Characterization of the Orbit of Standing Waves of Hartree Type Equations with External Coulomb Potential

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Abstract

We prove the orbital stability and then characterize the orbit of standing waves of Schrödinger equation of Hartree type.

1 Introduction

The Schrödinger equation of Hartree type appears in many problems arising in physics, [1] and references therein. It involves the non-local effects of the Coulomb potential and the local potential $p(|x|)$:

$$i\partial_t \Phi(t,x) + \Delta_{xx}\Phi(t,x) + \int_{\mathbb{R}^3} \frac{|\Phi(t,y)|^2}{|x-y|}dy \quad \Phi(t,x) + \Phi(t,x)p(|x|) = 0 \quad (1.1)$$

$$\Phi(0,x) = \Phi_0(x).$$

Standing waves are solutions of (1.1) that can be written as $\Phi(t,x) = e^{-i\lambda t}u(x)$. Thus (1.1) becomes equivalent to the following elliptic equation :

$$\Delta u(x) - \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}dy \quad u(x) + p(|x|)u(x) + \lambda u(x) = 0. \quad (1.2)$$

Physically, the most important solutions of (1.2) are the so-called ground state solutions, i.e, which are the minimizers of the following constrained variational problem :

$$I_c = \inf\{E(u) : u \in S_c\}. \quad (1.3)$$

$$E(u) = \frac{1}{2}|\nabla u|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}dxdy - \frac{1}{2}\int_{\mathbb{R}^3} p(|x|)|u(x)|^2 dx$$

$$S_c = \{u \in H^1(\mathbb{R}^3, \mathbb{R}) : \int_{\mathbb{R}^3} u^2(x)dx = c^2\},$$

where c is prescribed number .
In a recent paper, [2], Georgiev and Venkov have established the radial symmetry (up to a translation) and the uniquness of the minimizers of (1.3). In [3], Kawohl and Krömer have given a tricky and elegant proof of this important result by noticing that $J(|u|^2) = E(u)$ is a strictly convex functional. It is worth to mention that the main difficulty to solve (1.3) is that the convolution term has a non-negative sign, which excludes the use of rearrangement and symmetrization tools. The main novelty of [3] was that the authors were able to show that "the bad sign" of the convolution term is in fact "very good" because it has a strict convexity property.
In the present work, we will take advantage of their uniqueness result to characterize the orbit of standing waves of (1.1) in the sense introduced by Cazenave and Lions in [1].
For the convenience of the reader, we will first give all the definitions we need and the results we will use to prove our main theorems which are stated in section 3.

2 Notations and preliminary results

$H^1(\mathbb{R}^3, \mathbb{R})$ is the usual Hilbert space.

$H^1(\mathbb{R}^3, \mathbb{C}) = \{z = (u, v) \in H^1(\mathbb{R}^3, \mathbb{R}) \times H^1(\mathbb{R}^3, \mathbb{R})\}$.

We shall sometimes identify $z = (u, v)$ with $u + iv \in H^1(\mathbb{R}^3, \mathbb{C})$.

For $z \in H^1(\mathbb{R}^3, \mathbb{C})$; $\|z\|^2_{H^1(\mathbb{R}^3,\mathbb{C})} = \|z\|_2^2 + \|\nabla z\|_2^2$, where $\|z\|_2^2 = |u|_2^2 + |v|_2^2$ and $\|\nabla z\|_2^2 = |\nabla u|_2^2 + |\nabla v|_2^2$.
Here and elsewhere $| \, . \, |_p$ denotes the usual norm of $L^2(\mathbb{R}^3, \mathbb{R})$ and $\| \, \|_p$ is the usual norm of $L^p(\mathbb{R}^3, \mathbb{C})$.
We define the energy functional :

$$\hat{E}(z) = \frac{1}{2}|\nabla z|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|z(x)|^2|z(y)|^2}{|x-y|}dxdy - \frac{1}{2}\int_{\mathbb{R}^3} p(|x|)|z(x)|^2 dx.$$

For $c > 0$, we set $\hat{S}_c = \{z \in H^1(\mathbb{R}^3, \mathbb{C}), \|z\|_2^2 = c^2\}$.

$$\hat{I}_c = \inf\{\hat{E}(z) = z \in \hat{S}_c\}.$$

$$Z_c = \{z \in \hat{S}_c : \hat{E}(z) = \hat{I}_c\}$$
$$W_c = \{u \in S_c \cap C^1(\mathbb{R}^3) : E(u) = I_c \text{ and } u > 0\}.$$

