On the Cauchy problem of fractional Schrödinger equation with Hartree type nonlinearity

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Abstract

We study the Cauchy problem for the fractional Schrödinger equation

$$i\partial_t u = (m^2 - \Delta)^{\frac{\alpha}{2}} u + F(u) \ \text{ in } \ \mathbb{R}^{1+n},$$

where $n \geq 1$, $m \geq 0$, $1 < \alpha < 2$, and F stands for the nonlinearity of Hartree type:

$$F(u) = \lambda \left(\frac{\psi(\cdot)}{|\cdot|^\gamma} * |u|^2 \right) u$$

with $\lambda = \pm 1, 0 < \gamma < n$, and $0 \leq \psi \in L^\infty(\mathbb{R}^n)$. We prove the existence and uniqueness of local and global solutions for certain α, γ, λ, ψ. We also remark on finite time blowup of solutions when $\lambda = -1$.

Key Words and Phrases. fractional Schrödinger equation, Hartree type nonlinearity, Strichartz estimates, finite time blowup

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1 Introduction

In this paper we consider the following Cauchy problem:

$$\begin{cases} i\partial_t u = D_m^\alpha u + F(u), \ \text{ in } \ \mathbb{R}^{1+n} \times \mathbb{R}, \ n \geq 1 \\ u(x,0) = \varphi(x) \ \text{ in } \ \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $D_m = (m^2 - \Delta)^{\frac{1}{2}}$, $1 < \alpha < 2$, and $F(u)$ is nonlinear functional of Hartree type such that $F(u) = \lambda \left(\frac{\psi(\cdot)}{|\cdot|^\gamma} * |u|^2 \right) u \equiv \lambda K_\gamma(|u|^2)u$, where $*$ denotes the convolution in $\mathbb{R}^n$, $\lambda = \pm 1$, $\mu \geq 0$, $0 < \gamma < n$ and $0 \leq \psi \in L^\infty(\mathbb{R}^n)$.

When $m = 0$, the equation (1.1) is called fractional Schrödinger equation which was used to describe particles in Lévy stochastic process, and when

$m > 0$, generalized semirelativistic equation. See [19, 20, 21, 22] and the references therein.

If $m = 0$, then similarly to the Schrödinger case ($\alpha = 2$) the equation (1.1) has scaling invariance property. In fact the function $u_a(t,x) = a^{\frac{n-\gamma+\alpha}{2}} u(a^\alpha t, ax)$ ($a > 0$) is also a solution of (1.1). The associated invariant space is $\dot H^{\frac{\gamma-\alpha}{2}}$. So, we call the equation $\dot H^{\frac{\gamma-\alpha}{2}}$-subcritical if we pursue the solution $u \in H^s$ for $s > \frac{\gamma-\alpha}{2}$, $\dot H^{\frac{\gamma-\alpha}{2}}$-critical for $s = \frac{\gamma-\alpha}{2}$ and $\dot H^{\frac{\gamma-\alpha}{2}}$-supercritical for $s < \frac{\gamma-\alpha}{2}$.

The purpose of this paper is to establish the local and global existence theory to the equation (1.1) and also finite time blowup. In this paper we study the Cauchy problem (1.1) in the form of the integral equation:

$$u(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')dt', \tag{1.2}$$

where

$$U(t)\varphi(x) = (e^{-it(m^2-\Delta)^{\frac{\alpha}{2}}}\varphi)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - t(m^2+|\xi|^2)^{\frac{\alpha}{2}})}\widehat{\varphi}(\xi)\,d\xi.$$

Here $\widehat{\varphi}$ denotes the Fourier transform of φ such that $\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx$.

One of the key tools for the global theory is the conservation law. If the solution u of (1.1) has sufficient decay at infinity and smoothness, it satisfies two conservation laws:

$$\begin{aligned} \|u(t)\|_{L^2} &= \|\varphi\|_{L^2}, \\ E(u) \equiv K(u) + V(u) &= E(\varphi), \end{aligned} \tag{1.3}$$

where $K(u) = \frac{1}{2}\langle (m^2-\Delta)^{\frac{\alpha}{2}}\, u, u\rangle$, $V(u) = \frac{1}{4}\langle F(u), u\rangle$ and $\langle\ ,\ \rangle$ is the complex inner product in L^2. The energy space is $H^{\frac{\alpha}{2}}$. So, the equation (1.1) is referred to be energy critical if $\gamma = 2\alpha$, subcritical if $\gamma < 2\alpha$ and supercritical if $\gamma > 2\alpha$, respectively. Similarly we use the terminology mass critical, subcritical, supercritical for the case $\gamma = \alpha$, $\gamma < \alpha$, $\gamma > \alpha$, respectively. For the proof of (1.3) a regularizing method is simply applicable as in [22] in the case of $0 < \gamma \le \alpha$. For local solutions constructed by a contraction argument based on the Strichartz estimate stated below, the case of $\alpha < \gamma \le 2\alpha$ is treated by exactly the same method as in [23] without using approximate or regularizing approach. The second tool is the Strichartz estimates. In Section 2 we recall three versions which will be used in the argument of the paper.

In Section 3, without resort to Strichartz estimates local and global existence results are treated for $m \ge 0$ through the contraction argument and the conservation laws above. This result is an extension of the work of Lenzmann [22] and [8] to fractional NLS. In particular, we show the global existence in the focusing mass critical case, that is, $\gamma = \alpha, \lambda = -1$, for the initial norm with $\|\varphi\|_{L^2} < \|Q\|_{L^2}/\|\psi\|_{L^\infty}^{\frac{1}{2}}$, where Q is the solution of $(-\Delta)^{\frac{\alpha}{2}}Q - (|x|^{-\gamma} * |Q|^2)Q = -Q$. We also show the solution norm can be estimated uniformly in terms of m in finite time, which enables us to consider two types of limiting problems ($m \to 0$ and $m \to \infty$). See Remark 1 below.

In Section 4, we consider the local and global existence via standard Strichartz estimates (2.1) and (2.2) below. The advantage of Strichartz estimate is to give a chance for existence results of lower regularity than ones without using Strichartz estimates. However, owing to the regularity loss of Strichartz estimates, it is hard to handle the critical problem. On the other hand, such estimates enable us to get a small data global existence results and scattering for the case $2\alpha < \gamma < n$.

In Section 5, we treat the critical problem. To handle the critical regularity one needs Strichartz estimate without regularity loss as Schrödinger case. Recently, such estimates have been developed independently in [16] and [6], when radial symmetry or angular regularity is assumed. See (2.3) and (2.4) below. Using these, we show the global existence of radial solutions in $H^{\frac{\gamma-\alpha}{2}}$ for suitable γ and α. In [16], the authors considered the equation with $m = 0$ and power type nonlinearity.

Section 6 is devoted to the global existence of small data in critical solution space below L^2, that is $\dot{H}^{\frac{\gamma-\alpha}{2}}, \gamma < \alpha$ without radial symmetry. For this we use weighted Strichartz estimates (2.5) and (2.6) in the same way as in [5]. When $m > 0$, we could not control the homogeneous $\dot{H}^s$ norm by the weighted Strichartz estimates. Thus we only consider the case $m = 0$. It would be so interesting to show the global existence when $m > 0$. For the simplicity of presentation we try 3-d case in Section 6. We leave the general case to the readers.

In the last section, we study a finite time blowup for the focusing case. For this we consider a massive mass critical Hartree nonlinearity given by the mass $m > 0$ and the potential $-\psi(x)/|x|^\alpha$ where $\psi' \le 0$ and $|\psi'| \lesssim \frac{1}{\rho}$, and a initial data with $E(\varphi) < 0$. Then by adapting the Virial argument of [15] and [4] we show the nonnegative quantity $\langle u, x \cdot D_m^{2-\alpha} xu\rangle$ is estimated as follows: for any $m \ge 0$ and $t \in [0, T^*)$

$$\langle u, x \cdot D_m^{2-\alpha} xu\rangle \le 2\alpha^2 E(\varphi)t^2 + 2\alpha(\langle \varphi, A\varphi\rangle + C\|\varphi\|_{L^2}^4)t + \langle \varphi, M\varphi\rangle. \quad (1.4)$$

Since $E(\varphi) < 0$, the maximal existence time T^* of solution should be finite. In [4], the authors considered massless case and they obtained finite time blowup for mass critical equations. We extended their results to massive case and show that the constant C in (1.4) does not depend on $m > 0$. For the proof of (1.4) we show L^2 operator norm of the commutator $[D_m^\alpha, |x|^2 K_\alpha(|u|^2)]$ is bounded by $\|\varphi\|_{L^2}^4$ for which we need to assume that radial symmetry of solution. We also establish some propagation estimates of moment at the end of Section 7.

Now we close this section by introducing some notations. The mixed norm $\|F\|_{L^qX}$ means $(\int_{\mathbb{R}} \|F(t,\cdot)\|_X^q\, dt)^{\frac{1}{q}}$. We will use the notations $|\nabla| = \sqrt{-\Delta}$, $\dot{H}_r^s = |\nabla|^{-s}L^r$ $(\dot{H}_2^s = \dot{H}^s)$ and $H_r^s = (1-\Delta)^{-s/2}L^r$ $(H^s = H_2^s)$. Hereafter, we denote the space $L_T^q(B)$ by $L^q(0,T;B)$ and its norm by $\|\cdot\|_{L_T^qB}$ for some Banach space B, and also $L^q(B)$ with norm $\|\cdot\|_{L^qB}$ by $L^q(0,\infty;B)$,

$1 \le q \le \infty$. If not specified, throughout this paper, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \le CB$ and $A \ge C^{-1}B$, respectively. Different positive constants possibly depending on n, α and γ might be denoted by the same letter C. $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

2 Strichartz estimates

In this paper we will treat three versions of Strichartz estimates. We first consider the standard Strichartz estimate for the unitary group $U(t)$ (see [10]):

$$\|U(t)\varphi\|_{L_T^{q_1}L^{r_1}} \le Cc_\alpha^{\frac12-\frac1{r_1}}\|D_m^{\frac{n(2-\alpha)}{2}\left(\frac12-\frac1{r_1}\right)}\varphi\|_{L^2}, \tag{2.1}$$

$$\|\int_0^t U(t-t')F(t')\,dt'\|_{L_T^{q_1}L^{r_1}} \le Cc_\alpha^{1-\frac1{r_1}-\frac1{r_2}}\|\mathcal{D}_m^{\frac{n(2-\alpha)}{2}\left(1-\frac1{r_1}-\frac1{r_2}\right)}F\|_{L_T^{q_2'}L^{r_2'}}, \tag{2.2}$$

where $c_\alpha = (\alpha-1)^{-1}$ and the constant C does not depend on m. These estimates hold for $n \ge 1$ and the pairs $(q_i, r_i), i = 1, 2$ satisfying that $2 \le q_i, r_i \le \infty$, $\frac{2}{q_i} + \frac{n}{r_i} = \frac{n}{2}$ and $(q_i, r_i) \ne (2, \infty)$. The constant c_α shows the sharpness of the estimates near $\alpha = 1$. We will use the estimates (2.1) and (2.2) for the existence of H^s solutions for some $s < \frac{\gamma}{2}$ in Section 5.

Next we will use the recently developed radial Strichartz estimates [6, 16] as follows: for radial functions φ and F

$$\|U(t)\varphi\|_{L_T^{q_1}L^{r_1}} \le Cc_\alpha^{\frac12-\frac1{r_1}}\|D_m^\theta\varphi\|_{L^2}, \tag{2.3}$$

$$\|\int_0^t U(t-t')F(t')\,dt'\|_{L_T^{q_1}L^{r_1}} \le Cc_\alpha^{1-\frac1{r_1}-\frac1{r_2}}\|F\|_{L_T^{q_2'}L^{r_2'}}, \tag{2.4}$$

where C does not depend on m. Here $\theta \in \mathbb{R}$ and $n \ge 2$. The pairs $(q_i, r_i), i = 1, 2$, satisfy the range conditions $2 \le q_i, r_i \le \infty, q_2 \ne 2$,

$$\frac{n}{2}\left(\frac12 - \frac1{r_i}\right) \le \frac1{q_i} \le \frac{2n-1}{2}\left(\frac12 - \frac1{r_i}\right),$$

$$(n, q_i, r_i) \neq (2, 2, \infty), \quad (q_i, r_i) \neq (2, \frac{2(2n-1)}{2n-3}),$$

and the gap condition

$$\frac{\alpha}{q_1} + \frac{n}{r_1} = \frac{n}{2} - \theta, \qquad \frac{\alpha}{q_2} + \frac{n}{r_2} = \frac{n}{2} + \theta.$$

These will be used for global well-posedness of radial solution with critical regularity in Section 6.

