{R}^N} |\nabla \varphi|^2\, dx \text{ since } \varphi_n \to \varphi \text{ in } H^1.$$

Thus

$$\|\nabla z\|_2^2 \le \lim_{n\to\infty} \|\nabla z_n\|_2^2 = |\nabla \varphi|_2^2,$$

whereas, replacing z_n by z in (3.20), we see that

$$\|\nabla z\|_2^2 \ge |\nabla \varphi|_2^2.$$

Since $z_n \rightharpoonup z$ weakly in H, this proves that in fact, $z_n \to z$ in H, completing the proof of (1.12) and part (i) of the theorem.

For part (ii), we simply retrace the proof of (i) with $z_n = z$ for all n. □

4 Characterization of Z_c

Theorem 4.1 *Under the assumptions of Theorem 3.1, let $z = (u, v) \in Z_c$. Then*
(i) either $u \equiv 0$ or $u(x) \neq 0$ for all $x \in \mathbb{R}^N$,
(ii) either $v \equiv 0$ or $v(x) \neq 0$ for all $x \in \mathbb{R}^N$,
(iii) $Z_c = \{e^{i\sigma} w : \sigma \in \mathbb{R}$ and $w \in W_c\}$.

Proof. Let $z = (u,\, v) \in Z_c$ and set $\varphi = (u^2 + v^2)^{\frac{1}{2}}$. By Theorem 3.1(ii), we know that $\varphi \in W_c$. Furthermore, putting $z_n = z$ in (3.20) and (3.22), we have that

$$\int_{\mathbb{R}^N} \sum_{i=1}^{N} \frac{(u\partial_i v - v\partial_i u)^2}{u^2 + v^2}\, dx = 0. \tag{4.25}$$

On the other hand, $\tilde{E}(z) = M_c$ implies that there exists a Lagrange multiplier $\mu \in \mathbb{C}$ such that

$$\tilde{E}'(z)\eta = \frac{\mu}{2} \int_{\mathbb{R}^N} z\overline{\eta} + \eta\overline{z} dx \text{ for all } \eta \in H.$$

Putting $\eta = z$, one finds that $\mu \in \mathbb{R}$ and that

$$\begin{cases} \int_{\mathbb{R}^N} \nabla u \cdot \nabla w - g(x,\, u^2 + v^2) uw dx = \mu \int_{\mathbb{R}^N} uw dx \\ \int_{\mathbb{R}^N} \nabla v \cdot \nabla w - g(x,\, u^2 + v^2) vw dx = \mu \int_{\mathbb{R}^N} vw dx \end{cases}$$

for all $w \in H^1$. Using $(g0)$ and elliptic regularity theory, we easily deduce that u and $v \in C^1(\mathbb{R}^N) \cap H^2$ and satisfy the system (1.3).

Let

$$\Omega = \{x \in \mathbb{R}^N : u(x) = 0\}.$$

The continuity of u implies that Ω is closed. Suppose now that $x_0 \in \Omega$. Since $\varphi(x_0) > 0$, there is an open ball B with centre x_0 such that $v(x) \neq 0$ for all $x \in B$. Hence, for $x \in B$,

$$\frac{(u\partial_i v - v\partial_i u)^2}{u^2 + v^2} = [\partial_i(\frac{u}{v})]^2 \frac{v^4}{u^2 + v^2} \text{ for } i = 1, ..., N$$

and it follows from (4.25) that

$$\int_B \left|\nabla(\frac{u}{v})\right|^2 \frac{v^4}{u^2 + v^2} dx = 0.$$

Thus $\nabla(\frac{u}{v}) \equiv 0$ on B and so there exists a constant C such that $\frac{u}{v} = C$ on B. In fact, since $x_0 \in B$, we have that $C = 0$, showing that Ω is also an open subset of $\mathbb{R}^N$. This proves part (i) and part (ii) is similar.

Now we turn to the characterization of Z_c:
First we set $z = e^{i\sigma} w$, where $\sigma \in \mathbb{R}$ and $w \in W_c$. Then $z \in \tilde{S}_c$ and

$$\begin{aligned}\tilde{E}(z) &= \frac{1}{2}\{\|\nabla(e^{i\sigma}w)\|_2^2 - \int_{\mathbb{R}^N} G(x,|e^{i\sigma}w|)\,dx\} \\ &= \frac{1}{2}\{|\nabla w|_2^2 - \int_{\mathbb{R}^N} G(x,w)\,dx\} = E(w) = m_c = M_c.\end{aligned}$$

Thus $\{e^{i\sigma}w : \sigma \in \mathbb{R} \text{ and } w \in W_c\} \subset Z_c$.

Conversely, for $z = (u,v) \in Z_c$ we set $w \equiv |z|$. By Theorem 3.1(ii), we have that $E(w) = \tilde{E}(z) = M_c = m_c$ and $w \in W_c$.

If $v \equiv 0$, $w = |u| > 0$ on $\mathbb{R}^N$ and so $z = e^{i\sigma}w \in W_c$ where $\sigma = 0$ if $u > 0$ and $\sigma = \pi$ if $u < 0$ on $\mathbb{R}^N$. Otherwise, by part (ii) we conclude that $v(x) \neq 0$ for all $x \in \mathbb{R}^N$. Thus

$$\frac{(u\partial_i v - v\partial_i u)^2}{u^2+v^2} = [\partial_i(\frac{u}{v})]^2\frac{v^4}{u^2+v^2} \quad \text{for } i = 1,...,N$$

for all $x \in \mathbb{R}^N$ and it follows from (4.25) that $\nabla(\frac{u}{v}) \equiv 0$ on $\mathbb{R}^N$. Hence there exists a constant $\alpha \in \mathbb{R}$ such that $u \equiv \alpha v$ on $\mathbb{R}^N$. Thus, in complex notation, $z = (\alpha + i)v$ and $w = |\alpha + i|\,|v|$. Let $\theta \in \mathbb{R}$ be such that $\alpha + i = |\alpha + i|\, e^{i\theta}$ and let $\psi = 0$ if $v > 0$ and $\psi = \pi$ if $v < 0$ on $\mathbb{R}^N$. Setting $\sigma = \theta + \psi$ we have that $z = (\alpha + i)v = |\alpha + i|\, e^{i\theta}\,|v|\, e^{i\psi} = we^{i\sigma}$ where $w \in W_c$, completing the proof. □

5 Property $(P3)$

In this section, we give conditions on the function g under which $(P3)$ holds. We adopt the notation

$$2^* = \infty \text{ for } N = 1,2 \text{ and } 2^* = \frac{2N}{N-2} \text{ for } N \geq 3.$$

Lemma 5.1 *Let g satisfy the condition $(g0)$. Then $(P3)$ is satisfied provided that any sequence $\{u_n\} \subset H^1$ such that $|u_n|_2 \to c$ and $E(u_n) \to m_c$ has a subsequence converging in L^2.*

Proof. Consider a sequence $\{u_n\} \subset H^1$ such that $|u_n|_2 \to c$, $E(u_n) \to m_c$ and $u_n \to u$ in L^2. By Proposition 2.3(ii), this sequence is bounded in H^1 and so, passing to a subsequence, we may suppose that $\{u_n\}$ converges weakly to an element v in H^1. Since u_n converges to v in $L^2(B(0,R))$ for all $R > 0$, we have that $u = v$. Thus, for $2 \leq p < 2^*$, $u \in S_c \subset H^1 \subset L^p$ and $\{u_n\}$ is bounded in L^p. It follows that $u_n \to u$ in L^p.