From now on $c > 0$ is fixed :
Following the terminology introduced in [1], we say that Z_c is stable if :
$Z_c \neq \emptyset$ and $\forall\ w \in Z_c$, $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that for any $\Phi_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ satisfying $\|\Phi_0 - w\|_{H^1(\mathbb{R}^3,\mathbb{C})}$ it follows that $\inf_{z\in Z_c} \|\Phi(t,.) - z\|_{H^1(\mathbb{R}^3,\mathbb{C})} < \varepsilon$ for all $t \in \mathbb{R}$, where $\Phi(t,.)$ is the solution of (1.1) corresponding to the initial condition Φ_0. (of course this definition requirers that (1.1) has a unique global solution for every initial value $\Phi_0 \in H^1(\mathbb{R}^3, \mathbb{C})$, which was done in a more general context in [4]).
Note that if $w \in Z_c$, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that :

$$\Delta w(x) - \int_{\mathbb{R}^3} \frac{|w(y)|^2}{|x-y|} dy \quad w(x) + p(|x|)w(x) + \lambda w(x) = 0. \tag{2.1}$$

Thus $w = (u, v)$ solves the 2×2 elliptic system :

$$\left.\begin{aligned} \Delta u - \int_{\mathbb{R}^3} (\frac{u^2(y) + v^2(y)}{|x-y|}) dy \quad u(x) + p(|x|)u(x) + \lambda u(x) \quad &= 0 \\ \Delta v - \int_{\mathbb{R}^3} (\frac{u^2(y) + v^2(y)}{|x-y|}) dy \quad v(x) + p(|x|)v(x) = +\lambda v(x) \quad &= 0 \end{aligned}\right\} \tag{2.2}$$

Moreover $e^{-i\lambda t}w$ is a solution of (1.1) with initial value $\Phi_0 = w$. Also $e^{-i\lambda t}w$ can be viewed as a periodic solution of (1.1) and its orbit $\theta(w) = \{e^{-i\lambda t}w, t \in \mathbb{R}\} \subset Z_c$. Thus the usual definition for the orbital stability in $H^1(\mathbb{R}^3, \mathbb{C})$ of the periodic solution $e^{-i\lambda t}w$ is that :
$\forall\ \varepsilon > 0, \exists\ \delta > 0$ such that for any $\Phi_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ such that $\|\Phi_0 - w\|_{H^1(\mathbb{R}^3,\mathbb{C})} < \delta$, it follows that $\inf_{\varphi\in\theta(w)} \|\Phi(t,.) - \varphi\|_{H^1(\mathbb{R}^3,\mathbb{C})} < \varepsilon$ for all $t \in \mathbb{R}$, where $\Phi(t,.)$ is the solution of (1.1) corresponding to the initial condition Φ_0.
Finally, observe that the two definitions of stability coïncide when $Z_c = \phi(w)$ for some $c > 0$. Generally Z_c is larger than $\theta(w)$.
In the next section, we will need the following results that we collect for the convenience of the reader.

(P0) $\hat{E} \in C^1(H^1(\mathbb{R}^3, \mathbb{C}), \mathbb{R})$ and all minimizing sequence for $\hat{I}_c$ are bounded in $H^1(\mathbb{R}^3, \mathbb{C})$.

$E \in C^1(H^1(\mathbb{R}^3, \mathbb{R}), \mathbb{R})$ and all minimizing sequences for I_c are bounded in $H^1(\mathbb{R}^3, \mathbb{R})$.

(P1) For any $\Phi_0 \in H^1(\mathbb{R}^3, \mathbb{C})$, (1.1) admits a unique global solution $\Phi \in C(\mathbb{R}, H^1(\mathbb{R}^3, \mathbb{C})$ satisfying :

$$\|\Phi(t,.)\|_2 = \|\Phi_0\|_2 \quad \text{and } \hat{E}(\Phi(t,.)) = \hat{E}(\Phi_0) \quad \forall\, t \in \mathbb{R}$$

For a fixed $c > 0$:

(P2) For any sequence c_n, $\lim I_{c_n} = I_c$.

(P3) Any sequence $\{u_n\} \subset H^1(\mathbb{R}^3, \mathbb{R})$ such that $|u_n|_2 \to c$ and $E(u_n) \to I_c$ is relatively compact in $H^1(\mathbb{R}^3, \mathbb{R})$ (up to a translation).