Finally to treat the well-posedness in the case of below L^2 we will use the weighted Strichartz estimates:

(1) Let $0 < a < \frac{n-1}{2}$ and $\beta_1 \leq \frac{n-1}{2} - a$. Then we have

$$\| |x|^a |\nabla|^{a-\frac{n}{2}} d_\omega^{\beta_1} U(t)\varphi \|_{L_t^\infty L_r^\infty L_\omega^2} \leq C \|\varphi\|_{L_x^2}. \tag{2.5}$$

(2) Let $-\frac{n}{2} < b < -\frac{1}{2}$ and $\beta_2 \leq -\frac{1}{2} - b$. Then we have

$$\| |x|^b |\nabla|^{1+b} D_m^{-\frac{2-\alpha}{2}} d_\omega^{\beta_2} U(t)\varphi \|_{L_{t,x}^2} \leq C \|\varphi\|_{L_x^2}. \tag{2.6}$$

Here $d_\omega = \sqrt{1 - \Delta_\omega}$, Δ_ω is the Laplace-Beltrami operator on the unit sphere S^{n-1} and C does not depend on m. We have used the notation $\|f\|_{L_r^{r_1} L_\omega^{r_2}} = (\int_0^\infty (\int_{S^{n-1}} |f(\rho\omega)|^{r_2}\, d\omega)^{\frac{r_1}{r_2}} \rho^{n-1}\, d\rho)^{\frac{1}{r_1}}$. For the part(1) see [8] and [2]. For (2.6) we refer to [2] and also to [11, 7] for earlier and more general versions, respectively.

3 Existence I

In this section, we study the local and global existence without resort to Strichartz estimates.

Let us first introduce the following local existence result.

Proposition 3.1. *Let $m \geq 0$, $0 < \gamma < n$ and $n \geq 1$. Suppose $\varphi \in H^s(\mathbb{R}^n)$ with $s \geq \frac{\gamma}{2}$. Then there exists a positive time T such that* (1.2) *has a unique solution $u \in C([0,T]; H^s)$ with $\|u\|_{L_T^\infty H^s} \leq C\|\varphi\|_{H^s}$, where C does not depend on $m \geq 0$.*

Proof. Let $(X(T,\rho), d)$ be a complete metric space with metric d defined by

$$X(T,\rho) = \{u \in L_T^\infty(H^s(\mathbb{R}^n)) : \ \|u\|_{L_T^\infty H^s} \le \rho\}, \ \ d_X(u,v) = \|u-v\|_{L_T^\infty L^2}.$$

Now we define a mapping $\mathcal{N} : u \mapsto \mathcal{N}(u)$ on $X(T,\rho)$ by

$$\mathcal{N}(u)(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')\,dt'. \tag{3.1}$$

Our strategy is to use the standard contraction mapping argument. To do so, let us introduce a generalized Leibniz rule (see Lemma A1 $\sim$ Lemma A4 in Appendix of [18]).

Lemma 3.2. *For any* $s \ge 0$ *we have*

$$\||\nabla|^s(uv)\|_{L^r} \lesssim \||\nabla|^s u\|_{L^{r_1}}\|v\|_{L^{q_2}} + \|u\|_{L^{q_1}}\||\nabla|^s v\|_{L^{r_2}},$$

where $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{r_2}, \quad r_i \in (1,\infty),\ q_i \in (1,\infty], \quad i = 1,2.$

Then for $u \in X(T,\rho)$ and $s \ge \frac{\gamma}{2}$ we have

$$\begin{aligned}
\|\mathcal{N}(u)\|_{L_T^\infty H^s} &\le \|\varphi\|_{H^s} + T\|F(u)\|_{L_T^\infty H^s} \\
&\lesssim \|\varphi\|_{H^s} + T\Big(\|K_\gamma(|u|^2)\|_{L_T^\infty L^\infty}\|u\|_{L_T^\infty H^s} \\
&\qquad\qquad + \|K_\gamma(|u|^2)\|_{L_T^\infty H^s_{\frac{2n}{\gamma}}}\|u\|_{L_T^\infty L^{\frac{2n}{n-\gamma}}}\Big) \\
&\lesssim \|\varphi\|_{H^s} + T\left(\|u\|^2_{L_T^\infty H^{\frac{\gamma}{2}}}\|u\|_{L_T^\infty H^s} + \|u\|^2_{L_T^\infty L^{\frac{2n}{n-\gamma}}}\|u\|_{L_T^\infty H^s}\right) \\
&\lesssim \|\varphi\|_{H^s} + T\|u\|^2_{L_T^\infty H^{\frac{\gamma}{2}}}\|u\|_{L_T^\infty H^s} \lesssim \|\varphi\|_{H^s} + T\rho^3.
\end{aligned} \tag{3.2}$$

Here we have used the trivial inequality

$$K_\gamma(v) = \int_{\mathbb{R}^n}\frac{\psi(x-y)}{|x-y|^\gamma}v(y)\,dy \le \|\psi\|_{L^\infty}\int_{\mathbb{R}^n}|x-y|^{-\gamma}v(y)\,dy$$

for $v \geq 0$, the Hardy-Littlewood-Sobolev inequality, Lemma 3.2, the Hardy type inequality

$$\sup_{x\in\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\frac{|u(x-y)|^2}{|y|^\gamma}\,dy\right| \lesssim \|u\|^2_{\dot{H}^{\frac{\gamma}{2}}}, \tag{3.3}$$

and we used the Sobolev embedding $H^{\frac{\gamma}{2}} \hookrightarrow L^{\frac{2n}{n-\gamma}}$.

If we choose ρ and T such as $\|\varphi\|_{H^s} \leq \rho/2$ and $CT\rho^3 \leq \rho/2$, then $\mathcal{N}$ maps $X(T,\rho)$ to itself.

Now we show that $\mathcal{N}$ is a Lipschitz map for sufficiently small T. Let $u, v \in X(T,\rho)$. Then we have

$$\begin{aligned}
&d_X(\mathcal{N}(u),\mathcal{N}(v))\\
&\qquad\lesssim T\left\|K_\gamma(|u|^2)u - K_\gamma(|v|^2)v\right\|_{L_T^\infty L^2}\\
&\qquad\lesssim T\left(\left\|K_\gamma(|u|^2)(u-v)\right\|_{L_T^\infty L^2} + \left\|K_\gamma(|u|^2-|v|^2)v\right\|_{L_T^\infty L^2}\right)\\
&\qquad\lesssim T\left(\|u\|^2_{L_T^\infty H^{\frac{\gamma}{2}}} d(u,v) + \|K_\gamma(|u|^2-|v|^2)\|_{L_T^\infty L^{\frac{2n}{\gamma}}}\|v\|_{L_T^\infty L^{\frac{2n}{n-\gamma}}}\right)\\
&\qquad\lesssim T(\rho^2 d(u,v) + \rho\||u|^2-|v|^2\|_{L_T^\infty L^{\frac{2n}{2n-\gamma}}})\\
&\qquad\lesssim T(\rho^2 + \rho(\|u\|_{L_T^\infty L^{\frac{2n}{n-\gamma}}} + \|v\|_{L_T^\infty L^{\frac{2n}{n-\gamma}}}))d_X(u,v)\\
&\qquad\lesssim T\rho^2 d_X(u,v).
\end{aligned}$$

The above estimate implies that the mapping $\mathcal{N}$ is a contraction, if T is sufficiently small.

The uniqueness and time continuity follows easily from the equation (1.2) and a similar contraction argument. This completes the proof of Proposition 3.1. □

From the conservation laws (1.3), we get the following global well-posedness.

Theorem 3.3. *Let $m \geq 0$, $0 < \gamma \leq \alpha$ for $n \geq 2$, $0 < \gamma < 1$ for $n = 1$, and $s \geq \frac{\gamma}{2}$. Let T^* be the maximal existence time of the solution u as in*

Proposition 3.1. *Then if $\lambda = +1$, or if $\lambda = -1$ and $\|\varphi\|_{L^2}$ is sufficiently small, then $T^* = \infty$. Moreover $\|u(t)\|_{H^s} \le C\|\varphi\|_{H^s} e^{C(|E(\varphi)|+\|\varphi\|_{L^2}^2)t}$, where C does not depend on $m \ge 0$.*

Proof. From the estimate (3.3) and L^2 conservation, we have

$$|V(u)| \lesssim \|u\|_{\dot H^{\frac{\gamma}{2}}}^2 \|u\|_{L^2}^2. \tag{3.4}$$

Thus if $\lambda = +1$ or if $\lambda = -1$ and $\|\varphi\|_{L^2}$ is sufficiently small, then since $\gamma \le \alpha$

$$\|u(t)\|_{\dot H^{\frac{\gamma}{2}}}^2 \le C(|E(u)| + \|\varphi\|_{L^2}^2) = C(|E(\varphi)| + \|\varphi\|_{L^2}^2). \tag{3.5}$$

From (3.5) and a similar estimate to (3.2), we have

$$\begin{aligned} \|u(t)\|_{H^s} &\lesssim \|\varphi\|_{H^s} + \int_0^t \|u\|_{H^{\frac{\gamma}{2}}}^2 \|u\|_{H^s}\, dt' \\ &\lesssim \|\varphi\|_{H^s} + (|E(\varphi)| + \|\varphi\|_{L^2}^2) \int_0^t \|u\|_{H^s}\, dt'. \end{aligned} \tag{3.6}$$

Gronwall's inequality shows that

$$\|u(t)\|_{H^s} \le C\|\varphi\|_{H^s} \exp(C(|E(\varphi)| + \|\varphi\|_{L^2}^2)t).$$

This completes the proof of Theorem 3.3. □

If $\psi = 1$, $m \ge 0$, $\gamma = \alpha$ and $\lambda = -1$, then (1.1) is L^2-critical focusing FNLS and more precise statement is possible for global existence. In fact, (1.1) has a ground state Q in $H^{\frac{\alpha}{2}}$ (see Theorem 1.8 of [17]), which satisfies

$$(-\Delta)^{\frac{\alpha}{2}} Q - (|x|^{-\gamma} * |Q|^2) Q = -Q$$

and is a decreasing minimizer of the problem

$$2\|Q\|_{L^2}^2 = \inf_{u \in H^{\frac{\alpha}{2}} \setminus \{0\}} \frac{\|u\|_{\dot H^{\frac{\alpha}{2}}}^2 \|u\|_{L^2}^2}{|V_1(u)|}, \tag{3.7}$$

where $V_1(u) = -\frac14 \iint |x-y|^{-\gamma} |u(x)|^2 |u(y)|^2\, dxdy$. Then we have the following.

Theorem 3.4. *Let $m \geq 0$, $\gamma = \alpha$, $n \geq 2$ and $s \geq \frac{\gamma}{2}$. Suppose T^* be the maximal existence time of the solution u as in Theorem* 3.1. *Then if $\lambda = -1$ and $\|\varphi\|_{L^2} < \|Q\|_{L^2}/\|\psi\|_{L^\infty}^{\frac{1}{2}}$, then $T^* = \infty$.*

Proof. From (3.7) we estimate $E(u)$ as follows.

$$\begin{aligned}
E(\varphi) = E(u) &= \frac{1}{2}\|D_m^{\frac{\alpha}{2}} u\|_{L^2}^2 - |V(u)| \\
&\geq \frac{1}{2}\||\nabla|^{\frac{\alpha}{2}} u\|_{L^2}^2 - \frac{\|\psi\|_{L^\infty}}{2\|Q\|_{L^2}^2}\||\nabla|^{\frac{\alpha}{2}} u\|_{L^2}^2 \|u\|_{L^2}^2 \\
&= \frac{1}{2}\left(1 - \frac{\|\psi\|_{L^\infty}\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)\||\nabla|^{\frac{\alpha}{2}} u\|_{L^2}^2 \\
&= \frac{1}{2}\left(1 - \frac{\|\psi\|_{L^\infty}\|\varphi\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)\||\nabla|^{\frac{\alpha}{2}} u\|_{L^2}^2.
\end{aligned}$$

Thus $\|u\|_{H^{\frac{\alpha}{2}}}^2 \lesssim E(\varphi) + \|\varphi\|_{L^2}^2$, provided $\|\varphi\|_{L^2} < \|Q\|_{L^2}/\|\psi\|_{L^\infty}^{\frac{1}{2}}$. In the same way as in the proof of Theorem 3.3 we conclude the global existence. That is $T^* = \infty$. ∎

Remark 1. *Let $T_{fnls}^* = \inf_{m \geq 0} T^*$, where T^* is the maximal existence time of local solution u_m in Proposition 3.1. Then from the uniform estimate of solution in H^s norm with respect to $m \geq 0$ it follows that $T_{fnls}^* > 0$. This gives two types of limit problems as Propositions 2.4 and 2.5 of [8]. For each $m > 0$ let $u_m \in C([0, T_{fnls}^*); H^s)$ be the solution of* (1.1) *for $s \geq \frac{\gamma}{2}$ and u_0 be the H^s solution to the Cauchy problem:*

$$i\partial_t u_0 = (-\Delta)^{\frac{\alpha}{2}} u_0 + F(u_0), \quad u_0(0) = \varphi.$$

Then it immediately follows that for any $T < T_{fnls}^*$ $u_m \to u_0$ in $C([0, T]; H^s)$ as $m \to 0$.