Using the notation introduced for the proof of Proposition 2.3 (i), we have that $G = G_1 + G_2$ where

$$\begin{aligned}G_i(x,s) &= \int_0^{s^2} g_i(x,t)dt \text{ and hence, for all } s \in \mathbb{R}, \\ |G_1(x,s)| &\leq (1+2^l)Ks^2 \text{ and } |G_2(x,s)| \leq 2K\,|s|^{\frac{4}{N}+2}.\end{aligned}$$

Thus

$$T_1 \in C(L^2, \mathbb{R}) \text{ and } T_2 \in C(L^{\frac{4}{N}+2}, \mathbb{R}) \text{ where } T_i(u) = \int_{\mathbb{R}^N} G_i(x, u(x))dx.$$

Now, for all n,

$$\begin{aligned} |\nabla u_n|_2^2 - |\nabla u|_2^2 &= 2\{E(u_n) - E(u)\} + \int_{\mathbb{R}^N} G(x, u(x))dx - \int_{\mathbb{R}^N} G(x, u_n(x))dx \\ &\leq 2\{E(u_n) - m_c\} + \{T_1(u) - T_1(u_n)\} + \{T_2(u) - T_2(u_n)\} \end{aligned}$$

from which it follows that $\limsup |\nabla u_n|_2^2 \leq |\nabla u|_2^2$ since $\frac{4}{N} + 2 < 2^*$. But the weak convergence of $\{u_n\}$ to u in H^1 implies that $|\nabla u|_2^2 \leq \liminf |\nabla u_n|_2^2$. Therefore, $|\nabla u|_2^2 = \lim |\nabla u_n|_2^2$ and, in fact, $\|u - u_n\|_{H^1} \to 0$ as required. □

In establishing the property $(P3)$, we shall make use of the following assumption about g.

(H1) $g(x, s)$ is a non-decreasing function of s on $[0, \infty)$ for a.e. $x \in \mathbb{R}^N$ and we set

$$L(x) = \lim_{s \to \infty} g(x, s).$$

Remark It follows from (H1) that, for $s \in \mathbb{R}$ and $\alpha \geq 1$,

$$G(x, \alpha s) = \int_0^{\alpha^2 s^2} g(x, t)dt = \int_0^{s^2} g(x, \alpha^2 t)\alpha^2 dt \geq \alpha^2 \int_0^{s^2} g(x, t)dt = \alpha^2 G(x, s) \tag{5.26}$$

and hence, for all $u \in H^1$,

$$E(\alpha u) = \frac{1}{2} \int_{\mathbb{R}^N} \alpha^2 |\nabla u|^2 - G(x, \alpha u)dx \leq \alpha^2 E(u). \tag{5.27}$$

Hence (H1) implies that, for all $u \in H^1, t^{-2}E(tu)$ is a non-increasing function of t on $(0, \infty)$.

Let

$$Q(\infty) = \inf\{\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - L(x)u^2 dx : u \in S_1\} \tag{5.28}$$

where $L(x)u^2(x)$ is interpreted as 0 when $u(x) = 0$, even if $L(x) = \infty$.

Lemma 5.2 *Let g satisfy the conditions $(g0)$ and (H1). Then $c^{-2}m_c$ is a non-increasing function of c on $(0, \infty)$ and $\lim_{c \to \infty} c^{-2}m_c = Q(\infty)$.*

Proof. The monotonicity of $c^{-2}m_c$ follows easily from the monotonicity of $c^{-2}E(cu)$ that is implied by (H1). Now consider any $u \in S_1$ and $n \in \mathbb{N}\backslash\{0\}$,

$$n^{-2}E(nu) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - n^{-2}G(x, nu(x))dx.$$

Setting $w_n(x) = n^{-2}G(x, nu(x))$, we see that $\{w_n\}$ is a non-decreasing sequence of integrable functions on $\mathbb{R}^N$ and

$$n^{-2}G(x, nu(x)) = u(x)^2 \int_0^1 g(x, n^2\sigma u(x)^2)d\sigma \to L(x)u(x)^2 \text{ a.e. on } \mathbb{R}^N,$$

provided that we interpret $L(x)u(x)^2$ as 0 when $u(x) = 0$ even if $L(x) = \infty$. Hence we have that

$$n^{-2}E(nu) \to \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - L(x)u(x)^2 dx \text{ as } n \to \infty.$$

Thus $\lim_{n\to\infty} n^{-2}m_n \leq \lim_{n\to\infty} n^{-2}E(nu) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - L(x)u(x)^2 dx$ for all $u \in S_1$, showing that $\lim_{n\to\infty} n^{-2}m_n \leq Q(\infty)$.

On the other hand, for any $u \in S_n$, we have that $v = \frac{1}{n}u \in S_1$ and

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - G(x, u(x))dx = n^2\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - n^{-2}G(x, nv(x))dx \\ &\geq n^2\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - L(x)v^2 dx \geq n^2 Q(\infty) \end{aligned}$$

since $n^{-2}G(x, nv(x)) \leq L(x)v^2(x)$ by the monotonicity of the sequence $\{w_n\}$. Hence $m_n \geq n^2 Q(\infty)$ for all $n \in \mathbb{N}\backslash\{0\}$. □

For the rest of the discussion it is convenient to introduce the following notation for any function g satisfying $(g0)$.

$$\begin{aligned} V(x) &= g(x,0) \text{ and } h(x,s) = g(x,s) - g(x,0), \\ H(x,s) &= \int_0^{s^2} h(x,t)dt = G(x,s) - V(x)s^2. \end{aligned}$$

5.1 Compact nonlinearity

In this part we suppose that

$$(C) \text{ For some } \gamma \in [0, 2^* - 2), \lim_{|x|\to\infty} \frac{|h(x,s)|}{1 + s^{\gamma/2}} = 0 \text{ uniformly for } s \geq 0,$$

and we use the following notation.

$$\begin{aligned} V(\infty) &= \limsup_{|x|\to\infty} V(x), \\ \Phi(u) &= E(u) + \frac{1}{2}V(\infty)\int_{\mathbb{R}^N} u^2 dx, \\ \mu_c &= \inf\{\Phi(u) : u \in S_c\} = m_c + \frac{1}{2}V(\infty)c^2 \end{aligned}$$

Note that $(g0)$ implies that $V \in L^\infty$ and that by $V(\infty)$ we really mean

$$\lim_{R\to\infty} ess \sup_{|x|\geq R} V(x).$$

Then, for any $u \in H^1$,

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - G(x,u(x))dx + \frac{1}{2}V(\infty)\int_{\mathbb{R}^N} u^2 dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - [V(x)-V(\infty)]u^2 - H(x,u)dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + [V(x)-V(\infty)]_- u^2 - [V(x)-V(\infty)]_+ u^2 - H(x,u)dx\end{aligned}$$

where $s_+ = \max\{s,0\}$ and $s_- = \max\{-s,0\}$ for all $s \in \mathbb{R}$.

The reason for introducing the auxiliary functional Φ is that it is weakly sequentially lower semicontinuous on H^1 whereas E may not have this property. On the other hand, Φ and E only differ by a constant when restricted to S_c and so the property $(P3)$ for E can be established by showing that it holds for Φ.

Lemma 5.3 *Let g satisfy the conditions $(g0)$ and (C). Then the functional $\Phi : H^1 \to \mathbb{R}$ is weakly sequentially lower semi-continuous.*

Proof. For $u \in H^1$,

$$2\Phi(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + [V(x)-V(\infty)]_- u^2 - [V(x)-V(\infty)]_+ u^2 - H(x,u)dx$$

and, since $[V(x)-V(\infty)]_- \geq 0$, the functional

$$\int_{\mathbb{R}^N} |\nabla u|^2 + [V(x)-V(\infty)]_- u^2 dx$$

is weakly sequentially lower semi-continuous.

Since $[V(x)-V(\infty)]_+ \to 0$ as $|x| \to \infty$, it is easy to see that

$$\int_{\mathbb{R}^N} [V(x)-V(\infty)]_+ u^2 dx$$

is weakly sequentially continuous.