(P4) If $u \in H^1(\mathbb{R}^3, \mathbb{R})$, $v \in H^1(\mathbb{R}^3, \mathbb{R}), (u^2 + v^2)^{1/2} \in H^1(\mathbb{R}^3, \mathbb{R})$.

The proofs of (P0) and (P1) are classical, they can be found in a more general centext in [4]. (P2) follows immediately thanks to the properties of our functional. (P3) can be easily obtained using Lions compactness lemma in the same way as [5].

In what follows, a positive constant whose value is un important for the purposes of the discussion may vary from a line to another one.

- We will say that (z_n) converges to z even if it is only up to a translation.
- $\lim\limits_{|x|\to\infty} p(|x|) = 0$.

3 Main Reuslts

Theorem 3.1 : **orbital Stability**

1. For any $c > 0, I_c = \hat{I}_c$, $Z_c \neq \emptyset$ and Z_c is stable.
2. If $z \in Z_c$, then $|z| \in W_c$.

Theorem 3.2 : **Characterization of the orbit**

If $z = (u, v) \in Z_c$ then :

a) $u \equiv 0$ or $u(x) \neq 0$ for all $x \in \mathbb{R}^3$,

b) $v \equiv 0$ or $v(x) \neq 0$ for all $x \in \mathbb{R}^3$,

c) $Z_c = \{e^{i\sigma} w(. + y), \sigma \in \mathbb{R}, y \in \mathbb{R}^3\}$, where w the unique minimizer of (1.3).

Proof :

1. We first prove the stability of the orbit.
   We will argue by contradiction : Suppose that Z_c is not stable, then either $Z_c = \emptyset$ or there exists $w \in Z_c, \varepsilon_0 > 0$ and a sequence $\{\Phi_0^n\} \subset H^1(\mathbb{R}^3, \mathbb{C})$ such that

$$\begin{array}{l} \|\Phi_0^n - w\|_{H^1(\mathbb{R}^3,\mathbb{C})} \to 0 \quad \text{ as } n \to \infty \text{ but} \\ \inf\limits_{z \in Z_c} \|\Phi^0(t_n, .) - z\|_{H^1(\mathbb{R}^3,\mathbb{C})} \geq \varepsilon_0, \text{ for some sequence } \{t_n\} \subset \mathbb{R}, \\ \text{where } \Phi^n \text{ is the solution of (1.1) corresponding} \\ \text{to the initial condition } \Phi_0^n. \end{array} \tag{3.1}$$

Let $z_n = \Phi^n(t_n, .)$. Since $w \in \hat{S}_c$ and $\hat{E}(w) = \hat{I}_c$, it follows by the continuity of $\| \, \|_2$ and $\hat{E}$ on $H^1(\mathbb{R}^3, \mathbb{C})$ that $\|\Phi_0^n\|_2 \to c$ and $\hat{E}(\Phi_0^n) \to \hat{I}_c$.
Let $z_n = \Phi^n(t_n, .)$. Since $w \in \hat{S}_c$ and $\hat{E}(w) = \hat{I}_c$, it follows by the continuity of $\| \, \|_2$ and $\hat{E}$ on $H^1(\mathbb{R}^3, \mathbb{C})$ that $\|\Phi_0^n\|_2 \to c$ and $\hat{E}(\Phi_0^n) \to \hat{I}_c$.
If $\{z_n\}$ contains a subsequence converging in $H^1(\mathbb{R}^3, \mathbb{C})$ to an element f, we find that $\|f\|_2 = c$ and $\hat{E}(f) = \hat{I}_c$, showing that $f \in Z_c$. Thus $\inf\limits_{z \in Z_c} \|\Phi^n(t_n, .) - z\|_{H^1(\mathbb{R}^3,\mathbb{C})} \leq \|z_n - f\|_{H^1(\mathbb{R}^3,\mathbb{C})}$, contradicting (3.1).
Hence to establish the orbital stability of Z_c, it is enough to show that : $Z_c \neq \emptyset$ and for any sequence $\{z_n\} \subset H^1(\mathbb{R}^3, \mathbb{C})$, such that

