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On the Optimality of the Assumptions Used to Prove the Existence and Symmetry of Minimizers of Some Fractional Constrained Variational Problems

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Abstract. In this paper, we discuss the optimality of the assumptions used, in a previous paper, to prove existence and symmetry of minimizers of the fractional constrained variational problem:

$$\inf \left\{ \frac{1}{2} \int |\nabla_s u|^2 - \int F(|x|, u) : \; u \in H^s(\mathbb{R}^N) \text{ and } \int u^2 = c^2 \right\},$$

where c is a prescribed number.
More precisely, we will show that if one of the conditions, used to prove that all minimizers of the above constrained variational problem, are radial and radially decreasing for all c, do not hold true, then there are several interesting situations:

– There is no minimizer at all.
  The infimum is achieved but no minimizer is radial.
– For some values of c there is no minimizer. For large values, the minimizer is radial and radially decreasing.

In the fractional setting, such a study is more subtle than in the classical one. We take advantage of some brilliant results obtained recently in Cabre and Sire (Anal. PDEs, 2012), Dyda (Fractional calculus for power functions, 2012) and Hajaiej et al. (Necessary and sufficient conditions for the fractional Gagliardo–Nirenberg inequalities and applications to Navier–Stokes and generalized Boson equations, 2012).

1. Introduction

For a prescribed $c > 0, 0 < s < 1$ and $N > 2s$, the author has considered [3], the following fractional variational problem:

$$I_c = \inf \{E(u) : u \in S_c\} \tag{1.1}$$

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