For any $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that $|h(x,s)| \leq \varepsilon(1+s^{\gamma/2})$ for all $|x| \geq R(\varepsilon)$ and $s \geq 0$. Hence $|H(x,s)| \leq \varepsilon(s^2 + s^{\gamma+2})$ for all $|x| \geq R(\varepsilon)$ and $s \geq 0$, from which it follows easily that

$$\int_{\mathbb{R}^N} H(x,u)dx$$

is weakly sequentially continuous. □

Proposition 5.1 *Let g satisfy the conditions $(g0)$, (H1) and (C). If $\mu_d < 0$, then $(P3)$ holds for all $c \geq d$.*

(a) If $2Q(\infty) + V(\infty) < 0$, there exists $c_0 \geq 0$ such that $\mu_c < 0$ for all $c > c_0$.

(b) We have that $\mu_c < 0$ for all $c > 0$ under the following hypotheses.

$$(i) \qquad \exists\ A > 0 \text{ such that } |x|^2 [V(x) - V(\infty)]_- \leq A \text{ for all } x \in \mathbb{R}^N.$$

$$(ii) \qquad \exists\ z \in \mathbb{R}^N \text{ and constants } B > 0, \delta > 0, \tau \in (0,2) \text{ and } \sigma \in (0, \frac{2(2-\tau)}{N})$$
$$\text{such that } H(x,s) \geq B\,|x|^{-\tau}\,|s|^{\sigma+2}$$
$$\text{for } |s| \leq \delta \text{ and } x \in \Omega = \{ty : t \geq 1 \text{ and } |y - z| \leq 1\}.$$

Remarks Note that this result covers linear problems ($g(x,s) \equiv V(x)$ for all $x \in \mathbb{R}^N$ and $s \geq 0$) and, in that case, the hypothesis in part (a) reduces to $V \in L^\infty(\mathbb{R}^N)$ and

$$2Q(\infty) = \inf\{\int_{\mathbb{R}^N} |\nabla u|^2 - V(x)u^2 dx : u \in S_1\} < -\lim \sup_{|x| \to \infty} V(x).$$

The latter condition is equivalent to the property that $-\Delta - V$ has an eigenvalue in L^2 below its essential spectrum which is contained in $[-V(\infty), \infty)$.
For many nonlinear problems $L(x) = \infty$ on a set of positive measure and then the inequality $2Q(\infty) + V(\infty) < 0$ is trivially satisfied since $Q(\infty) = -\infty$. However, even in such cases, it can happen that $(P3)$ is not satisfied for small values of $c > 0$. For example, if

$$\begin{aligned} g(x,s) &= q(x)s^{\sigma/2} \text{ where } \sigma \in (0, \frac{4}{N}) \text{ and also } \sigma > 2 \text{ if } N = 1 \\ \text{and } q &\in L^p \text{ where } p = \frac{2N}{4 - N\sigma}, \end{aligned}$$

the Gagliardo-Nirenberg inequality shows that there exists $c_1 > 0$ such that m_c is not attained for $c \in (0, c_1)$ and consequently the property $(P3)$ cannot hold for these values of c. See Example 4 in [11] for a calculation of this kind.

Proof. Fixing a value of $c \geq d$, it follows from Lemma 5.2 that $\mu_c < 0$. Consider a sequence $\{u_n\} \subset H^1$ such that $|u_n|_2 \to c$ and $E(u_n) \to m_c$. Then $\Phi(u_n) \to \mu_c = m_c + \frac{1}{2}V(\infty)c^2 < 0$. Furthermore, by Proposition 2.16(ii), the sequence $\{u_n\}$ is bounded in H^1 and so, by passing to a subsequence, we may suppose henceforth that $u_n \rightharpoonup u$ weakly in H^1. It follows from Lemma 5.3 that $\Phi(u) \leq \lim \Phi(u_n) = \mu_c < 0$. Since $\Phi(u) = 0$, this shows that $u \neq 0$ and hence $0 < |u|_2 \leq \lim |u_n|_2 = c$. Setting $t = c/|u|_2$ and $v = tu$, we have that $t \geq 1$ and $v \in S_c$. Hence $\Phi(v) = E(v) + \frac{1}{2}V(\infty)c^2 \geq m_c + \frac{1}{2}V(\infty)c^2 = \mu_c$. On the other hand, (5.27) implies that $E(v) \leq t^2 E(u)$, so

$$\Phi(v) = E(v) + \frac{1}{2}V(\infty)\,|v|_2^2 \leq t^2\{E(u) + \frac{1}{2}V(\infty)\,|u|_2^2\} = t^2\Phi(u) \leq t^2\mu_c.$$

Combining these estimates we obtain $\mu_c \le \Phi(v) \le t^2\mu_c$. Since $\mu_c < 0$ we must have $t^2 \le 1$, so in fact, $t = 1$ and $\Phi(v) = \mu_c$. But $t = 1$ means that $|u|_2 = \lim |u_n|_2 = c$. However, the weak convergence of $\{u_n\}$ to u in H^1 implies that $u_n \rightharpoonup u$ weakly in L^2 and, since we also have that $|u|_2 = \lim |u_n|_2$, we may conclude that $|u_n - u|_2$. Recalling Lemma 5.1, this establishes $(P3)$.

(i) By Lemma 5.2, $\lim_{c\to\infty} c^{-2}\mu_c = Q(\infty) + \frac{1}{2}V(\infty)$.

(ii) Fix $c > 0$. Set $u(x) = |x|\, e^{-|x|}$ and then, for $\alpha > 0$, let

$$u_\alpha(x) = \frac{c\alpha^{N/2}u(\alpha x)}{|u|_2}.$$

Then $u_\alpha \in S_c$ for all $\alpha > 0$ and there exists $\alpha_c \in (0, 1]$ such that $|u_\alpha| \le \delta$ for all $\alpha \in (0, \alpha_c]$. Now

$$\begin{aligned} 2\Phi(u_\alpha) &= \int_{\mathbb{R}^N} |\nabla u_\alpha|^2 + [V(x) - V(\infty)]_- u_\alpha^2 - [V(x) - V(\infty)]_+ u_\alpha^2 - H(x, u_\alpha)dx \\ &\le \int_{\mathbb{R}^N} |\nabla u_\alpha|^2 + A\,|x|^{-2}\, u_\alpha^2 dx - \int_\Omega H(x, u_\alpha)dx \end{aligned}$$

by (i) and the fact that $H \ge 0$ by (H1) where

$$\int_{\mathbb{R}^N} |\nabla u_\alpha|^2 + A\,|x|^{-2}\, u_\alpha^2 dx = \frac{c^2\alpha^2}{|u|_2^2}\{\int_{\mathbb{R}^N} |\nabla u|^2 + A\,|x|^{-2}\, u^2 dx\}, \text{ and}$$

$$\begin{aligned} \int_\Omega H(x, u_\alpha)dx \ge \int_\Omega B\,|x|^{-\tau}\,|u_\alpha|^{\sigma+2}\,dx &= B\alpha^{\frac{N\sigma}{2}+\tau}\left(\frac{c}{|u|_2}\right)^{\sigma+2}\int_{\alpha\Omega} |x|^{-\tau}\,|u|^{\sigma+2}\,dx \\ &\ge B\alpha^{\frac{N\sigma}{2}+\tau}\left(\frac{c}{|u|_2}\right)^{\sigma+2}\int_{\Omega} |x|^{-\tau}\,|u|^{\sigma+2}\,dx \end{aligned}$$

since $\Omega \subset \alpha\Omega$ for all $\alpha \in (0, 1]$. But $\frac{N\sigma}{2} + \tau < 2$ so there exists $\beta_c \in (0, \alpha_c)$ such that $\Phi(u_\alpha) < 0$ for all $\alpha \in (0, \beta_c)$, proving that $\mu_c < 0$. □

5.2 Non-compact nonlinearity

In this part we suppose that g satisfies the condition $(g0)$ and, in addition, that g is asymptotically periodic in the following sense:

(A1) There exists a function g^∞ satisfying $(g0)$ such that $g^\infty(x + z, s) = g^\infty(x, s)$ for all $x \in \mathbb{R}^N, z \in \mathbb{Z}^N$ and $s \ge 0$, and for some $\gamma \in [0, 2^* - 2)$,

$$\lim_{|x|\to\infty} \frac{g(x, s) - g^\infty(x, s)}{1 + s^{\gamma/2}} = 0,$$

uniformly for $s \ge 0$.