On the other hand, let $v_m = e^{itm} u_m$, the phase modulation of the solution u_m to (1.1). *Then v_m is the solution in $C([0, T_{fnls}^*); H^s)$ to the problem*

$$i\partial_t v_m = ((m^2 - \Delta)^{\frac{\alpha}{2}} - m^\alpha) v_m + F(v_m), \quad v_m(0) = \varphi,$$

and if w_m be the solution in $C([0,T^*_{nls});H^s)$ to

$$i\partial_t w_m = -\frac{\alpha}{2m^{2-\alpha}}\Delta w_m + F(w_m), \quad w_m(0) = \varphi.$$

Here T^*_{nls} is the infimum of maximal existence time of w_m with respect to m and the uniform estimate of w_m similar to u_m implies that $T^*_{nls} > 0$. Then by the same argument as in the proof of Proposition 2.5 of [8] one can also show that $\|v_m - w_m\|_{L^\infty(0,T;H^s)} \to 0$ as $m \to \infty$ for any $T < \min(T^*_{fnls}, T^*_{nls})$.

4 Existence II: via Strichartz estimates

In this section, we show the existence results with slightly lower regularity than the previous by using Strichartz estimates (2.1) and (2.2). The following is on the local existence.

Proposition 4.1. *Let $n \ge 1$, $m \ge 0$ and $s > \frac{\gamma}{2} - \min(\gamma,2)\frac{\alpha}{4}$ for $1 < \alpha < 2$ and $0 < \gamma < n$. If $\varphi \in H^s$ then there exists a positive time T such that (1.2) has a unique solution $u \in C([0,T];H^s) \cap L^q_T(H^{s-\sigma}_r)$, where $q = \frac{4}{\delta}$, $r = \frac{2n}{n-\delta}$ and $\sigma = \frac{\delta(2-\alpha)}{4}$ for some δ with $0 < \delta < \min(\gamma,2)$ and $s > \frac{\gamma}{2} - \frac{\delta\alpha}{4}$.*

Proof. Given n, α, γ and s, choose a number δ with $0 < \alpha < \min(\gamma,2)$ and $s > \frac{\gamma}{2} - \frac{\delta\alpha}{4}$. Then for some positive number T to be chosen later, let us define a complete metric space $(Y(T,\rho), d_Y)$ with metric d_Y by

$$\begin{aligned}
&Y(T,\rho) = \left\{ v \in L^\infty_T(H^s) \cap L^q_T(H^{s-\sigma}_r) : \|v\|_{L^\infty_T H^s} + \|v\|_{L^q_T H^{s-\sigma}_r} \le \rho \right\}, \\
&d_Y(u,v) = \|u - v\|_{L^\infty_T H^s \cap L^q_T H^{s-\sigma}_r},
\end{aligned}$$

where q, r, σ are the same indices as in Proposition 4.1.

We will show that the mapping $\mathcal{N}$ defined by (3.1) is a contraction on $Y(T,\rho)$, provided T is sufficiently small. For this purpose we introduce a useful lemma.

Lemma 4.2 (Lemma 3.2 of [8])**.** *Let* $0 < \gamma < n$*. Then for any* $0 < \varepsilon < n - \gamma$ *we have*

$$\left\| K_\gamma(|u|^2) \right\|_{L^\infty} \lesssim \|u\|_{L^{\frac{2n}{n-\gamma-\varepsilon}}} \|u\|_{L^{\frac{2n}{n-\gamma+\varepsilon}}}.$$

If we use the Strichartz estimates (2.1) and (2.2) with the pair

$$(q_1, r_1, q_2, r_2) = \left(q = \frac{4}{\delta}, r = \frac{2n}{n-\delta}, \infty, 2 \right)$$

together with Plancherel theorem, Lemma 4.2, and generalized Leibniz rules (Lemma 3.2), then since $\sigma = \frac{\delta(2-\alpha)}{4}$ we have

$$\begin{aligned}
&\|\mathcal{N}(u)\|_{L_T^\infty H^s \cap L_T^q H_r^{s-\sigma}} \\
&\quad \lesssim \|\varphi\|_{H^s} + \|D_m^{\frac{2-\alpha}{2} n(\frac{1}{2}-\frac{1}{r})} F(u)\|_{L_T^1 H^{s-\sigma}} \\
&\quad \lesssim \|\varphi\|_{H^s} + \|K_\gamma(|u|^2)\|_{L_T^1 L^\infty} \|u\|_{L_T^\infty H^s} \\
&\qquad\qquad + \int_0^T \|K_\gamma(|u|^2)\|_{H^s_{\frac{2n}{\gamma+\varepsilon}}} \|u\|_{L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \, dt \\
&\quad \lesssim \|\varphi\|_{H^s} + \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma-\varepsilon)}}} \|u\|_{L_T^\infty H^s} \\
&\qquad\qquad + \int_0^T \||u|^2\|_{H^s_{\frac{2n}{2n-(\gamma-\varepsilon)}}} \|u\|_{L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \, dt \\
&\quad \lesssim \|\varphi\|_{H^s} + \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma-\varepsilon)}}} \|u\|_{L_T^\infty H^s}
\end{aligned} \tag{4.1}$$

for sufficiently small ε. Here the involved constant is uniform on m if $0 \le m \le m_0$.

Using Hölder's inequality for time integral, we have

$$\begin{aligned}
&\|\mathcal{N}(u)\|_{L_T^\infty H^s \cap L_T^q H_r^{s-\sigma}} \\
&\quad \lesssim \|\varphi\|_{H^s} + T^{1-\frac{2}{q}} \|u\|_{L_T^q L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \|u\|_{L_T^q L^{\frac{2n}{n-(\gamma-\varepsilon)}}} \|u\|_{L_T^\infty H^s}.
\end{aligned} \tag{4.2}$$

Now if we choose $\varepsilon > 0$ so small that $\varepsilon < \min(\gamma - \delta, 2(s-\sigma) - \gamma)$, then since

$$\frac{2n}{n-\delta} \le \frac{2n}{n-(\gamma-\varepsilon)} < \frac{2n}{n-(\gamma+\varepsilon)} \le \frac{2n}{n-\delta-2(s-\sigma)},$$

we have from (4.2) and Sobolev embedding $H_r^{s-\sigma} \hookrightarrow L^r \cap L^{\frac{2n}{n-\delta-2(s-\sigma)}}$ that

$$\begin{aligned}\|\mathcal{N}(u)\|_{L_T^\infty H^s \cap L_T^q H_r^{s-\sigma}} &\le C(\|\varphi\|_{H^s} + T^{1-\frac{2}{q}} \|u\|_{L_T^\infty H^s} \|u\|_{L_T^q H_r^{s-\sigma}}^2) \\ &\le C(\|\varphi\|_{H^s} + T^{1-\frac{2}{q}} \rho^3)\end{aligned}$$

for some constant C. Here we used the conventional embedding that if $2(s-\sigma) \ge n-\delta$ then $H_r^{s-\sigma} \hookrightarrow L^{r_1}$ for any $r_1 \ge r$. Thus if we choose ρ and T so that $C\|\varphi\|_{H^s} \le \frac{\rho}{2}$ and $CT^{1-\frac{2}{q}}\rho^3 \le \frac{\rho}{2}$, then we conclude that $\mathcal{N}$ maps from $Y(T,\rho)$ to itself.

For any $u, v \in Y(T,\rho)$, we have

$$\begin{aligned} &d_Y(\mathcal{N}(u), \mathcal{N}(v)) \\ &\qquad \lesssim \|F(u) - F(v)\|_{L_T^1 H^s} \\ &\qquad \lesssim \|K_\gamma(|u|^2 - |v|^2)u\|_{L_T^1 H^s} + \|K_\gamma(|v|^2)(u-v)\|_{L_T^1 H^s}.\end{aligned} \tag{4.3}$$

By Lemma 4.2 and Hölder's inequality, we have for sufficiently small $\varepsilon > 0$

$$\begin{aligned} &\|K_\gamma(|u|^2 - |v|^2)u\|_{L_T^1 H^s} \\ &\quad \lesssim \|K_\gamma(|u|^2 - |v|^2)\|_{L_T^2 L^\infty} \|u\|_{L_T^\infty H^s} \\ &\qquad\qquad + \|K_\gamma(|u|^2 - |v|^2)\|_{L_T^2 H^s_{\frac{2n}{\gamma+\varepsilon}}} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \\ &\quad \lesssim \rho \||u|^2 - |v|^2\|_{L_T^1 L^{\frac{n}{n-(\gamma+\varepsilon)}}}^{\frac{1}{2}} \||u|^2 - |v|^2\|_{L_T^1 L^{\frac{n}{n-(\gamma-\varepsilon)}}}^{\frac{1}{2}} \\ &\qquad + \rho\|u - v\|_{L_T^\infty H^s} (\|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma-\varepsilon)}}} + \|v\|_{L_T^2 L^{\frac{2n}{n-(\gamma-\varepsilon)}}}) \\ &\qquad + \rho\|u - v\|_{L_T^2 L^{\frac{2n}{n-(\gamma-\varepsilon)}}} (\|u\|_{L_T^\infty H^s} + \|v\|_{L_T^\infty H^s}).\end{aligned} \tag{4.4}$$

Now by another Hölder's inequality with respect to the time variable, we have

$$\|K_\gamma(|u|^2 - |v|^2)u\|_{L_T^1 H^s} \lesssim T^{1-\frac{2}{q}} \rho^2 d_Y(u,v).$$

Similarly,

$$\begin{aligned}
&\|K_\gamma(|v|^2)(u-v)\|_{L^1_T H^s} \\
&\qquad \lesssim \|K_\gamma(|v|^2)\|_{L^1_T L^\infty}\|u-v\|_{L^\infty_T H^s} \\
&\qquad\qquad + \|K_\gamma(|v|^2)\|_{L^2_T H^s_{\frac{2n}{\gamma+\varepsilon}}}\|u-v\|_{L^2_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \\
&\qquad \lesssim \|v\|_{L^2_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}}\|v\|_{L^2_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}} d_T(u,v) \\
&\qquad\qquad + \|v\|_{L^\infty_T H^s}\|v\|_{L^2_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}}\|u-v\|_{L^2_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}}.
\end{aligned} \tag{4.5}$$

Thus we get

$$\|K_\gamma(|v|^2)(u-v)\|_{L^1_T H^s} \lesssim T^{1-\frac{2}{q}}\rho^2 d_Y(u,v).$$

Substituting these two estimates into (4.3) and then using the fact $CT^{1-\frac{2}{q}}\rho^2 \le \frac{1}{2}$ for small T, we conclude that $\mathcal{N}$ is a contraction mapping. □

Now we show the local solutions can be extended globally in time by using the energy conservation law. We first consider defocusing case.

Theorem 4.3. *Let $m \ge 0$, $0 < \gamma < \min(2\alpha, n)$, $n \ge 1$. If $\lambda = +1$, then for any $\varphi \in H^{\frac{\alpha}{2}}$, then (1.2) has a unique solution $u \in C([0,\infty); H^{\frac{\alpha}{2}}) \cap L^q_{loc}(H_r^{\frac{\alpha}{2}-\sigma})$, where $q = \frac{4}{\delta}$, $r = \frac{2n}{n-\delta}$ and $\sigma = \frac{\delta(2-\alpha)}{4}$ for some δ with $0 < \delta < \min(\gamma, 2)$ and $\frac{\alpha}{2} > \frac{\gamma}{2} - \frac{\delta\alpha}{4}$.*

Proof. Let T^* be the maximal existence time. We will prove that T^* is infinite by contradiction. Suppose that $T^* < \infty$. Then the local theory shows that $\|u\|_{L^q_{T^*} H_r^{\frac{\alpha}{2}-\sigma}} = \infty$. Since $\gamma < 2\alpha$, from the local existence Proposition 4.1, we see that the energy conservation law (1.3) holds. Thus if $\lambda = +1$, then at any $t < T^*$, the solution u satisfies that

$$\frac{1}{2}\|u(t)\|^2_{H^{\frac{\alpha}{2}}} \le \frac{1}{2}\|u(t)\|^2_{L^2} + E(u) = \frac{1}{2}\|\varphi\|^2_{L^2} + E(\varphi).$$

From the estimate (4.2) which is used with $s = \frac{\alpha}{2}$, we have

$$\|u\|_{L^q_T H_r^{\frac{\alpha}{2}-\sigma}} \lesssim \|\varphi\|^2_{L^2} + E(\varphi) + T^{1-\frac{2}{q}}(\|\varphi\|^2_{L^2} + E(\varphi))^{\frac{1}{2}}\|u\|^2_{L^q_T H_r^{\frac{\alpha}{2}-\sigma}}.$$