$$\|z_n\|_2 \to c \text{ and } \hat{E}(z_n) \to \hat{I}_c \tag{3.2}$$

is relatively compact (up to a translation).
Let $z_n = \{u_n, v_n) \subset H^1(\mathbb{R}^3, \mathbb{C})$ be a subsequence such that $\|z_n\|_2^2 \to c^2$ and $\hat{E}(z_n) \to \hat{I}_c$. Our first objective is to prove that $\{z_n\}$ has a subsequence converging in $H^1(\mathbb{R}^3, \mathbb{C})$.
By (P0), $\{z_n\}$ is bounded in $H^1(\mathbb{R}^3, \mathbb{C})$, thus passing to a subsequence, there exists $z = (u, v) \in H^1(\mathbb{R}^3, \mathbb{C})$, such that u_n converges weakly to u in $H^1(\mathbb{R}^3, \mathbb{R})$ and v_n converges weakly to v in $H^1(\mathbb{R}^3, \mathbb{R})$ and

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2 \text{ exists .} \tag{3.3}$$

Setting $\rho_n = |z_n|$, (P4) assures that $\{\rho_n\} \subset H^1(\mathbb{R}^3, \mathbb{R})$ and that for all $n \in \mathbb{N}$ and $1 \leq i \leq 3$,

$$\partial_i \rho_n(x) = \begin{cases} \dfrac{u_n(x)\partial_i u_n(x) + v_n(x)\partial_i v_n(x)}{\{u_n^2(x) + v_n^2(x)\}^{1/2}} & \text{if } u_n^2 + v_n^2 > 0 \\ 0 & \text{otherwise .} \end{cases}$$

Hence

$$\begin{aligned}\hat{E}(z_n) - E(\rho_n) &= \frac{1}{2}\{\|\nabla z_n\|_2^2 - |\nabla \rho_n|_2^2\} \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2 - |\nabla(u_n^2 + v_n^2)^{1/2}|^2. \\ &= \frac{1}{2}\int_{\{u_n^2+v_n^2>0\}} \sum_{i=1}^{3} \frac{(u_n\partial_i v_n - v_n \partial_i u_n)^2}{u_n^2+v_n^2} dx\end{aligned} \tag{3.4}$$

proving that $\hat{I}_c = \lim_{n\to\infty} \hat{E}(z_n) \geq \limsup E(\rho_n)$.

On the other hand, $\|z_n\|_2^2 = |\rho_n|_2^2 = c_n^2 \to c^2$. (3.5)

And hence by (P2), we have that $\liminf E(\rho_n) \geq \liminf I_{c_n} \geq I_c \geq \hat{I}_c$.

Thus $\lim_{n\to\infty} E(\rho_n) = \lim_{n\to\infty} \hat{E}(z_n) = I_c = \hat{I}_c$. (3.6)

Moreover (3.4) and (3.6) imply that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2 - |\nabla(u_n^2+v_n^2)^{1/2}|^2 dx = 0.$$

Furthermore, it follows from (3.3) that :

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2 dx &= \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla(u_n^2+v_n^2)^{1/2}|^2 dx \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla \rho_n|^2 dx\end{aligned} \tag{3.7}$$

Combining (3.5), (3.6) and (P3), we may suppose (after passing to a subsequence and up to a translation) that there exists $\rho \in H^1(\mathbb{R}^3, \mathbb{R})$ such that $\rho_n \to \rho$ in $H^1(\mathbb{R}^3, \mathbb{R})$. Certainly $\rho \in S_c$ and $E(\rho) = I_c$. Hence ρ is a weak solution of the equation $\Delta\rho(x) - \int_{\mathbb{R}^3} \frac{\rho^2(y)}{|x-y|} dy \rho(x) + p(|x|)\rho(x) + \lambda\rho(x) = 0$. Thus using elliptic regularity theory and maximum principal we obtain that $\rho \in C^1(\mathbb{R}^3)$ and $\rho > 0$. Hence $\rho \in W_c \subset Z_c$ since $I_c = \hat{I}_c$. (3.8)

On the other hand $z = (u, v)$ and $(u^2 + v^2)^{1/2} = \rho$. This follows immediately from the fact that $u_n \to u$ and $v_n \to v$ in any $L^2(B(0, R))$ for any $R > 0$. Then observing that $[(u_n^2 + v_n^2)^{1/2} - (u^2 + v^2)^{1/2}]^2 \leq |u_n - u|^2 + |v_n - v|^2$, we can conclude that $(u_n^2 + v_n^2)^{1/2} \to (u^2 + v^2)^{1/2}$ in $L^2(B(0, R))$ for any $R > 0$. But $(u_n^2 + v_n^2)^{1/2} = \rho_n \to \rho$ in L^2 so we must have $(u^2 + v^2)^{1/2} = \rho$ a.e in $\mathbb{R}^3$. $\|z_n\|_2 = |\rho_n|_2 \to |\rho|_2 = \|z\|_2$. To end the proof, we need to show that $\lim_{n\to\infty} \|\nabla z_n\|_2^2 = \|\nabla z\|_2^2$.