Let

$$\begin{aligned}
V^\infty(x) &= g^\infty(x,0) \text{ and } h^\infty(x,s) = g^\infty(x,s) - V^\infty(x) \\
H^\infty(x,s) &= \int_0^{s^2} h^\infty(x,t)dt, \\
\Lambda^\infty &= \inf\{\int_{\mathbb{R}^N} |\nabla u|^2 - V^\infty(x)u^2 dx : u \in S_1\} \\
J(u) &= E(u) - \frac{1}{2}\Lambda^\infty \int_{\mathbb{R}^N} u^2 dx \\
\mu_c &= \inf\{J(u) : u \in S_c\} = m_c - \frac{1}{2}\Lambda^\infty c^2.
\end{aligned}$$

The condition (A1) implies that

$$\lim_{|x|\to\infty}\{V(x) - V^\infty(x)\} = 0 \text{ and } \lim_{|x|\to\infty} \frac{h(x,s) - h^\infty(x,s)}{1 + s^{\gamma/2}} = 0$$

uniformly for $s \geq 0$. Noting that (A1) implies that V^∞ is periodic, we see that $V(\infty) = \limsup_{|x|\to\infty} V(x) = \sup_{x\in\mathbb{R}^N} V^\infty(x)$ and hence that $-\Lambda^\infty \leq V(\infty)$, with equality in the case where V^∞ is constant. Under the assumptions $(g0)$ and (A1), we have that $h^\infty(x,0) = 0$ and there exists a constant $C > 0$ such that

$$|h^\infty(x,s)| \leq C(1 + s^{l/2}) \text{ and } |H^\infty(x,s)| \leq C(s^2 + s^{l+2}) \text{ for all } s \geq 0 \qquad (5.29)$$

since g^∞ satisfies $(g0)$, where $H^\infty(x,s) = \int_0^{s^2} h^\infty(x,t)dt$.

We introduce the following additional assumptions about the limit problem:

(A2) There exist $r \in L^\infty$ and $\sigma \in (0, \frac{4}{N})$ such that

$\dfrac{h^\infty(x,s)}{s^{\sigma/2}}$ is a non-decreasing function of s on $(0,\infty)$ for a.e. $x \in \mathbb{R}^N$, and $\displaystyle\lim_{s\to 0} \frac{h^\infty(x,s)}{s^{\sigma/2}} = r(x) \geq 0$ but $\not\equiv 0$, uniformly for $x \in \mathbb{R}^N$.

It follows that

$$\begin{aligned}
H^\infty(x,\theta s) &= \int_0^{\theta^2 s^2} h^\infty(x,t)dt = \theta^2 \int_0^{s^2} h^\infty(x,\theta^2\tau)d\tau \\
&\geq \theta^{\sigma+2} \int_0^{s^2} h^\infty(x,\tau)d\tau = \theta^{\sigma+2} H^\infty(x,s) \text{ and} \qquad (5.30) \\
H^\infty(x,s) &\geq \int_0^{s^2} r(x)\tau^{\sigma/2}d\tau \geq \frac{2r(x)\,|s|^{\sigma+2}}{\sigma+2}, \text{ for all } s \in \mathbb{R} \text{ and } \theta \geq 1 \qquad (5.31)
\end{aligned}$$

5.2.1 Periodic nonlinearity

We begin with a result concerning a nonlinearity that is a periodic function of x.

Lemma 5.4 *Let g satisfy the condition $(g0)$ and, in addition,*

$$g(x+z,s)=g(x,s) \text{ for all } x\in\mathbb{R}^N, z\in\mathbb{Z}^N \text{ and } s\geq 0. \tag{5.32}$$

Thus (A1) is satisfied with $g^\infty\equiv g$ and we assume that (A2) holds for $h^\infty\equiv h$. Then for all $c>0$, $m_c<\frac{1}{2}\Lambda^\infty c^2$ and $W_c\neq\emptyset$.

Remark Here also $V\equiv V^\infty$.

Proof. We apply the concentration-compactness principle (see Appendix) to the functional J.
Step 1 We set $I_c=m_{\sqrt{c}}-\frac{1}{2}\Lambda^\infty c$ and we begin by showing that

$$I_c<0 \text{ and } I_{\theta c}<\theta I_c \text{ for all } c>0 \text{ and } \theta>1,$$

from which it follows that $m_c<\frac{1}{2}\Lambda^\infty c^2$ and, by Lemma II.1 of [16] Part 1,

$$I_\lambda<I_\alpha+I_{\lambda-\alpha} \text{ for } 0<\alpha<\lambda. \tag{5.33}$$

Since

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2-[\Lambda^\infty+V(x)]u^2-H(x,u)dx$$

we obtain

$$J(u)\leq\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2-[\Lambda^\infty+V(x)]u^2-\frac{2r(x)\left|u\right|^{\sigma+2}}{\sigma+2}dx.$$

It follows from its definition that $\Lambda^\infty=\inf\sigma(-\Delta-V)$, and so as is shown in Lemmas 3.2 and 3.3 of [12] and Lemma 9.5(b) of [20], there is a sequence $\{z_k\}\subset H^1$ such that, as $k\to\infty$,

$$\begin{aligned}
&\left|z_k\right|_2\to A>0,\\
&k^2\int_{\mathbb{R}^N}|\nabla z_k|^2-[\Lambda^\infty+V(x)]{z_k}^2dx\to B>0,\\
&k^{\frac{N\sigma}{2}}\int_{\mathbb{R}^N}r(x)\left|z_k\right|^{\sigma+2}dx\to C>0.
\end{aligned}$$

Given $c>0$, we set $v_k=cz_k/\left|z_k\right|_2$. Then $v_k\in S_c$ and

$$\begin{aligned}
&k^2\int_{\mathbb{R}^N}|\nabla v_k|^2-[\Lambda^\infty+V(x)]v_k^2-\frac{2r(x)\left|v_k\right|^{\sigma+2}}{\sigma+2}dx\\
=\;&\frac{c^2k^2}{\left|z_k\right|_2^2}\int_{\mathbb{R}^N}|\nabla z_k|^2-[\Lambda^\infty+V(x)]z_k^2dx-\frac{2c^{\sigma+2}k^2}{(\sigma+2)\left|z_k\right|_2^{\sigma+2}}\int_{\mathbb{R}^N}r(x)\left|z_k\right|^{\sigma+2}dx\\
\to\;&-\infty,
\end{aligned}$$

showing that $J(v_k) < 0$ for large enough k. Hence $\mu_c = I_{c^2} < 0$ for all $c > 0$.

Next, for any $c > 0$ and $\theta > 1$, we can choose $\varepsilon > 0$ such that $\varepsilon < -I_c(1-\theta^{-\sigma})$ and there exists $v \in S_{\sqrt{c}}$ such that $I_c \leq J(v) < I_c + \varepsilon$. Hence

$$\begin{aligned}
I_{\theta^2 c} &\leq J(\theta v) = \theta^2 \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - [\Lambda^\infty + V(x)]v^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} H(x,\theta v)dx \\
&\leq \theta^2 \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - [\Lambda^\infty + V(x)]v^2 dx - \frac{\theta^{\sigma+2}}{2}\int_{\mathbb{R}^N} H(x,v)dx \\
&\leq \theta^2 \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - [\Lambda^\infty + V(x)]v^2 dx \\
&\quad -\theta^{\sigma+2}\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - [\Lambda^\infty + V(x)]v^2 dx - J(v)\} \\
&= (\theta^2 - \theta^{\sigma+2})\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - [\Lambda^\infty + V(x)]v^2 dx + \theta^{\sigma+2}J(v) \leq \theta^{\sigma+2}J(v) \\
&\leq \theta^{\sigma+2}\{I_c + \varepsilon\} < \theta^2 I_c, \text{ by the choice of } \varepsilon.
\end{aligned}$$

This establishes (5.33).