Thus for sufficiently small T depending on $\|\varphi\|_{L^2}^2 + E(\varphi)$,

$$\|u\|_{L^q(T_{j-1},T_j;H_r^{\frac{\alpha}{2}-\sigma})} \le C(\|\varphi\|_{L^2}^2 + E(\varphi)),$$

where $T_j - T_{j-1} = T$ for $j \le k-1$ and $T_k = T^*$ This means that

$$\|u\|_{L^q(0,T^*;H_r^{\frac{\alpha}{2}-\sigma})}^q \le \sum_{1\le j\le k} \|u\|_{L^q(T_{j-1},T_j;H_r^{\frac{1}{2}-\sigma})}^q \le (kC(\|\varphi\|_{L^2}^2 + E(\varphi)))^q < \infty.$$

This is the contradiction to the hypothesis $T^* < \infty$. This completes the proof of Theorem 4.3. ☐

To treat the focusing problem we need more elaboration. Let us first observe that for any $f \in H^{\frac{\alpha}{2}}$

$$|V(f)| \le \|\psi\|_{L^\infty} \||x|^{-\gamma} * |f|^2\|_{L^{\frac{r}{r-2}}} \|f\|_{L^2}^2 \le \|\psi\|_{L^\infty} \|f\|_{L^{\widetilde{r}}}^2 \|f\|_{L^r}^2,$$

where $\frac{1}{\widetilde{r}} = 1 - \frac{1}{r} - \frac{\gamma}{2n}$. If $\alpha < \gamma < 2\alpha$ and $2 < r < \frac{2n}{n-\alpha}$, then $2 < \widetilde{r} < \frac{2n}{n-\alpha}$. Thus from Sobolev embedding it follows that

$$|V(f)| \lesssim \|\psi\|_{L^\infty} \|f\|_{L^2}^{2(2-\frac{\gamma}{\alpha})} \|f\|_{\dot H^{\frac{\alpha}{2}}}^{\frac{2\gamma}{\alpha}}. \tag{4.6}$$

From (4.6) we can treat a variational problem. Let us invoke from [9] that the embedding $H_{rad}^{\frac{\alpha}{2}} \hookrightarrow L^r$ is compact if $n \ge 2$, $1 < \alpha < 2$ and $2 < r < \frac{2n}{n-\alpha}$. Here $H_{rad}^{\frac{\alpha}{2}}$ is the Sobolev space $H^{\frac{\alpha}{2}}$ of radial functions. From this one can easily get the existence of nontrivial radial solution in $H^{\frac{\alpha}{2}}$ to the problem

$$J = \sup_{u \in H^{\frac{\alpha}{2}} \setminus \{0\}} \frac{|V(u)|}{\|u\|_{\dot H^{\frac{\alpha}{2}}}^{\frac{2\gamma}{\alpha}} \|u\|_{L^2}^{2(2-\frac{\gamma}{\alpha})}}.$$

Now we consider the focusing case.

Theorem 4.4. *Let $m \ge 0$, $\lambda = -1$, $\alpha < \gamma < \min(2\alpha, n)$ and $n \ge 2$. If $\varphi \in H^{\frac{\alpha}{2}}$ and $\|\varphi\|_{\dot H^{\frac{\alpha}{2}}}$ is sufficiently small, (1.2) has a unique solution $u \in C([0,\infty); H^{\frac{\alpha}{2}}) \cap L_{loc}^q(H_r^{\frac{\alpha}{2}-\sigma})$, where $q = \frac{4}{\delta}$, $r = \frac{2n}{n-\delta}$ and $\sigma = \frac{\delta(2-\alpha)}{4}$ for some δ with $0 < \delta < \min(\gamma, 2)$ and $\frac{\alpha}{2} > \frac{\gamma}{2} - \frac{\delta\alpha}{4}$.*

Proof. From (4.6) we deduce that $|E(\varphi)| = O(\|\varphi\|^2_{\dot{H}^{\frac{\alpha}{2}}})$ as $\|\varphi\|_{\dot{H}^{\frac{\alpha}{2}}} \to 0$. Thus we have

$$E(\varphi) = E(u) \geq \frac{1}{2}\|u\|^2_{\dot{H}^{\frac{\alpha}{2}}} - J\|u\|_{L^2}^{2(2-\frac{\gamma}{\alpha})}\|u\|_{\dot{H}^{\frac{\alpha}{2}}}^{\frac{2\gamma}{\alpha}}.$$

By the continuity argument we see that for any φ with sufficiently small $\|\varphi\|_{\dot{H}^{\frac{\alpha}{2}}}$ such as

$$|E(\varphi)| < 4^{-\frac{\gamma}{\gamma-\alpha}}\left(J\|\varphi\|_{L^2}^{2(2-\frac{\gamma}{\alpha})}\right)^{-\frac{\alpha}{\gamma-\alpha}},$$

the corresponding solution u satisfies the estimate

$$\|u\|^2_{\dot{H}^{\frac{\alpha}{2}}} \leq 4|E(\varphi)|.$$

Then the conclusion follows in the same way as in the proof of Theorem 4.3. □

Now we consider the small data global existence and scattering for $2\alpha \leq \gamma < n$.

Theorem 4.5. *Let $m \geq 0$, $2\alpha \leq \gamma < n$, $n \geq 3$ and $s > \frac{\gamma}{2} - \frac{\alpha}{2}$. Then there exists $\rho > 0$ such that for any $\varphi \in H^s$ with $\|\varphi\|_{H^s} \leq \rho$, (1.2) has a unique solution $u \in C_b([0,\infty); H^s) \cap L^2(0,\infty; H^{s-\frac{2-\alpha}{2}}_{\frac{2n}{n-2}})$. Moreover there is $\varphi^+ \in H^s$ such that*

$$\|u(t) - U(t)\varphi^+\|_{H^s} \to 0 \ \text{ as } \ t \to \infty.$$

Proof. Let us define a complete metric space $(Y(\rho), d)$ with metric d_Y by

$$Y(\rho) = \left\{v \in Y \equiv C_b([0,\infty); H^s) \cap L^2(0,\infty; H^{s-\frac{2-\alpha}{2}}_{\frac{2n}{n-2}}) : \|v\|_Y \leq \rho\right\},$$
$$d_Y(u,v) = \|u - v\|_Y.$$

Then from the estimate (4.2), we have

$$\|\mathcal{N}(u)\|_Y \leq C\|\varphi\|_{H^s} + C\|u\|^2_{L^2(0,\infty;H^{s-\frac{2-\alpha}{2}}_{\frac{2n}{n-2}})}\|u\|_{L^\infty(0,\infty;H^s)}.$$

If we choose sufficiently small ρ such that $C\|\varphi\|_{H^s} \le \frac{\rho}{2}$ and $C\rho^3 \le \frac{\rho}{2}$, then $\mathcal{N}$ maps $Y(\rho)$ to itself. Similarly, from (4.3)–(4.5), one can show that $d(\mathcal{N}(u), \mathcal{N}(v)) \le \frac{1}{2} d(u, v)$. This proves the existence part.

To prove the scattering, let us define a function φ^+ by

$$\varphi^+ = \varphi - i \int_0^\infty U(-t')F(u)(t')\, dt'.$$

Then since the solution u is in $Y(\rho)$, $\varphi^+ \in H^s$, and therefore

$$\begin{aligned}\|u(t) - u^+(t)\|_{H^s} &\lesssim \int_t^\infty \|F(u)\|_{H^s}\, dt' \\ &\lesssim \|u\|_{L^\infty(0,\infty;H^s)} \int_t^\infty \|u\|^2_{H^{s-\frac{2-\alpha}{2}}_{\frac{2n}{n-2}}}\, dt' \to 0 \ \text{ as } \ t \to \infty.\end{aligned}$$

□

5 Existence III: radial case

In this section we establish the global existence theory of radial solution of (1.1) without regularity loss. We denote the Banach space X of radial functions by X_{rad}. We always assume that $m \ge 0$ and ψ is radially symmetric.

5.1 Subcritical case

We first consider the mass-(and energy-)subcritical problems.

Theorem 5.1. (1) *Let* $\frac{2n}{2n-1} \le \alpha < 2$ *and* $0 < \gamma < \alpha$*. If* $\varphi \in L^2_{rad}$*, then there exists a unique solution* u *of* (1.1) *such that* $u \in C_b([0,\infty); L^2_{rad}) \cap L^{\frac{3\alpha}{\gamma}}_{loc}(0,\infty; L^{\frac{2n}{n-\frac{2\gamma}{3}}})$*.*
(2) *Let* $\frac{2n}{2n-1} \le \alpha < 2$ *and* $\alpha < \gamma < \min(2\alpha, n)$*. If* $\varphi \in H^{\frac{\alpha}{2}}_{rad}$ *(*$\|\varphi\|_{\dot H^{\frac{\alpha}{2}}}$ *is sufficiently small if* $\lambda = -1$*), then there exists a unique solution* u *of* (1.1) *such that* $u \in C_b([0,\infty); H^{\frac{\alpha}{2}}_{rad}) \cap L^{\frac{3\alpha}{\gamma-\alpha}}_{loc}(0,\infty; H^{\frac{\alpha}{2}}_{\frac{2n}{n-\frac{2(\gamma-\alpha)}{3}}})$*.*

Contrary to Theorems 4.3 and 4.4, the mass-critical case is treated in the part (1) and a better Strichartz norm is obtained in the energy-subcritical case, part (2).

Proof. **Case (1)**. Let us define a complete metric space $(Z(T,\rho), d_Z)$ with metric d_Z by

$$Z(T,\rho) = \left\{ v \in Z \equiv C_b([0,T]; L^2_{rad}) \cap L_T^{\frac{3\alpha}{\gamma}} L^{\frac{2n}{n-\frac{2\gamma}{3}}}) : \|v\|_Z \le \rho \right\},$$
$$d_Z(u,v) = \|u-v\|_Z.$$

For some T and ρ we will show that the mapping $\mathcal{N}$ is a contraction on $Z(T,\rho)$.

From (2.3) and (2.4) with $\theta = 0$ and $(q_1, r_1) = (\frac{3\alpha}{\gamma}, \frac{2n}{n-\frac{3\gamma}{2}})$, $(q_2, r_2) = (\infty, 2)$ (thus $1 - \frac{1}{r_2} = \frac{3}{r_1} - \frac{n-\gamma}{n}$) we have for any $u \in Z(T,\rho)$

$$\begin{aligned} \|\mathcal{N}(u)\|_Z &\lesssim \|\varphi\|_{L^2} + \|K_\gamma(|u|^2)u\|_{L_T^1 L^2} \lesssim \|\varphi\|_{L^2} + \|u\|^3_{L_T^3 L^{r_1}} \\ &\lesssim \|\varphi\|_{L^2} + T^{1-\frac{\gamma}{\alpha}} \|u\|^3_{L_T^{q_1} L^{r_1}} \lesssim \|\varphi\|_{L^2} + T^{1-\frac{\gamma}{\alpha}} \rho^3. \end{aligned}$$

The involved constant is uniform on m if $0 \le m \le m_0$. From the gap condition it follows that $\frac{2n}{2n-1} \le \alpha < 2$.

Similarly one can easily show that for any $u, v \in Z(T,\rho)$

$$d_Z(\mathcal{N}(u), \mathcal{N}(v)) \lesssim T^{1-\frac{\gamma}{\alpha}} \rho^2 d_Z(u,v).$$

For suitable ρ and T, $\mathcal{N}$ becomes a contraction mapping, which means there is a unique solution $u \in Z(T,\rho)$. Now by the L^2 conservation and time iteration scheme, u turns out to be a global solution of (1.1).

Case (2). In this case we define the metric space $(Z(T,\rho), d_Z)$ by

$$Z(T,\rho) = \left\{ v \in Z \equiv C_b([0,T]; H^{\frac{\alpha}{2}}_{rad}) \cap L_T^{\frac{3\alpha}{\gamma-\alpha}} H^{\frac{\alpha}{2}}_{\frac{2n}{n-\frac{2(\gamma-\alpha)}{3}}}) : \|v\|_Z \le \rho \right\},$$
$$d_Z(u,v) = \|u-v\|_Z.$$

As above we choose $\theta = 0$, $(q_1, r_1) = (\frac{3\alpha}{\gamma-\alpha}, \frac{2n}{n-\frac{2(\gamma-\alpha)}{3}})$ and $(q_2, r_2) = (\infty, 2)$. Then

$$1 - \frac{1}{r_2} = 2(\frac{1}{r_1} - \frac{\alpha}{2n}) - \frac{n-\gamma}{n} + \frac{1}{r_1}$$

and we have

$$\begin{aligned}\|\mathcal{N}(u)\|_Z &\lesssim \|\varphi\|_{H^{\frac{\alpha}{2}}} + \|K_\gamma(|u|^2)u\|_{L_T^1 H^{\frac{\alpha}{2}}} \\ &\lesssim \|\varphi\|_{L^2} + T^{2-\frac{\gamma}{\alpha}}\|u\|^3_{L_T^{q_1} H_{r_1}^{\frac{\alpha}{2}}} \\ &\lesssim \|\varphi\|_{L^2} + T^{2-\frac{\gamma}{\alpha}}\rho^3\end{aligned}$$

and

$$d_Z(\mathcal{N}(u), \mathcal{N}(v)) \lesssim \|K_\gamma(|u|^2)u - K_\gamma(|v|^2)v\|_{L_T^1 H^{\frac{\alpha}{2}}} \lesssim T^{2-\frac{\gamma}{\alpha}}\rho^2 d_Z(u,v).$$

We now have only to choose T, ρ for contraction of $\mathcal{N}$. This yields the local existence.