Step 2 Now we consider a sequence $\{u_n\} \subset S_c$ such that $E(u_n) \to m_c$. By Proposition 2.3(ii) this sequence is bounded and, passing to a subsequence we may assume henceforth that $u_n \rightharpoonup u$ weakly in H^1.

Vanishing does not occur: If vanishing occurs, it follows from Lemma I.1 in [16], Part 2 (see also Lemma 8.4 in [14]) that $|u_n|_p \to 0$ as $n \to \infty$ for $p \in (2, 2^*)$. By (g0) and (A2), there is a constant $K > 0$ such that $|h(x,s)| \leq K(s^{l/2} + s^{\sigma/2})$ for all $s \geq 0$ and all $x \in \mathbb{R}^N$. Therefore, $|H(x,s)| \leq C(s^{l+2} + s^{\sigma+2})$ for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^N$ so

$$\int_{\mathbb{R}^N} |H(x,u_n)|\, dx \leq C\{|u_n|_{l+2}^{l+2} + |u_n|_{\sigma+2}^{\sigma+2}\} \to 0$$

as $n \to \infty$ and this means that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u_n|^2 - \{\Lambda^\infty + V(x)\}u_n^2 dx &= 2J(u_n) + \int_{\mathbb{R}^N} H(x,u_n)dx \\
&\to 2\mu_c = 2I_{c^2}
\end{aligned}$$

which is clearly impossible since $I_{c^2} < 0$ and $\int_{\mathbb{R}^N} |\nabla u_n|^2 - \{\Lambda^\infty + V(x)\}u_n^2 dx \geq 0$ for all n.

Dichotomy does not occur: We use the notation introduced in the Appendix for this case.

For $n \geq n_0$,

$$\begin{aligned}
& 2\{J(u_n) - J(u_{n,1}) - J(u_{n,2})\} \\
= & \int_{\mathbb{R}^N} |\nabla u_n|^2 - |\nabla u_{n,1}|^2 - |\nabla u_{n,2}|^2 \, dx \\
& - \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_{n,1}) - G(x,u_{n,2}) dx - \Lambda^\infty \int_{\mathbb{R}^N} (u_n^2 - u_{n,1}^2 - u_{n,2}^2) dx \\
= & \int_{\mathbb{R}^N} |\nabla u_n|^2 - |\nabla u_{n,1}|^2 - |\nabla u_{n,2}|^2 \, dx - \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_{n,1}+u_{n,2}) dx \\
& -\Lambda^\infty\{c^2 - \int_{\mathbb{R}^N} (u_{n,1}^2 + u_{n,2}^2) dx\} \text{ (since supp} u_{n,1} \cap \text{supp} u_{n,2} \text{ is empty)} \\
\geq & -2\varepsilon - \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_{n,1}+u_{n,2}) dx - |\Lambda^\infty| \, 2\varepsilon.
\end{aligned}$$

Since the sequences $\{u_n\}, \{u_{n,1}\}$ and $\{u_{n,2}\}$ are bounded in H^1, it follows from (2.15) that there exist $C, K > 0$ such that

$$\begin{aligned}
& \left| \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_{n,1}+u_{n,2}) dx \right| \\
\leq & \sup_{\|u\|_{H^1} \leq K} \|\Gamma(u)\|_{H^{-1}} \|u_n - (u_{n,1}+u_{n,2})\|_{H^1} \\
\leq & \sup_{\|u\|_{H^1} \leq K} \|\Gamma_1(u)\|_{L^2} \|u_n - (u_{n,1}+u_{n,2})\|_{L^2} \\
& + \sup_{\|u\|_{H^1} \leq K} \|\Gamma_2(u)\|_{L^p} \|u_n - (u_{n,1}+u_{n,2})\|_{L^{p'}} \\
\leq & C \sup_{\|u\|_{H^1} \leq K} \|u\|_{L^2} \|u_n - (u_{n,1}+u_{n,2})\|_{L^2} \\
& + C \sup_{\|u\|_{H^1} \leq K} \|u\|_{L^q}^{1+\frac{4}{N}} \|u_n - (u_{n,1}+u_{n,2})\|_{L^{p'}} \\
\leq & C_1 K \|u_n - (u_{n,1}+u_{n,2})\|_{L^2} + C_2 K^{1+\frac{4}{N}} \|u_n - (u_{n,1}+u_{n,2})\|_{L^{p'}} . \quad (5.34)
\end{aligned}$$

Thus, given any $\delta > 0$ we can choose $\varepsilon = \varepsilon_\delta \in (0, \delta)$ such that

$$-2\varepsilon - \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_{n,1}+u_{n,2}) dx - 2\varepsilon |V(\infty)| \geq -\delta$$

and hence

$$2\{J(u_n) - J(u_{n,1}) - J(u_{n,2})\} \geq -\delta$$

Let

$$a_{n,i}(\delta)^2 = \int_{\mathbb{R}^N} u_{n,i}^2 dx \text{ for } i = 1, 2.$$

Passing to a subsequence we may suppose that

$$a_{n,i}(\delta)^2 \to a_i(\delta)^2, \text{ where } \left|a_1(\delta)^2 - a^2\right| \leq \varepsilon_\delta \leq \delta \text{ and } \left|a_2(\delta)^2 - (c^2 - a^2)\right| \leq \varepsilon_\delta \leq \delta.$$

Recalling that μ_c is a continuous function of c, we find that

$$\begin{aligned}\mu_c &= \lim_{n\to\infty} J(u_n) \geq \lim\inf_{n\to\infty}\{J(u_{n,1})+J(u_{n,2})\} - \frac{\delta}{2}\\ &\geq \lim\inf_{n\to\infty}\{\mu_{a_{n,1}(\delta)}+\mu_{a_{n,2}(\delta)}\} - \frac{\delta}{2}\\ &= \mu_{a_1(\delta)}+\mu_{a_2(\delta)} - \frac{\delta}{2} \text{ for all } \delta>0.\end{aligned}$$

Letting $\delta \to 0$, we obtain $\mu_c \geq \mu_a + \mu_{\sqrt{c^2-a^2}}$ which can be expressed as $I_{c^2} \geq I_{a^2} + I_{c^2-a^2}$ contradicting (5.33). Thus dichotomy cannot occur.

Hence there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that, for all $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$\int_{B(y_n,\, R(\varepsilon))} u_n^2(x)\, dx \geq c^2 - \varepsilon.$$

For each $n \in N$, we can choose $z_n \in \mathbb{Z}^N$ such that $y_n - z_n \in Q = [0,1]^N$. Setting $v_n(x) = u_n(x + z_n)$ we have that $\|v_n\|_{H^1} = \|u_n\|_{H^1}$ is bounded and so, passing to a subsequence we may suppose that $v_n \rightharpoonup v$ weakly in H^1. In particular, $v_n \rightharpoonup v$ weakly in L^2 and $|v_n|_2^2 = c^2$. But

$$\begin{aligned}\int_{\mathbb{R}^N} v^2 dx &\geq \int_{B(0,\, R(\varepsilon)+\sqrt{N})} v^2 dx = \lim_{n\to\infty}\int_{B(0,\, R(\varepsilon)+\sqrt{N})} v_n^2 dx\\ &= \lim_{n\to\infty}\int_{B(z_n,\, R(\varepsilon)+\sqrt{N})} u_n^2 dx\end{aligned}$$

where

$$\int_{B(z_n,\, R(\varepsilon)+\sqrt{N})} u_n^2 dx \geq \int_{B(y_n,\, R(\varepsilon))} u_n^2 dx \geq c^2 - \varepsilon$$

since $|y_n - z_n| \leq \sqrt{N}$. Hence $|v|_2^2 \geq c^2 - \varepsilon$ for all $\varepsilon > 0$ and this implies that $|v_n - v|_2 \to 0$ as $n \to \infty$. Furthermore, by the periodicity of g, $E(v_n) = E(u_n)$ so that $\{v_n\} \subset S_c$, $E(v_n) \to m_c$ and $\{v_n\}$ converges in L^2. As in Lemma 5.1, this implies that $\{v_n\}$ converges to v in H^1, showing that $v \in S_c$ and $E(v) = m_c$. By Theorem 3.1, $|v| \in W_c$. □

Remark 5.1 Under the hypotheses of Lemma 5.4 the property $(P3)$ is holds, up to $\mathbb{Z}^N-$translation, for all $c > 0$, and since Z_c is invariant under $\mathbb{Z}^N-$translations, this is sufficient to ensure that Z_c is stable for all $c > 0$. Note also that this periodic situation includes the autonomous case treated in [5] and [6].

More precisely we have the following result.

Proposition 5.2 *Under the hypotheses of Lemma 5.4 the following conclusions are valid for any* $c > 0$.

(a) For any $\{u_n\} \subset H^1$ *such that* $|u_n|_2 \to c$ *and* $E(u_n) \to m_c$*, there exists a sequence* $\{y_n\} \subset \mathbb{Z}^N$ *such that* $\{v_n\}$ *is relatively compact in* H^1*, where* $v_n(x) = u_n(x + y_n)$ *for all* $x \in \mathbb{R}^N$.

(b) If in addition $(g1)$ *is satisfied, then* Z_c *is stable for all* $c > 0$.

Proof. (a) This is established in Step 2 of the proof of Lemma 5.4.

(b) By Lemma 5.4 we already have that $Z_c \neq \varnothing$. Let us suppose that (1.11) occurs and set $z_n = \Phi^n(t_n, \cdot)$. Then $\{z_n\} \subset H$ is a sequence such that $\|z_n\|_2 \to c$ and $\widetilde{E}(z_n) \to M_c$. By Proposition 2.3(ii), $\{z_n\}$ is bounded in H, and by passing to a subsequence we may assume that (3.19) holds. Setting $\varphi_n = |z_n|$ and following the arguments leading to (3.21) and (3.22), we can conclude from part (a) that there exists a sequence $\{y_n\} \subset \mathbb{Z}^N$ and an element $\psi \in H^1$ such that $\widetilde{\varphi}_n \to \psi$ in H^1, where $\widetilde{\varphi}_n(x) = \varphi_n(x + y_n)$ for all $x \in \mathbb{R}^N$. Let $\widetilde{z}_n(x) = z_n(x + y_n)$. Then $\{\widetilde{z}_n\}$ is also a sequence in H such that $\|\widetilde{z}_n\|_2 \to c$ and $\widetilde{E}(\widetilde{z}_n) \to M_c$. Now $\widetilde{\varphi}_n = |\widetilde{z}_n|$ and the proof of Theorem 3.1 shows that $\widetilde{z}_n \to w$ in H. Thus $w \in Z_c$ and

$$\inf_{z \in Z_c} \|z_n - z\|_H = \inf_{z \in Z_c} \|\widetilde{z}_n - z\|_H$$

by the periodicity of g. But then

$$\inf_{z \in Z_c} \|\Phi^n(t_n, \cdot) - z\|_H = \inf_{z \in Z_c} \|z_n - z\|_H = \inf_{z \in Z_c} \|\widetilde{z}_n - z\|_H \leq \|\widetilde{z}_n - w\|_H \to 0,$$

contradicting (1.11).

5.2.2 Asymptotically periodic nonlinearity

We now come to the main point of this part which concerns asymptotically periodic problems. Under the assumption (A1), we set

$$\begin{aligned}
E^\infty(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - V^\infty(x)u^2 - H^\infty(x,u)dx \\
J^\infty(u) &= E^\infty(u) - \frac{1}{2}\Lambda^\infty \int_{\mathbb{R}^N} u^2 dx \\
m_c^\infty &= \inf\{E^\infty(u) : u \in S_c\} \text{ and } \mu_c^\infty = \inf\{J^\infty(u) : u \in S_c\} = m_c^\infty - \frac{1}{2}\Lambda^\infty c^2.
\end{aligned}$$

(A3) For almost all $x \in \mathbb{R}^N$ and all $s \geq 0$, $g(x,s) = V(x) + h(x,s) \geq V^\infty(x) + h^\infty(x,s)$ and the inequality is strict for all x in a set of positive measure.

Lemma 5.5 *Let g satisfy the conditions $(g0)$, (H1) and (A1) to (A3). Then*

$$\begin{aligned}
\mu_c &< \mu_c^\infty < 0 \textit{ for all } c > 0 \textit{ and} \\
\mu_c &\leq \mu_a + \mu_{\sqrt{c^2-a^2}} < \mu_a + \mu^\infty_{\sqrt{c^2-a^2}} < \mu_a \textit{ for } 0 < a < c.
\end{aligned}$$

Proof. Noting that the functional E^∞ satisfies the hypotheses of Lemma 5.5, we have that $\mu_c^\infty < 0$ for all $c > 0$, and there exists a function $u_c \in S_c \cap C^1(\mathbb{R}^N)$ such that $u_c > 0$ and $J^\infty(u_c) = \mu_c^\infty$. Then $J(u_c) < J^\infty(u_c) < 0$ by (A3) showing that $\mu_c < \mu_c^\infty < 0$ for all $c > 0$.

From the condition (H1), we conclude that $\mu_{tc} \leq t^2\mu_{tc} < 0$ for all $c > 0$ and $t \geq 1$, from which it follows that $\mu_c \leq \mu_a + \mu_{\sqrt{c^2-a^2}}$ for $0 < a < c$. □

Proposition 5.3 *Let g satisfy the conditions $(g0)$, (H1) and (A1) to (A3). Then the property $(P3)$ is satisfied for all $c > 0$.*

Proof. Fix $c > 0$ and consider a sequence $\{u_n\} \subset H^1$ such that $|u_n|_2 \to c$ and $E(u_n) \to m_c$. By Proposition 2.3(ii), we know that $\{u_n\}$ is bounded in H^1. By Lemma 5.1, it is enough to show that $\{u_n\}$ contains a subsequence converging in L^2. First we observe that by replacing u_n by $\frac{c}{c_n}u_n$, we may assume that

$$|u_n|_2 = c \text{ for all } n \in \mathbb{N},\ u_n \rightharpoonup u \text{ weakly in } H^1 \text{ and } J(u_n) \to \mu_c. \tag{5.35}$$

The remainder of the proof is organized as follows. Applying the concentration-compactness lemma to $\{u_n\}$ we show that compactness must occur and from this we deduce the existence of a sequence converging in L^2.