Using energy conservation and time iteration scheme for $\lambda = +1$ and smallness argument as in Theorem 4.4 for $\lambda = -1$, we get a unique global solution. This completes the proof of Theorem 5.1.

□

5.2 Critical case

Theorem 5.2. (1) *Let* $\frac{2n}{2n-1} \le \alpha < 2$ *and* $\alpha \le \gamma < n$. *If* $\varphi \in H_{rad}^{\frac{\gamma-\alpha}{2}}$ *and* $\|\varphi\|_{H^{\frac{\gamma-\alpha}{2}}}$ *is sufficiently small, then there exists a unique solution* u *of* (1.1) *such that* $u \in C_b([0,\infty); H_{rad}^{\frac{\gamma-\alpha}{2}}) \cap L^3(0,\infty; H^{\frac{\gamma-\alpha}{2}}_{\frac{2n}{n-\frac{2\alpha}{3}}})$.

(2) *Let* $\frac{2n}{2n-1} \le \alpha < 2$ *and* $\frac{\alpha}{3} \le \gamma < \alpha$. *If* $\varphi \in \dot{H}_{rad}^{\frac{\gamma-\alpha}{2}}$ *and* $\|\varphi\|_{\dot{H}^{\frac{\gamma-\alpha}{2}}}$ *is sufficiently small, then there exists a unique solution* u *of* (1.1) *such that* $u \in C_b([0,\infty); H_{rad}^{\frac{\gamma-\alpha}{2}}) \cap L^3(0,\infty; L^{\frac{2n}{n-(\gamma-\frac{\alpha}{3})}})$.

Proof. **Case (1)**. We define the metric space $(Z(\rho), d_Z)$ by

$$Z(\rho) = \left\{ v \in Z \equiv C_b([0,\infty); H_{rad}^{\frac{\gamma-\alpha}{2}}) \cap L^3(0,\infty; L^{\frac{2n}{n-(\gamma-\frac{\alpha}{3})}}) : \|v\|_Z \leq \rho \right\},$$
$$d_Z(u,v) = \|u-v\|_Z.$$

By the same way as the part (2) of Theorem 5.2 we choose $\theta = 0$, $(q_1, r_1) = (3, \frac{2n}{n-\frac{2\alpha}{3}})$ and $(q_2, r_2) = (\infty, 2)$ so that

$$1 - \frac{1}{r_2} = 2(\frac{1}{r_1} - \frac{\gamma-\alpha}{2n}) - \frac{n-\gamma}{n} + \frac{1}{r_1}.$$

Then we have

$$\begin{aligned}
\|\mathcal{N}(u)\|_Z &\lesssim \|\varphi\|_{H^{\frac{\gamma-\alpha}{2}}} + \|K_\gamma(|u|^2)u\|_{L_T^1 H^{\frac{\gamma-\alpha}{2}}} \\
&\lesssim \|\varphi\|_{H^{\frac{\gamma-\alpha}{2}}} + \|u\|^3_{L_T^{q_1} H_{r_1}^{\frac{\gamma-\alpha}{2}}} \\
&\lesssim \|\varphi\|_{L^2} + \rho^3
\end{aligned}$$

and also

$$d_Z(\mathcal{N}(u), \mathcal{N}(v)) \lesssim \rho^2 d_Z(u,v).$$

If $C\|\varphi\|_{H^{\frac{\gamma-\alpha}{2}}} \leq \frac{\rho}{2}$ and $C\rho^2 \leq \frac{1}{2}$, then $\mathcal{N}$ is a contraction.

Case (2). Take metric space $Z(\rho)$ as

$$Z(\rho) = \left\{ v \in Z \equiv C_b([0,\infty); \dot{H}_{rad}^{\frac{\gamma-\alpha}{2}}) \cap L^3(0,\infty; L^{\frac{2n}{n-(\gamma-\frac{\alpha}{3})}}) : \|v\|_Z \leq \rho \right\},$$
$$d_Z(u,v) = \|u-v\|_Z.$$

Then it follows from (2.3) and (2.4) with $\theta = \frac{\gamma-\alpha}{2}$, $(q_1, r_1) = (3, \frac{2n}{n-(\gamma-\frac{\alpha}{3})})$ and $(q_2, r_2) = (\infty, \frac{2n}{n-(\alpha-\gamma)})$ that for any $u \in Z(\rho)$

$$\|\mathcal{N}(u)\|_Z \lesssim \|\varphi\|_{\dot{H}^{\frac{\gamma-\alpha}{2}}} + \int_0^\infty \|K_\gamma(|u|^2)u\|_{\dot{H}^{\frac{\gamma-\alpha}{2}} \cap L^{\frac{2n}{n+\alpha-\gamma}}}\, dt.$$

Since $\|\psi\|_{\dot{H}^{\frac{\gamma-\alpha}{2}}} \lesssim \|\psi\|_{L^{r_2'}}$ and $\frac{1}{r_2'} = \frac{2}{r_1} - \frac{n-\gamma}{n} + \frac{1}{r_1}$,

$$\|\mathcal{N}(u)\|_Z \lesssim \|\varphi\|_{\dot{H}^{\frac{\gamma-\alpha}{2}}} + \rho^3$$

and for any $u, v \in Z(\rho)$

$$d_Z(\mathcal{N}(u), \mathcal{N}(v)) \lesssim \rho^2 d_Z(u, v).$$

Taking small $\|\varphi\|_{\dot{H}^{\frac{\gamma-\alpha}{2}}}$ and ρ completes the proof of (2) of Theorem 5.2. ☐

6 Existence IV: via weighted Strichartz estimates

In this section we show the global well-posedness below L^2. To avoid complexity we only consider the case $m = 0$ and $n = 3$. We utilize the weighted Strichartz estimates (2.5) and (2.6) and have the following.

Theorem 6.1. *Let $\psi \in L^\infty_{rad}$ and $m = 0$. Suppose that $n = 3$, $\frac{21+\sqrt{21}}{15} < \alpha \le 2$ and $\frac{15\alpha-\alpha^2}{12+2\alpha} < \gamma < \alpha$. Then there exists a positive constant ρ depending on n, α, γ and λ such that if $\varphi \in \dot{H}^{s_c} H^{s_1+s_2}_\omega$ and $\||\nabla|^{s_c} d^{s_1+s_2}_\omega \varphi\|_{L^2_x} < \rho$, then the integral equation (1.2) has a unique solution $u \in C_b([0,\infty); \dot{H}^{s_c} H^{s_1+s_2}_\omega)$, where $s_1 = \frac{2}{q} - \frac{\gamma+1-\alpha}{2}$ and s_2 satisfies that*

$$\max\left(\frac{n+1}{q_1} - \frac{\alpha}{2}, \gamma + 3 - \alpha + \frac{n+1}{q_1} - \frac{4}{q}\right) < s_2 < \min\left(\frac{n-1}{q}, \gamma - \frac{n}{q_1}\right).$$

Moreover, there exists $\varphi_+ \in \dot{H}^{s_c} H^{s_1+s_2}_\omega$ such that

$$\|u(t) - U(t)\varphi_+\|_{\dot{H}^{s_c} H^{s_1+s_2}_\omega} \to 0 \text{ as } t \to \infty.$$

The proof of the theorem consists of several subsections.

6.1 Weighted estimates

In this subsection we assume that $n \geq 2$. We introduce several weighted estimates based on the Strichartz estimates (2.5) and (2.6). In fact, from interpolation of (2.5) and (2.6) we get the following:

Lemma 6.2. *Let $n \geq 2$ and $2 \leq q \leq \infty$. Then*
(1) *For each c and δ_1 such that*

$$-\frac{n}{q} < c < -\frac{n}{q} + \frac{n-1}{2},$$
$$\delta_1 \leq -\frac{n}{q} + \frac{n-1}{2} - c,$$

we have

$$\||x|^c |\nabla|^{c+\frac{n+\alpha}{q}-\frac{n}{2}} d_\omega^{\delta_1} U(t)\varphi\|_{L_t^q L_r^q L_\omega^2} \lesssim \|\varphi\|_{L_x^2}. \tag{6.1}$$

(2) *For each c with $-\frac{n}{q} < c < -\frac{1}{q}$ and $\delta_2 \leq -c - \frac{1}{q}$ we have*

$$\||x|^c |\nabla|^{c+\frac{\alpha}{q}} d_\omega^{\delta_2} U(t)\varphi\|_{L_t^q L_x^2} \lesssim \|\varphi\|_{L_x^2}. \tag{6.2}$$

Proof. Interpolating (2.5) and (2.6), we obtain (6.1) after arranging interpolation indices with respect to c of interpolated weight $|x|^c$. For (6.2) one can use (2.6) and trivial estimate $\|U(t)\varphi\|_{L_t^\infty L_x^2} = \|\varphi\|_{L_x^2}$. □

To handle the Hartree nonlinearity we consider the following weighted convolution estimates (see [11] and [12]).

Lemma 6.3. *Let $1 \leq p, q \leq \infty$, $0 \leq d_1 < d_2 < \frac{n-1}{p'}$ and $\frac{1}{q} \leq 1 - \frac{d_2}{n-1}$. Then we have*

$$\||x|^{d_1}(|x|^{-\frac{n}{p}-d_2} * f)\|_{L_x^p} \lesssim \||x|^{-(d_2-d_1)} f\|_{L_r^1 L_\omega^{q,1}}. \tag{6.3}$$

Moreover, if $p = \infty$, then $d_1 = d_2$ is also allowed. Here $L_\omega^{q,1}$ denotes the Lorentz space on the unit sphere.

Throughout the section the triplet (c_0, c_1, c_2) denotes

$$\left(\tfrac{\gamma+n-\alpha}{2} - \tfrac{n+\alpha}{q}, \quad \tfrac{n+\alpha}{q_1} - \alpha, \quad \tfrac{n+\alpha}{q_2} + \tfrac{\gamma-n-\alpha}{2}\right).$$

Here we use the explicit exponents

$$\begin{aligned}
\frac{1}{q_1} &= \frac{1}{2}\left(\frac{\alpha+\gamma}{\alpha-1} + \frac{\alpha+2\gamma}{4n+2}\right),\\
\frac{1}{q_2} &= \frac{1}{2}\left(\frac{\alpha-\gamma+1}{2\alpha} + \frac{\alpha+1-\gamma+\frac{2n-4}{q1}}{4}\right),\\
\frac{1}{q} &= 1 - \frac{1}{q_1} - \frac{1}{q_2}.
\end{aligned}$$

Note that $c_0 = c_1 + c_2$.

6.2 Duhamel formula

One can verify that q, q_1 and q_2 defined above satisfy all the assumption in the following lemmas.

We first consider $\dot{H}^{s_c} H_\omega^{s_1+s_2}$ estimate for the Duhamel part $U(t)\Phi_t$, where

$$\Phi_t \equiv -i\lambda \int_0^t U(-t') K_\gamma(|u|^2) u(t')\, dt'.$$

Lemma 6.4. *Let* $s_1 = \frac{2}{q} - \frac{\gamma+1-\alpha}{2}$ *and* $0 \le s_2 \le \min(\gamma - \frac{n}{q_1}, \frac{n-1}{q_1})$*. Suppose that* q_1 *satisfies* $\frac{\alpha-\gamma}{\alpha} < \frac{1}{q_1} \le \frac{\alpha}{n+\alpha}$*, then we have*

$$\||\nabla|^{s_c} d_\omega^{s_1+s_2} U(t)\Phi_t\|_{L_t^\infty L_x^2} \lesssim \||x|^{-c_0} d_\omega^{s_2}[K_\gamma(|u|^2)u]\|_{L_t^{q'} L_r^{q'} L_\omega^2} \lesssim \widetilde{W}_1(u)^2 \widetilde{W}_2(u),$$

where

$$\begin{aligned}
\widetilde{W}_1(u) &= \||x|^{-(\gamma-\frac{n}{q_1}+c_1)/2} d_\omega^{(\gamma-\frac{n}{q_1}+s_2)/2} u\|_{L_t^{2q_1} L_x^2},\\
\widetilde{W}_2(u) &= \||x|^{-c_2} d_\omega^{\frac{n-1}{q_1}} u\|_{L_t^{q_2} L_r^{q_2} L_\omega^2}.
\end{aligned}$$

Proof. By the dual estimate of (6.1) and Strichartz estimate (2.1) we have

$$\|\int_0^t U(-t')K_\gamma(|u|^2)u(t')\,dt'\|_{L_t^\infty L_x^2} \lesssim \||x|^{-c_0}|\nabla|^{-s_c}d_\omega^{-s_1}(K_\gamma(|u|^2)u)\|_{L_t^{q'}L_r^{q'}L_\omega^2},$$

which implies

$$\||\nabla|^{s_c}d_\omega^{s_1+s_2}U(t)\Phi_t\|_{L_t^\infty L_x^2} \lesssim \||x|^{-c_0}d_\omega^{s_2}(K_\gamma(|u|^2)u)\|_{L_t^{q'}L_r^{q'}L_\omega^2}.$$

Since $d_\omega^{s_2}$ commutes with radial function ψ and $|x|^{-c_0}$, we obtain

$$\||x|^{-c_0}d_\omega^{s_2}[(|x|^{-\gamma}*|u|^2)u]\|_{L_t^{q'}L_r^{q'}L_\omega^2} \lesssim \|d_\omega^{s_2}[|x|^{-c_1}(|x|^{-\gamma}*|u|^2)|x|^{-c_2}u]\|_{L_t^{q'}L_r^{q'}L_\omega^2}.$$

Now by Leibniz rule on the unit sphere with $1/q' = 1/q_1 + 1/q_2$

$$\begin{aligned}&\||x|^{-c_0}d_\omega^{s_2}[(|x|^{-\gamma}*|u|^2)u]\|_{L_t^{q'}L_r^{q'}L_\omega^2}\\ &\lesssim \||x|^{-c_1}d_\omega^{s_2}(|x|^{-\gamma}*|u|^2)\|_{L_{t,x}^{q_1}}\||x|^{-c_2}d_\omega^{s_2}u\|_{L_t^{q_2}L_r^{q_2}L_\omega^{\widetilde{q_2}}},\end{aligned} \tag{6.4}$$

where $1/2 = 1/q_1 + 1/\widetilde{q_2} - s_2/(n-1)$. Here we need $0 \le s_2 \le \frac{n-1}{q_1}$. By using Sobolev imbedding on the unit sphere, we obtain

$$\||x|^{-c_2}d_\omega^{s_2}u\|_{L_t^{q_2}L_r^{q_2}L_\omega^{\widetilde{q_2}}} \lesssim \||x|^{-c_2}d_\omega^{\frac{n-1}{q_1}}u\|_{L_t^{q_2}L_r^{q_2}L_\omega^2}. \tag{6.5}$$

Since $d_\omega^{s_2}$ also commutes with the convolution operator $|x|^{-\gamma}*$, we have

$$\||x|^{-c_1}d_\omega^{s_2}(|x|^{-\gamma}*|u|^2)\|_{L_x^{q_1}} = \||x|^{-c_1}(|x|^{-\gamma}*(d_\omega^{s_2}(|u|^2)))\|_{L_x^{q_1}} \quad \text{a.e. } t.$$

By using the weighted convolution estimate (6.3), we get

$$\||x|^{-c_1}(|x|^{-\gamma}*(d_\omega^{s_2}(|u|^2)))\|_{L_x^{q_1}} \lesssim \||x|^{-\widetilde{c_1}}d_\omega^{s_2}(|u|^2)\|_{L_r^1 L_\omega^{\frac{n-1}{n-1-(\gamma-\frac{n}{q_1})},1}},$$

where $\widetilde{c_1} = \gamma - \frac{n}{q_1} + c_1$. Since $s_2 \le \gamma - \frac{n}{q_1} < n-1-\gamma+\frac{n}{q_1}$, the Leibniz rule on the unit sphere gives

$$\begin{aligned}&\left\||x|^{-\widetilde{c_1}}d_\omega^{s_2}(|u|^2)\right\|_{L_r^1 L_\omega^{\frac{n-1}{n-1-(\gamma-\frac{n}{q_1})},1}}\\ &\lesssim \left\||x|^{-\frac{\widetilde{c_1}}{2}}d_\omega^{s_2}u\right\|_{L_r^2 L_\omega^{\frac{2(n-1)}{n-1-(\gamma-\frac{n}{q_1}-s_2)},2}}\left\||x|^{-\frac{\widetilde{c_1}}{2}}u\right\|_{L_r^2 L_\omega^{\frac{2(n-1)}{n-1-(\gamma-\frac{n}{q_1}+s_2)},2}}.\end{aligned}$$

Using the Sobolev embedding on the sphere again, we obtain

$$\big\||x|^{-\widetilde{c_1}} d_\omega^{s_2}(|u|^2)\big\|_{L_t^{q_1} L_r^1 L_\omega^{\frac{n-1}{n-1-(\gamma-\frac{n}{q_1})},1}} \lesssim \Big\||x|^{-\frac{\widetilde{c_1}}{2}} d_\omega^{(\gamma-\frac{n}{q_1}+s_2)/2} u\Big\|_{L_t^{2q_1} L_x^2}^2.$$

Combining this with (6.4) and (6.5), we get the desired estimate. □

If we further restrict the range of q_1, q_2, then we can handle the weighted norms of (6.4) in a closed form through the Christ-Kiselev lemma (for instance see [13, 28, 1]), which is stated as follows:

Lemma 6.5 (Christ-Kiselev lemma)**.** *Let $1 \le r < q \le \infty$, and X, Y be Banach spaces. Suppose that*

$$\|U(t)\phi\|_{L_t^q(Y)} \lesssim \|\phi\|_{L_x^2} \quad and \quad \|\int_0^\infty U(-t')g(t')dt'\|_{L_x^2} \lesssim \|g\|_{L_t^r(X)}.$$

Then

$$\|\int_0^t U(t-t')g(t')dt'\|_{L_t^q(Y)} \lesssim \|g\|_{L_t^r(X)}.$$

Now we consider weighted estimates for Duhamel part.

Lemma 6.6. *Let $s_1 = \frac{2}{q} - \frac{\gamma+1-\alpha}{2}$ and $\max(\gamma - \frac{n+1}{q_1} + 3 - \alpha - \frac{4}{q}, \frac{n+1}{q_1} - \frac{\alpha}{2}) \le s_2 \le \min(\gamma - \frac{n}{q_1}, \frac{n-1}{q_1})$. Suppose $\frac{\alpha-\gamma}{\alpha-1} < \frac{1}{q_1} \le \frac{\alpha}{n+\alpha}$ and $\frac{\alpha-\gamma+1}{2\gamma} < \frac{1}{q_2} \le \frac{1}{2}$. Then we have*

$$\widetilde{W}_1(U(t)\Phi_t) + \widetilde{W}_2(U(t)\Phi_t) \lesssim \widetilde{W}_1(u)^2 \widetilde{W}_2(u).$$

Proof. From the dual estimates of (6.1) with $c = c_0$ it follows that

$$\|\int_0^\infty U(-t')K_\gamma(|u|^2)u(t')\,dt'\|_{L_x^2} \lesssim \||x|^{-c_0}|\nabla|^{-s_c} d_\omega^{-s_1}[K_\gamma(|u|^2)u]\|_{L_t^{q'} L_r^{q'} L_r^2}. \tag{6.6}$$

Since $q' < q_2$, by Lemma 6.5 together with, (6.1) with $c = -c_2$ and (6.6) we have

$$\begin{aligned} &\||x|^{-c_2}|\nabla|^{-s_c} d_\omega^{\frac{2}{q_2}+\frac{\gamma-3}{2}} U(t)\Phi_t\|_{L_t^{q_2} L_r^{q_2} L_\omega^2} \\ &\lesssim \||x|^{-c_0}|\nabla|^{-s_c} d_\omega^{-s_1}[K_\gamma(|u|^2)u]\|_{L_t^{q'} L_r^{q'} L_r^2}, \end{aligned} \tag{6.7}$$

which implies

$$\||x|^{-c_2} d_\omega^{\frac{2}{q_2}+\frac{\gamma-3}{2}+s_1+s_2} U(t)\Phi_t\|_{L_t^{q_2} L_r^{q_2} L_\omega^2} \lesssim \||x|^{-c_0} d_\omega^{s_2} [K_\gamma(|u|^2)u]\|_{L_t^{q'} L_r^{q'} L_r^2}.$$

Since $\frac{n-1}{q_1} \le \frac{2}{q_2} + \frac{\gamma-3}{2} + s_1 + s_2$, we get $\widetilde{W}_2(U(t)\Phi_t) \lesssim \widetilde{W}_1(u)^2 \widetilde{W}_2(u)$.

By a similar way to get (6.6) and (6.7) with the estimates (6.2) instead of (6.1) we get

$$\begin{aligned} &\||x|^{-(\gamma-\frac{n}{q_1}+c_1)/2} |\nabla|^{-s_c} d_\omega^{-(\frac{2-\gamma}{2}-\frac{1}{2q_1})} U(t)\Phi_t\|_{L_t^{2q_1} L_x^2} \\ &\lesssim \||x|^{-c_0} |\nabla|^{-s_c} d_\omega^{-s_1} [K_\gamma(|u|^2)u]\|_{L_t^{q'} L_r^{q'} L_\omega^2}. \end{aligned}$$

Then by angular regularity shift, we also have

$$\begin{aligned} &\||x|^{-(\gamma-\frac{n}{q_1}+c_1)/2} d_\omega^{-(\frac{2-\gamma}{2}-\frac{1}{2q_1})+s_1+s_2} U(t)\Phi_t\|_{L_t^{2q_1} L_x^2} \\ &\lesssim \||x|^{-c_0} d_\omega^{s_2} [K_\gamma(|u|^2)u]\|_{L_t^{q'} L_r^{q'} L_\omega^2}, \end{aligned}$$

which implies $\widetilde{W}_1(U(t)\Phi_t) \lesssim \widetilde{W}_1(u)^2 \widetilde{W}_2(u)$ because $(\gamma - \frac{n}{q_1} + s_2)/2 < -(\frac{2-\gamma}{2} - \frac{1}{2q_1}) + s_1 + s_2$ for s_2 as stated. This completes the proof of Lemma 6.6. $\square$

We note that $\max(\frac{n+1}{q_1} - \frac{\alpha}{2}, \gamma + 3 - \alpha + \frac{n+1}{q_1} - \frac{4}{q})$ is strictly less than $\min(\frac{n-1}{q}, \gamma - \frac{n}{q_1})$. So, one can find a common s_2 which meets the condition of Theorem 6.1 and the requirements of Lemmas 6.4, 6.6.

Now we are ready to prove Theorem 6.1.

6.3 Proof of Theorem 6.1

For $\varepsilon > 0$, let us define function space B_ρ by

$$B_\rho \equiv \{u \in C(\mathbb{R}; \dot{H}^{s_c} H_\omega^{s_1+s_2}) \ : \ \|u\|_B \le \rho\},$$

where

$$\|u\|_B = \||\nabla|^{s_c} d_\omega^{s_1+s_2} u\|_{L_t^\infty L_x^2} + \widetilde{W}_1(u) + \widetilde{W}_2(u).$$

Then the set B_ρ is a complete metric space endowed with the metric

$$d_B(u,v) \equiv \||\nabla|^{s_c} d_\omega^{s_1+s_2}(u-v)\|_{L_t^\infty L_x^2} + \widetilde{W}_1(u-v) + \widetilde{W}_2(u-v).$$

Now we define

$$\mathcal{N}(u) = U(t)(\varphi + \Phi_t) \text{ on } B_\rho.$$

and show the mapping $\mathcal{N}$ is a contraction mapping from B_ρ to itself for a sufficiently small ρ.

First, from Lemma 6.2 it follows that

$$\begin{aligned}&\||\nabla|^{s_c} d_\omega^{s_1+s_2} U(t)\varphi\|_{L_t^\infty L_x^2} + \||x|^{-c_2} d_\omega^{\overline{\frac{n-1}{q_1}}} U(t)\varphi\|_{L_t^{q_2} L_r^{q_2} L_\omega^2} \\ &+ \||x|^{-(-\frac{n}{q_1}+\gamma+c_1)/2} d_\omega^{(\gamma-\frac{n}{q_1}+s_2)/2} U(t)\varphi\|_{L_t^{2q_1} L_x^2} \lesssim \||\nabla|^{s_c} d_\omega^{s_1+s_2}\varphi\|_{L_x^2}.\end{aligned} \tag{6.8}$$

On the other hand, for any $u, v \in B_\rho$ we have for any $a, \beta \in \mathbb{R}$

$$\begin{aligned}&\left||x|^a d_\omega^\beta[|x|^{-\gamma} * (|u|^2)u)] - |x|^a d_\omega^\beta[|x|^{-\gamma} * (|v|^2)v)]\right| \\ \le& \left||x|^a d_\omega^\beta[|x|^{-\gamma} * (|u|^2)(u-v)]\right| \\ &+ \left||x|^a d_\omega^\beta[|x|^{-\gamma} * ((u-v)\bar{v})v]\right| + \left||x|^a d_\omega^\beta[|x|^{-\gamma} * (u\overline{(u-v)})v]\right|.\end{aligned}$$

Then by adopting the arguments such as duality, Strichartz estimate, and Christ-Kiselev lemma, as in the proofs of Lemmas 6.4, 6.6 we obtain the following.

$$d_B(\mathcal{N}(u), \mathcal{N}(v)) \lesssim (\widetilde{W}_1(u) + \widetilde{W}_2(u) + \widetilde{W}_1(v) + \widetilde{W}_2(v))^2 d_B(u,v). \tag{6.9}$$

Therefore

$$d_B(\mathcal{N}(u), \mathcal{N}(v)) \le C\rho^2 d_B(u,\, v) \tag{6.10}$$

for some constant C independent of u, v, ρ. Now choose ρ and the size of the norm $\|\varphi\|_{\dot{H}^{s_c} H_\omega^{s_1+s_2}}$ small enough to ensure that $C\rho^2 \le \frac{1}{2}$ and $C\|\varphi\|_{\dot{H}^{s_c} H_\omega^{s_1+s_2}} \le \frac{1}{2}\rho$. Then combining (6.8) and (6.10), we conclude that the mapping $\mathcal{N}$ becomes a contraction on B_ρ.