Vanishing does not occur: If vanishing occurs, it follows from Lemma I.1 in [16], Part2 (see also Lemma 8.4 in [14]) that $|u_n|_p \to 0$ as $n \to \infty$ for $p \in (2, 2^*)$. By (5.29) and (A2), there is a constant $K > 0$ such that $|h^\infty(x,s)| \le K(s^{l/2} + s^{\sigma/2})$ for all $s \ge 0$ and consequently, for any $\delta > 0$, there exists $R_\delta > 0$ such that $|h(x,s)| \le \delta(1+s^{\gamma/2}) + K(s^{l/2}+s^{\sigma/2})$ for all $s \ge 0$ and $|x| \ge R_\delta$. Hence, $|H(x,s)| \le \delta(s^2 + |s|^{\gamma+2}) + K(|s|^{l+2} + |s|^{\sigma+2})$ for all $s \in \mathbb{R}$ and $|x| \ge R_\delta$, showing that

$$\int_{|x|\ge R_\delta} |H(x,u_n)|\, dx \le \delta(|u_n|_2^2 + |u_n|_{\gamma+2}^{\gamma+2}) + K(|u_n|_{l+2}^{l+2} + |u_n|_{\sigma+2}^{\sigma+2})$$

and

$$\limsup_{n\to\infty} \int_{|x|\ge R_\delta} |H(x,u_n)|\, dx \le \delta c^2.$$

On the other hand, $|H(x,s)| \le C(s^2 + |s|^{l+2})$ for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^N$, so

$$\begin{aligned}\int_{|x|\le R_\delta} |H(x,u_n)|\, dx &\le C\int_{|x|\le R_\delta} u_n^2 + |u_n|^{l+2}\, dx \\ &\le C\{|u_n|_{l+2}^{2/(l+2)} |R_\delta|^{l/(l+2)} + |u_n|_{l+2}^{l+2}\} \to 0\end{aligned}$$

as $n \to \infty$. Hence, for any $\delta > 0$, we have that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |H(x,u_n)|\, dx \le \delta c^2 \text{ and so } \int_{\mathbb{R}^N} H(x,u_n)dx \to 0 \text{ as } n \to \infty.$$

Since $\lim_{|x|\to\infty}\{V(x) - V^\infty(x)\} = 0$, we also have that

$$\int_{\mathbb{R}^N} \{V(x) - V^\infty(x)\}u_n^2 dx \to 0 \text{ as } n \to \infty$$

by a similar argument. This means that

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u_n|^2 - \{\Lambda^\infty + V^\infty(x)\}u_n^2 dx \\ =\ & 2J(u_n) + \int_{\mathbb{R}^N} \{V(x) - V^\infty(x)\}u_n^2 + H(x,u_n)dx \\ \to\ & 2\mu_c\end{aligned}$$

which is clearly impossible since $\mu_c < 0$ and $\int_{\mathbb{R}^N} |\nabla u_n|^2 - \{\Lambda^\infty + V^\infty(x)\}u_n^2 dx \geq 0$ for all n.

Dichotomy does not occur: We use the notation introduced in the Appendix for this case.

Suppose first that the sequence $\{y_n\}$ is bounded.

$$\begin{aligned}
&2\{J(u_n) - J(u_{n,1}) - J^\infty(u_{n,2})\} \\
= &\int_{\mathbb{R}^N} |\nabla u_n|^2 - |\nabla u_{n,1}|^2 - |\nabla u_{n,2}|^2 \, dx \\
&- \int_{\mathbb{R}^N} \{\Lambda^\infty + V\}(u_n^2 - u_{n,1}^2 - u_{n,2}^2)dx - \int_{\mathbb{R}^N} H(x,u_n) - H(x,u_{n,1}) - H(x,u_{n,2})dx \\
&+ \int_{\mathbb{R}^N} H^\infty(x,u_{n,2}) - H(x,u_{n,2})dx - \int_{\mathbb{R}^N} \{V - V^\infty\}u_{n,2}^2 dx \\
\geq &-2\varepsilon - \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_{n,1}) - G(x,u_{n,2})dx - \Lambda^\infty \int_{\mathbb{R}^N} (u_n^2 - u_{n,1}^2 - u_{n,2}^2)dx \\
&+ \int_{\mathbb{R}^N} H^\infty(x,u_{n,2}) - H(x,u_{n,2})dx - \int_{\mathbb{R}^N} \{V - V^\infty\}u_{n,2}^2 dx \\
= &-2\varepsilon - \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_{n,1} + u_{n,2})dx - \Lambda^\infty \int_{\mathbb{R}^N} (u_n^2 - u_{n,1}^2 - u_{n,2}^2)dx \\
&+ \int_{\mathbb{R}^N} H^\infty(x,u_{n,2}) - H(x,u_{n,2})dx - \int_{\mathbb{R}^N} \{V - V^\infty\}u_{n,2}^2 dx \\
&\text{(since } \mathrm{supp} u_{n,1} \cap \mathrm{supp} u_{n,2} \text{ is empty)} \\
\geq &-2\varepsilon - \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_{n,1} + u_{n,2})dx - 2\varepsilon |\Lambda^\infty| \\
&+ \int_{|x-y_n| \geq R_n} H^\infty(x,u_{n,2}) - H(x,u_{n,2}) - \{V - V^\infty\}u_{n,2}^2 dx.
\end{aligned}$$

Thus, using (5.34) we see that, given any $\delta > 0$ we can choose $\varepsilon = \varepsilon_\delta \in (0,\delta)$ such that

$$-2\varepsilon - \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_{n,1} + u_{n,2})dx - 2\varepsilon |\Lambda^\infty| \geq -\delta$$

and hence

$$\begin{aligned}
&2\{J(u_n) - J(u_{n,1}) - J^\infty(u_{n,2})\} \\
\geq &-\delta + \int_{|x-y_n| \geq R_n} H^\infty(x,u_{n,2}) - H(x,u_{n,2}) - \{V - V^\infty\}u_{n,2}^2 dx.
\end{aligned}$$

Given any $\eta > 0$, there exists $R > 0$ such that, for all $s \in \mathbb{R}$ and $|x| \geq R$,

$$\begin{aligned}
|H^\infty(x,s) - H(x,s)| &\leq \eta(s^2 + |s|^{\gamma+2}) \\
|V(x) - V^\infty(x)| &\leq \eta.
\end{aligned}$$

Since $R_n \to \infty$ and we are supposing that $\{y_n\}$ is bounded, we have that

$$\{x : |x - y_n| \geq R_n\} \subset \{x : |x| \geq R\} \text{ for all large enough } n.$$

From this and the boundedness of the sequence $\{u_{n,2}\}$ in H^1, it follows from (H1) that

$$\lim_{n\to\infty}\int_{|x-y_n|\geq R_n} H^{\infty}(x,u_{n,2})-H(x,u_{n,2})-\{V-V^{\infty}\}u_{n,2}^2dx=0.$$

Let

$$a_{n,i}(\delta)^2=\int_{\mathbb{R}^N}u_{n,i}^2dx \text{ for } i=1,2.$$

Passing to a subsequence we may suppose that

$$a_{n,i}(\delta)^2\to a_i(\delta)^2 \text{ where } \left|a_1(\delta)^2-a^2\right|\leq\varepsilon_\delta\leq\delta \text{ and } \left|a_2(\delta)^2-(c^2-a^2)\right|\leq\varepsilon_\delta\leq\delta.$$

Recalling that μ_c and μ_c^{∞} are continuous functions of c, we find that

$$\begin{aligned}\mu_c &= \lim_{n\to\infty}J(u_n)\geq\liminf_{n\to\infty}\{J(u_{n,1})+J^{\infty}(u_{n,2})\}-\frac{\delta}{2}\\ &\geq \liminf_{n\to\infty}\{\mu_{a_{n,1}(\delta)}+\mu^{\infty}_{a_{n,2}(\delta)}\}-\frac{\delta}{2}\\ &= \mu_{a_1(\delta)}+\mu^{\infty}_{a_2(\delta)}-\frac{\delta}{2} \text{ for all } \delta>0.\end{aligned}$$

Letting $\delta\to 0$, we obtain $\mu_c\geq\mu_a+\mu^{\infty}_{\sqrt{c^2-a^2}}$ contradicting Lemma 5.5.

Thus the sequence $\{y_n\}$ cannot be bounded and, passing to a subsequence we may suppose that $|y_n|\to\infty$. Now we obtain a contradiction to Lemma 5.5 by using similar arguments applied to $J(u_n)-J^{\infty}(u_{n,1})-J(u_{n,2})$ to show that $\mu_c\geq\mu_a^{\infty}+\mu_{\sqrt{c^2-a^2}}$.

Thus dichotomy cannot occur.