Now we show the existence of scattering. Let us define functions φ_+ by

$$\varphi_+ = \varphi - i\lambda \int_0^\infty U(-t')[K_\gamma(|u|^2)u](t')\,dt'.$$

Then by the estimates (6.9), $\varphi_\pm \in \dot{H}^{s_c} H_\omega^{s_1+s_2}$ and

$$\|u(t) - U(t)\varphi_+\|_{\dot{H}^{s_c} H_\omega^{s_1+s_2}} \to 0 \text{ as } t \to \infty.$$

This completes the proof of Theorem 6.1.

7 Finite time blowup

In this section we consider the blowup dynamics of massive focusing mass critical FNLS ($m > 0$, $\gamma = \alpha, \lambda = -1$). For this purpose we adapt the Virial type argument of [15], in which the evolution of two quantities $\langle u, Au\rangle$ and $\langle u, Mu\rangle$ for

$$A = -\frac{i}{2}(\nabla \cdot x + x \cdot \nabla), \quad M = x \cdot D_m^{2-\alpha} x.$$

It is obvious from Proposition 3.1 that if $\varphi \in H^k, k = \max(3, \frac{\gamma}{2})$, then there exists a maximal existence time $T^* > 0$ and a unique solution $u \in C([0, T^*); H^k) \cap C^1([0, T^*); H^{k-1})$ of (1.1). If $T^* < \infty$, then $\lim_{t \nearrow T^*} \|u(t)\|_{H^{\frac{\gamma}{2}}} = \infty$. If further $x\varphi, |x|\nabla\varphi \in L^2$, then we can show the propagation of moment: $xu(t), |x|\nabla u(t) \in L^2$ for all $t \in [0, T^*)$. We postpone the proof to the end of this section.

Now let us introduce our blowup result.

Theorem 7.1. *Set $\gamma = \alpha$ and $m > 0$. Let $1 < \alpha < 2$ and $n \geq 4$. Suppose that ψ is smooth radial function with $\psi'(\rho) = \partial_r\psi(\rho) \leq 0$, $|\psi'(\rho)| \lesssim \frac{1}{\rho}$ for $\rho > 0$, and $\varphi \in H_{rad}^k$ and $x\varphi, |x|\nabla\varphi \in L_{rad}^2$ with $E(\varphi) < 0$, we have that for each m the maximal existence time $T_m^* \leq r_m$ and $\lim_{t \nearrow T_m^*} \|u(t)\|_{H^{\frac{\gamma}{2}}} = \infty$, where r_m is the positive root of*

$$2\alpha^2 E(\varphi)t^2 + 2\alpha(\langle\varphi, A\varphi\rangle + C\|\varphi\|_{L^2}^4)t + \langle\varphi, M\varphi\rangle.$$

Here C does not depend on m.

7.1 Proof of Theorem 7.1

Let us now show the theorem. We begin with the dilation operator

$$A = -\frac{i}{2}(\nabla \cdot x + x \cdot \nabla).$$

Since $u \in H^k$ and $xu, |x|\nabla u \in L^2$, $\langle u, Au\rangle$ is well-defined and so is

$$\frac{d}{dt}\langle u, Au\rangle = i\langle u, [H, A]u\rangle, \tag{7.1}$$

where $H = D_m^\alpha + \mathcal{V}$ and $\mathcal{V} = -K_\alpha(|u|^2) = -(\psi/|\cdot|^\alpha) * |u|^2$. Here $[H, A]$ denotes the commutator $HA - AH$. As a matter of fact we have the following.

Lemma 7.2. *Let u, φ and ψ be as above in Theorem 7.1. Then*

$$\frac{d}{dt}\langle u, Au\rangle \le 2\alpha E_2(\varphi). \tag{7.2}$$

Proof of Lemma 7.2. Using the identity $D_m^\alpha x = x D_m^\alpha - \alpha D_m^{\alpha-2}\nabla$, we have

$$[\, D_m^\alpha, A] = -i\alpha D_m^{\alpha-2} D_0^2. \tag{7.3}$$

Similarly,

$$[\, \mathcal{V}, A] = i(x \cdot \nabla)\mathcal{V}. \tag{7.4}$$

Substituting (7.3) and (7.4) into (7.1), we get

$$\frac{d}{dt}\langle u, Au\rangle = \alpha\langle u, D_m^\alpha u\rangle - \alpha m^2\langle u, D_m^{\alpha-2}u\rangle - \langle u, (x \cdot \nabla)\mathcal{V}u\rangle. \tag{7.5}$$

For the second term on RHS of (7.5) we obtain the following identities:

$$\begin{aligned}(x \cdot \nabla)\mathcal{V} &= \alpha \int \frac{\psi(|x-y|)}{|x-y|^\alpha}|u(y)|^2\, dy - \int \frac{\psi'(|x-y|)}{|x-y|^\alpha}|x-y||u(y)|^2\, dy \\ &\quad + \int \left(\alpha\frac{\psi(|x-y|)}{|x-y|^{\alpha+1}} - \frac{\psi'(|x-y|)}{|x-y|^\alpha}\right)\frac{y\cdot(x-y)}{|x-y|}|u(y)|^2\, dy,\end{aligned}$$

$$\langle u, (x\cdot\nabla)\mathcal{V}u\rangle = -4\alpha V(u) - \iint \frac{|x-y|\psi'(|x-y|)}{|x-y|^\alpha}|u(x)|^2|u(y)|^2\,dxdy$$
$$- \langle u, (x\cdot\nabla)\mathcal{V}u\rangle,$$

which implies

$$\langle u, (x\cdot\nabla)\mathcal{V}u\rangle = -2\alpha V(u) - \frac{1}{2}\iint \frac{|x-y|\psi'(|x-y|)}{|x-y|^\alpha}|u(x)|^2|u(y)|^2\,dxdy.$$

Substituting this into (7.5), we have

$$\frac{d}{dt}\langle u, Au\rangle \le 2\alpha E(\varphi) + \frac{1}{2}\iint (|x-y|\psi_1'(|x-y|))\,\frac{|u(x)|^2|u(y)|^2}{|x-y|^\alpha}\,dxdy.$$

Since $\psi'(|x|)\le 0$, we get (7.2). □

Next we consider the nonnegative quantity $\langle u, Mu\rangle$ with

$$M \equiv x\cdot D_m^{2-\alpha}x = \sum_{k=1}^n x_k D_m^{2-\alpha}x_k.$$

From the regularity and decay condition of u the quantity $\langle u(t), Mu(t)\rangle$ is well-defined and finite for all $t\in[0,T^*)$ since $|\langle u, Mu\rangle| \lesssim_m \|xu\|_{L^2}(\|xu\|_{L^2} + \|x\cdot\nabla u\|_{L^2})$, and so is

$$\frac{d}{dt}\langle u, Mu\rangle = i\langle u, [H, M]u\rangle = i\langle u, [\,D_m^\alpha, M]u\rangle - i\langle u, [K_\alpha(|u|^2), M]u\rangle. \quad (7.6)$$

We have the following.

Lemma 7.3. *With the same condition as in Theorem 7.1, we have*

$$\frac{d}{dt}\langle u, Mu\rangle \le 2\alpha\langle u, Au\rangle + C\|\varphi\|_{L^2}^4, \quad (7.7)$$

where C is a positive constant depending only on n, α but not on m.

Theorem 7.1 follows immediately from Lemmas 7.2 and 7.3.

Proof of Lemma 7.3. Using the identity $D_m^\alpha x = xD_m^\alpha - \alpha D_m^{\alpha-2}\nabla$, we first have the estimate:

$$[\, D_m^\alpha, M] = D_m^\alpha x D_m^{2-\alpha} x - x D_m^{2-\alpha} x D_m^\alpha = -\alpha(x\cdot\nabla + \nabla\cdot x).$$

For a smooth function v we get

$$\begin{aligned}[v, M] &= vxD_m^{2-\alpha}x - xD_m^{2-\alpha}xv \\ &= v|x|^2D_m^{2-\alpha} - (2-\alpha)vx\cdot\nabla D_m^{-\alpha} - D_m^{2-\alpha}|x|^2v + (2-\alpha)D_m^{-\alpha}\nabla\cdot xv \\ &= [\,|x|^2v, D_m^{2-\alpha}] + (\alpha-2)\left(vx\cdot\frac{\nabla}{|\nabla|}|\nabla|D_m^{-\alpha} + |\nabla|D_m^{-\alpha}\frac{\nabla}{|\nabla|}\cdot xv\right).\end{aligned}$$

By density we may replace v with $K_\alpha(|u|^2)$. We will show in the next section

$$|\langle u, [\,|x|^2K_\alpha(|u|^2), D_m^{2-\alpha}]u\rangle| \lesssim \|\varphi\|_{L^2}^4. \tag{7.8}$$

By the convolution estimate, Lemma 6.3 in case when $p=\infty$, $d_1 = d_2 = \gamma$ and f is radial, one have

$$\begin{aligned}&|\langle u, \left(vx\cdot\frac{\nabla}{|\nabla|}|\nabla|D_m^{-\alpha} + |\nabla|D_m^{-\alpha}\frac{\nabla}{|\nabla|}\cdot xv\right)u\rangle| \\ &\lesssim \|\psi\|_{L^\infty}\|\varphi\|_{L^2}^2\int|u(x)||x|^{-(\alpha-1)}\int|x-y|^{-(n-(\alpha-1))}|(\frac{\nabla}{|\nabla|}u)(y)|dydx\end{aligned} \tag{7.9}$$

To estimate this, we make use of the Stein-Weiss inequality [27]: for $f\in L^p$ with $1<p<\infty$, $0<\lambda<n$, $\beta<\frac{n}{p}$, and $n=\lambda+\beta$

$$\||x|^{-\beta}(|\cdot|^{-\lambda} * f)\|_{L^p} \lesssim \|f\|_{L^p}. \tag{7.10}$$

Applying (7.10) with $p=2$, $\beta=\alpha-1$ and $\lambda = n-(\alpha-1)$, (7.9) is bounded by $C\|\varphi\|_{L^2}^4$.

□

7.2 L^2 boundedness of commutator

We show the commutator estimate (7.8). We set $f = |x|^2 K_\alpha(|u|^2)$. From a simple calculation we observe that

$$[D_m^{2-\alpha}, f]u(x) = m^{n+2-\alpha}\left([D_1^{2-\alpha}, f\left(\frac{\cdot}{m}\right)]u_m\right)(mx), \tag{7.11}$$

where $u_m(x) = m^{-n}u(\frac{x}{m})$. Thus we have the identity of the operator norms

$$\|[D_m^{2-\alpha}, f]\|_{L^2\to L^2} = m^{2-\alpha}\|[D_1^{2-\alpha}, f(\cdot/m)]\|_{L^2\to L^2}.$$

Set $f(x/m) = g(x)$. We define T_i, a pseudodifferential operator of order $1-\alpha$, by $T_i = -D_1^{2-\alpha}(-\Delta)^{-1}\partial_i$ so that $D_1^{2-\alpha} = -\sum_{i=1}^n T_i\partial_i$. Denote the kernel of $T_i\partial_i$ by k_i. Then $[T_i\partial_i, g] = [T_i, g]\partial_i + T_i(\partial_i g)$ and the kernel $[T_i, g]$ is given by

$$K_i(x,y) = k_i(x,y)(g(y) - g(x)).$$

Suppose that g is in Lipschitz class $\dot\Lambda^{2-\alpha}$. Then K_i is easily shown to be a Calderón-Zygmund kernel. Here $\|g\|_{\dot\Lambda^{2-\alpha}} = \sup_{x,y}\frac{|g(x)-g(y)|}{|x-y|^{2-\alpha}}$. We show that $[T_i, g]\partial_i$ is bounded in L^2 and its norm is bounded by a constant multiple of $\|g\|_{\dot\Lambda^{2-\alpha}}$. By Theorem 3 in p. 294 of [26] and the duality of $[T_i, g]\partial_i$ we have only to show that

$$\|[T_i, g]\partial_i(\zeta(\cdot/N))\|_{L^2} \lesssim \|g\|_{\dot\Lambda^{2-\alpha}} N^{\frac{n}{2}} \tag{7.12}$$

for a fixed bump function ζ supported in the unit ball. From the kernel estimate $|k_i(x,y)| \lesssim |x-y|^{-n+\alpha-1}$ it follows that $|K_i(x,y)| \lesssim \|g\|_{\dot\Lambda^{2-\alpha}}|x-y|^{-(n-1)}$. If $|x| < 2N$, then

$$|[T_i, g]\partial_i(\zeta(\cdot/N))(x)| \lesssim \|g\|_{\dot\Lambda^{2-\alpha}}.$$

Thus $\|[T_i, g]\partial_i(\zeta(\cdot/N))\|_{L^2(\{|x|<2N\}} \lesssim \|g\|_{\dot\Lambda^{2-\alpha}} N^{\frac{n}{2}}$. If $|x| \ge 2N$, then

$$|[T_i, g]\partial_i(\zeta(\cdot/N))(x)| \lesssim \|g\|_{\dot\Lambda^{2-\alpha}} N^{n-1}|x|^{-(n-1)}.$$

Therefore

$$\|[T_i, g]\partial_i(\zeta(\cdot/N))\|_{L^2(\{|x|\geq 2N\}} \lesssim \|g\|_{\Lambda^{2-\alpha}} N^{n-1}\Big(\int_{|x|>2N} |x|^{-2(n-1)}\,dx\Big)^{\frac{1}{2}}$$
$$\lesssim \|g\|_{\dot{\Lambda}^{2-\alpha}} N^{\frac{n}{2}}.$$

This shows (7.12) and thus $\|[T_i, g]\partial_i\|_{L^2\to L^2} \lesssim \|g\|_{\dot{\Lambda}^{2-\alpha}} = m^{-(2-\alpha)}\|f\|_{\dot{\Lambda}^{2-\alpha}}$. If $x \neq y$, then

$$|f(x) - f(y)| \leq |x - y| \int_0^1 |\nabla f(z_s)|\,ds, \quad z_s = x + s(y - x).$$

Since $|\psi'(\rho)| \leq C\rho^{-1}$ for $\rho > 0$, from Lemma 6.3 and mass conservation it follows that

$$|\nabla f(z_s)| \lesssim |z_s|^{1-\alpha}\|u\|_{L^2}^2 = ||x| - s|x - y||^{1-\alpha}\|\varphi\|_{L^2}^2,$$

provided $\alpha < n - 2$. By a simple calculation we see that if $0 < \theta < 1$, then

$$\sup_{a>0} \int_0^1 |a - s|^{-\theta}\,ds \leq C_\theta.$$

Thus from this we get that

$$|f(x) - f(y)| \lesssim |x - y|^{2-\alpha}\|\varphi\|_{L^2}^2,$$

which implies that

$$\|[T_j, g]\partial_j\|_{L^2\to L^2} \lesssim m^{-(2-\alpha)}\|\varphi\|_{L^2}. \tag{7.13}$$

On the other hand, $T_i(\partial_i g)(u)(x) = \int k_i(x, y)\partial_i g(y)u(y)\,dy$ and

$$|T_i(\partial_i g)(u)(x)| \lesssim \int |x - y|^{-(n-(\alpha-1))}|\partial_i g(y)||u(y)|\,dy.$$

From the duality and Lemma 6.3

$$\begin{aligned}|\langle u, T_j((\partial_j g)u)\rangle| &= |\langle T_j^* u, (\partial_j g)u\rangle| \\ &\lesssim m^{-1}\|u\|_{L^2}\||(\partial_j)f(\cdot/m)| \int |\cdot - y|^{-(n-\alpha+1)}|u(y)|\,dy\|_{L^2} \\ &\lesssim m^{-(2-\alpha)}\|u\|_{L^2}^3\||\cdot|^{1-\alpha}| \int |\cdot - y|^{-(n-\alpha+1)}|u(y)|\,dy\|_{L^2},\end{aligned}$$

where T_j^* is the dual operator of T_j. Using the Stein-Weiss inequality (7.10) for $\beta = \alpha - 1$, $\lambda = n - \alpha + 1$ and $p = 2$, we get $|\langle u, T_j, \partial_j g u\rangle| \lesssim m^{-(2-\alpha)}\|u\|_{L^2}^4$. Thus

$$\|T_j, \partial_j g\|_{L^2 \to L^2} \lesssim m^{-(2-\alpha)}\|\varphi\|_{L^2}^2. \tag{7.14}$$

Therefore from (7.13) and (7.14) it follows that

$$\|[D_m^{2-\alpha}, f]\|_{L^2 \to L^2} = m^{2-\alpha}\|[D_1^{2-\alpha}, g]\|_{L^2 \to L^2} \le C\|\varphi\|_{L^2}^2.$$

Here it should be noted that the constant C does not depend on m.

7.3 Propagation of the moment

We finally show a propagation estimate of the moment. In what follows, Bessel potential estimates are used crucially. So, we introduce some basics of Bessel potential.

Let us denote the kernels of Bessel potential $D^{-\beta}$ ($\beta > 0$) and $|\nabla|^\alpha D^{-\alpha} D^{-\beta}$ by $G_\beta(x)$ and $K(x)$, respectively, where $D = \sqrt{1-\Delta}$. Then

$$K(x) = \sum_{k=0}^{\infty} A_k G_{2k+\beta}(x),$$

where the coefficients A_k is given by the expansion $(1-t)^{\frac{\alpha}{2}} = \sum_{k=0}^{\infty} A_k t^k$ for $|t| < 1$ with $\sum_{k \ge 0} |A_k| < \infty$. One can show that $(1+|x|)^\ell K \in L^1$ for $\ell \ge 1$. In fact, we have that for $2k + \beta < n$

$$G_{2k+\beta}(x) \le C(|x|^{-n+\beta}\chi_{\{|x|\le 1\}}(x) + e^{-c|x|}\chi_{\{|x|>1\}}(x)). \tag{7.15}$$

And also from the integral representation of $G_{2k+\beta}$ such that

$$G_{2k+\beta}(x) = \frac{1}{(4\pi)^{n/2}\Gamma(k+\beta/2)} \int_0^\infty \lambda^{(2k+\beta-n)/2-1} e^{-|x|^2/4\lambda} e^{-\lambda}\, d\lambda$$

we deduce that if $2k + \beta \ge n$, then

$$G_{2k+\beta}(x) \le C(\chi_{\{|x|\le 1\}}(x) + e^{-c|x|}\chi_{\{|x|>1\}}(x)). \tag{7.16}$$

Here the constants C of (7.15) and (7.16) are independent of k. The functions $(1+|x|)^\ell G_{2k+\beta}$ have a uniform integrable majorant on k for each $\ell \geq 1$ and so K does. For more details see p.132–135 of [25].

We introduce the moment estimate

Proposition 7.4. *Let $m > 0$ and T^* be the maximal existence time of solution $u \in C([0,T^*); H^k)$, $k = \max(\frac{\gamma}{2}, 4)$ to (1.1). If $x\varphi, |x|\nabla\varphi \in L^2$, then $xu(t), |x|\nabla u(t) \in L^2$ for all $t \in [0,T^*)$. Moreover, we have for $t \in [0,T^*)$*

$$\||x|u\|_{L^2} \leq \||x|\varphi\|_{L^2} + Cm^{\alpha-3}\int_0^t \|u(t')\|_{H^2}\,dt',$$
$$\||x|\nabla u\|_{L^2} \leq \||x|\nabla\varphi\|_{L^2} + Cm^{\alpha-3}\int_0^t \|u(t')\|_{H^3}\,dt',$$

where C does not depend on m.

For the proof for $\alpha = 1, 2$ see [3] for NLS and [15] for semirelativistic case.

Proof of Proposition 7.4. We first consider the case $m > 0$. Let us denote

$$\mathbf{m}_\varepsilon(t) = \langle u(t), |x|^2 e^{-2\varepsilon|x|} u(t)\rangle$$

for $0 < \varepsilon \leq m$. From the regularity of u and (7.11) it follows that

$$\begin{aligned}\mathbf{m}_\varepsilon'(t) &= im^{n+\alpha-2}\langle u_m, [D^\alpha, |x|^2 e^{-2\varepsilon|x|/m}]u_m\rangle \\ &= -2m^{n+\alpha-2}\,\mathrm{Im}\,\langle xe^{-\varepsilon|x|/m}u_m, [D^\alpha, xe^{-\varepsilon|x|/m}]u_m\rangle,\end{aligned} \tag{7.17}$$

where $D = D_1 = \sqrt{1-\Delta}$ and $u_m(x) = m^{-n}u(x/m)$. Then

$$\begin{aligned}\langle xe^{-\varepsilon|x|/m}&u_m, [D^\alpha, xe^{-\varepsilon|x|/m}]u_m\rangle \\ &= \langle xe^{-\varepsilon|x|/m}u_m, [D^{\alpha-2}, xe^{-\varepsilon|x|/m}]D^2u_m\rangle \\ &\qquad + \langle D^{\alpha-2}(xe^{-\varepsilon|x|/m}u_m), [D^2, xe^{-\varepsilon|x|/m}]u_m\rangle \\ &\equiv I + I\!I.\end{aligned}$$

To handle I set $\beta = 2 - \alpha$ and denote the kernel of Bessel potential $D^{-\beta}$ by G_β. Then by mean value inequality such that $|ye^{-\varepsilon|y|/m} - xe^{-\varepsilon|x|/m}| \lesssim |x-y|$, we have

$$\begin{aligned}
&|([D^{-\beta}, xe^{-\varepsilon|x|/m}]D^2 u_m)(x)| \\
&= \left| \int G_\beta(x-y) ye^{-\varepsilon|y|/m} D^2 u_m(y) dy - xe^{-\varepsilon|x|/m} \int G_\beta(x-y) D^2 u_m(y) dy \right| \\
&= \left| \int G_\beta(x-y)(ye^{-\varepsilon|y|/m} - xe^{-\varepsilon|x|/m}) D^2 u_m(y) dy \right| \\
&\lesssim \int G_\beta(x-y)|x-y||D^2 u_m(y)| dy.
\end{aligned}$$

Since $|x|G_\beta$ is integrable, from Cauchy-Schwarz inequality and Young's inequality it follows that

$$|I| \leq Cm^{-\frac{n}{2}-2} \|u\|_{H^2} \sqrt{\mathbf{m}_\varepsilon}, \tag{7.18}$$

where C is independent of ε and m.

Now using Cauchy-Schwarz inequality we estimate $I\!I$ as follows:

$$\begin{aligned}
|I\!I| &= |\langle D^{-\beta}(xe^{-\varepsilon|x|/m} u_m), [D^2, xe^{-\varepsilon|x|/m}] u_m \rangle| \\
&= |\langle D^{-\beta}(xe^{-\varepsilon|x|/m} u_m), (\Delta(xe^{-\varepsilon|x|/m}) + 2\nabla(xe^{-\varepsilon|x|/m}) \cdot \nabla) u_m \rangle| \\
&\leq Cm^{-n} \|u\|_{H^1} \sqrt{\mathbf{m}_\varepsilon},
\end{aligned} \tag{7.19}$$

where C is independent of ε and m. We have used the fact

$$|\Delta(xe^{-\varepsilon|x|/m}) + 2\nabla(xe^{-\varepsilon|x|/m}| \leq C.$$

Substituting the estimates for I and $I\!I$ into (7.17), we have

$$\mathbf{m}_\varepsilon \leq \mathbf{m}_\varepsilon(0) + Cm^{\alpha-3} \int_0^t \|u(t')\|_{H^2} \sqrt{\mathbf{m}_\varepsilon(t')} dt'.$$

Gronwall's inequality yields

$$\sqrt{\mathbf{m}_\varepsilon} \leq \sqrt{\mathbf{m}_\varepsilon(0)} + Cm^{\alpha-3}/2 \int_0^t \|u(t')\|_{H^2} dt'.$$

Thus letting $\varepsilon \to 0$, it follows that

$$\||x|u\|_{L^2} \le \||x|\varphi\|_{L^2} + Cm^{\alpha-3}/2 \int_0^t \|u(t')\|_{H^2}\, dt' \quad \text{for all} \quad t \in [0, T^*). \tag{7.20}$$

Now let us observe that $u \in H^3$, set $v = \partial_j u$. Then one can easily show that

$$\frac{d}{dt}\langle v\, |x|^2 e^{-2\varepsilon|x|} v\rangle = i\langle v, [D_m^\alpha, |x|^2 e^{-2\varepsilon|x|}]v\rangle.$$

So, by the same estimates as above we get

$$\||x|\nabla u\|_{L^2} \lesssim \||x|\nabla\varphi\|_{L^2} + m^{\alpha-3} \int_0^t \|u(t')\|_{H^3}\, dt' \quad \text{for all} \quad t \in [0, T^*). \tag{7.21}$$

□

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