Existence of a convergent subsequence Since we have shown that vanishing and dichotomy cannot occur, there exists a sequence $\{y_n\}\subset\mathbb{R}^N$ such that, for all $\varepsilon>0$, there exists $R(\varepsilon)>0$ such that

$$\int_{B(y_n,\,R(\varepsilon))}u_n^2(x)\,dx\geq c^2-\varepsilon \text{ for all } n\in\mathbb{N}. \tag{5.36}$$

Let us show that the sequence $\{y_n\}$ is bounded. Indeed, if this is not the case we can assume that $|y_n|\to\infty$ by passing to a subsequence. Then, as in the last part of the proof of Lemma 5.4, for each $n\in N$, we can choose $z_n\in\mathbb{Z}^N$ such that $y_n-z_n\in Q=[0,1]^N$ and, setting $v_n(x)=u_n(x+z_n)$ we may suppose that $v_n\rightharpoonup v$ weakly in $H^1$and $|v_n-v|_2\to 0$ as $n\to\infty$. From the boundedness of $\{v_n\}$ in H^1 it follows that $|v_n-v|_p\to 0$ as $n\to\infty$ for $2\leq p<2^*-2$. Furthermore, by the periodicity of V^{∞} and h^{∞}, $J^{\infty}(v_n)=J^{\infty}(u_n)$. But

$$\begin{aligned}J(u_n)-J^{\infty}(u_n) &= \int_{\mathbb{R}^N}\{V^{\infty}(x)-V(x)\}u_n^2+H^{\infty}(x,u_n)-H(x,u_n)dx\\ &= \int_{\mathbb{R}^N}\{V^{\infty}(x)-V(x-z_n)\}v_n^2+H^{\infty}(x,v_n)-H(x-z_n,v_n)dx.\end{aligned}$$

Now, given any $\varepsilon > 0$, it follows from (A1) that there exists $R > 0$ such that

$$\begin{aligned}
& \left| \int_{|x-z_n|\geq R} \{V^\infty(x) - V(x-z_n)\} v_n^2 + H^\infty(x, v_n) - H(x-z_n, v_n) dx \right| \\
= & \left| \int_{|x-z_n|\geq R} \{V^\infty(x-z_n) - V(x-z_n)\} v_n^2 + H^\infty(x-z_n, v_n) - H(x-z_n, v_n) dx \right| \\
\leq & \ \varepsilon \int_{|x-z_n|\geq R} v_n^2 + |v_n|^{\gamma+2}\, dx \leq \varepsilon C \{\|v_n\|_{H^1}^2 + \|v_n\|_{H^1}^{\gamma+2}\} \leq \varepsilon D
\end{aligned}$$

since $\{v_n\}$ is bounded in H^1. On the other hand, since $|z_n| \to \infty$, there exists $n_R > 0$ such that, for all $n \geq n_R$,

$$\begin{aligned}
& \left| \int_{|x-z_n|\leq R} \{V^\infty(x) - V(x-z_n)\} v_n^2 + H^\infty(x, v_n) - H(x-z_n, v_n) dx \right| \\
\leq & \left| \int_{|x|\geq \frac{1}{2}|z_n|} \{V^\infty(x) - V(x-z_n)\} v_n^2 + H^\infty(x, v_n) - H(x-z_n, v_n) dx \right| \\
\leq & \ 2 \int_{|x|\geq \frac{1}{2}|z_n|} |V|_\infty\, v_n^2 + C(v_n^2 + |v_n|^{l+2}) dx \\
\leq & \ K \int_{|x|\geq \frac{1}{2}|z_n|} v_n^2 + |v_n|^{l+2}\, dx \\
\leq & \ K \int_{|x|\geq \frac{1}{2}|z_n|} v^2 + |v|^{l+2}\, dx + K \int_{|x|\geq \frac{1}{2}|z_n|} (v-v_n)^2 + |v-v_n|^{l+2}\, dx \\
\leq & \ K \int_{|x|\geq \frac{1}{2}|z_n|} v^2 + |v|^{l+2}\, dx + K \int_{\mathbb{R}^N} (v-v_n)^2 + |v-v_n|^{l+2}\, dx
\end{aligned}$$

and hence

$$\lim_{n\to\infty} \left| \int_{|x-z_n|\leq R} \{V^\infty(x) - V(x-z_n)\} v_n^2 + H^\infty(x, v_n) - H(x-z_n, v_n) dx \right| = 0.$$

Thus,

$$\liminf_{n\to\infty} \{J(u_n) - J^\infty(u_n)\} \geq -\varepsilon D \text{ for all } \varepsilon > 0$$

and so $\mu_c = \lim J(u_n) \geq \liminf_{n\to\infty} J^\infty(u_n) \geq \mu_c^\infty$, contradicting Lemma 5.5. Hence the sequence $\{y_n\}$ is bounded as claimed.

Let $r = \sup_{n\in\mathbb{N}} |y_n|$. It follows from (5.36) that

$$\int_{B(0,\, R(\varepsilon)+r)} u_n^2(x)\, dx \geq \int_{B(y_n,\, R(\varepsilon))} u_n^2(x)\, dx \geq c^2 - \varepsilon \text{ for all } n \in \mathbb{N}.$$

Hence

$$\int_{\mathbb{R}^N} u^2 dx \geq \int_{B(0,\, R(\varepsilon)+r)} u^2(x)\, dx = \lim_{n\to\infty} \int_{B(0,\, R(\varepsilon)+r)} u_n^2(x)\, dx \geq c^2 - \varepsilon$$

for all $\varepsilon > 0$. This proves that $u \in S_c$ and, recalling (5.35) we can conclude that $|u - u_n|_2 \to 0$ as $n \to \infty$. By Lemma 5.1, this completes the proof. □

6 Appendix

The concentration - compactness lemma: see Lemmas I.1 and III.1in [16] Part1.

Let $\{u_n\}$ be a bounded sequence in H^1 such that $\int u_n^2 = c^2$, where $c > 0$. Then there exists a subsequence (we also denote it by $\{u_n\}$) satisfying one of the following properties

(1) *Vanishing*: $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,\,R)} u_n^2(x)\,dx = 0$ for every $R < \infty$,

(2) *Dichotomy*: There exists $a \in (0, c)$ such that $\forall\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ and two bounded sequences in H^1 denoted by $\{u_{n,\,1}\}$ and $\{u_{n,\,2}\}$ (all depending on ε) such that for every $n \geq n_0$,

$$\begin{aligned} \left|\int_{\mathbb{R}^N} u_{n,\,1}^2 dx - a^2\right| &< \varepsilon, \left|\int_{\mathbb{R}^N} u_{n,\,2}^2 dx - (c^2 - a^2)\right| < \varepsilon, \\ \int_{\mathbb{R}^N} |\nabla u_n|^2 - |\nabla u_{n,1}|^2 - |\nabla u_{n,2}|^2\,dx &\geq -2\varepsilon, \text{ and} \\ |u_n - (u_{n,\,1} + u_{n,\,2})|_p &\leq 4\varepsilon \text{ for all } p \in [2, 2^*). \end{aligned}$$

Furthermore, $\exists$ a sequences $\{y_n\} \subset \mathbb{R}^N$ and $\{R_n\} \in (0, \infty)$ such that $\lim_{n\to\infty} R_n = \infty$ and

$$\left\{\begin{array}{cc} u_{n,\,1} = u_n & \text{if } |x - y_n| \leq R_0 \\ |u_{n,\,1}| \leq |u_n| & \text{if } R_0 \leq |x - y_n| \leq 2R_0 \\ u_{n,\,1} = 0 & \text{if } |x - y_n| \geq 2R_0 \end{array}\right. \text{ and } \left\{\begin{array}{cc} u_{n,\,2} = 0 & \text{if } |x - y_n| \leq R_n \\ |u_{n,\,2}| \leq |u_n| & \text{if } R_n \leq |x - y_n| \leq 2R_n \\ u_{n,\,2} = u_n & \text{if } |x - y_n| \geq 2R_n \end{array}\right. .$$

(3) *Compactness:* There exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that, for all $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$\int_{B(y_n,\,R(\varepsilon))} u_n^2(x)\,dx \geq c^2 - \varepsilon \text{ for all } n \in \mathbb{N}.$$

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