Thanks to (3.7), we have that

$$\lim_{n\to\infty} \|\nabla z_n\|_2^2 = \lim_{n\to\infty} \int |\nabla \rho_n|^2 dx$$

and $\lim \int_{\mathbb{R}^3} |\nabla \rho_n|^2 dx = \int |\nabla \rho|^2 dx$ since $\rho_n \to \rho$ in $H^1(\mathbb{R}^3, \mathbb{R})$. Hence

$$\|\nabla z\|_2^2 \leq \lim_{n\to\infty} \|\nabla z_n\|_2^2 = \|\nabla \rho\|_2^2.$$

Replacing z_n by z in (3.4), we see that $\|\nabla z\|_2^2 \geq |\nabla \rho|_2^2$.
On the other hand z_n converges weakly to z in $H^1(\mathbb{R}^3, \mathbb{C})$, which implies that $z_n \to z$ completing the proof of the orbital stability of standing waves of (1.1).

Proof of Theorem 3.2

Let $z = (u, v) \in Z_c$ and set $\rho = (u^2 + v^2)^{1/2}$. By (P4), we know that $\rho \in W_c$. Furthermore, putting $z_n = z$ in (3.4) and (3.6), we have that

$$\int_{\mathbb{R}^3} \sum_{i=1}^{3} \left(\frac{u\partial_i v - v\partial_i u}{u^2 + v^2} \right)^2 dx = 0. \tag{3.9}$$

On the other hand $\hat{E}(z) = \hat{I}_c$, which implies that there exists a Lagrange multiplier $\alpha \in \mathbb{C}$ such that :

$$\hat{E}'(z)\xi = \frac{\alpha}{2} \int_{\mathbb{R}^3} z\bar{\xi} + \xi\bar{z} dx \text{ for all } \xi \in H^1(\mathbb{R}^3, \mathbb{C}).$$

Putting $\xi = z$, it follows immediately that $\alpha \in \mathbb{R}$ and :

$$\begin{cases} \int_{\mathbb{R}^3} \nabla u \nabla f + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(y) + v^2(y)}{|x-y|} dy f(x) dx - \frac{1}{2}\int_{\mathbb{R}^3} p(|x|) u^2(x) \\ = \alpha \int_{\mathbb{R}^3} u(x) f(x) dx \\ \int_{\mathbb{R}^3} \nabla v \nabla f + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(y) + v^2(y)}{|x-y|} dy f(x) dx - \frac{1}{2}\int_{\mathbb{R}^3} p(|x|) v^2(x) dx \\ = \alpha \int_{\mathbb{R}^3} v(x) f(x) dx. \end{cases} \tag{3.10}$$

for all $f \in H^1(\mathbb{R}^3, \mathbb{R})$. This implies that u and v solve the following elliptic system :

$$\begin{cases} \Delta u(x) - \int_{\mathbb{R}^3} \frac{u^2(y) + v^2(y)}{|x-y|} dy \;\; u(x) + p(|x|) u(x) + \alpha u(x) = 0 \\ \Delta(x) - \int_{\mathbb{R}^3} \frac{u^2(y) + v^2(y)}{|x-y|} dy \;\; v(x) + p(|x|) v(x) + \alpha v(x) = 0 \end{cases} \tag{3.11}$$

Elementary elliptic regularity theory implies that u and $v \in C^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3, \mathbb{R})$.
Let $\Omega = \{x \in \mathbb{R}^3 : u(x) = 0\}$.
Ω is closed since u is continuous. Let us prove that it is also open. Suppose that $x_0 \in \Omega$. Using the fact that $\varphi(x_0) > 0$, we can find an open ball B centered in x_0 such that $v(x) \neq 0$ for any $x \in B$. Thus for $x \in B$:

$$\frac{(u\partial_i v - v\partial_i u)^2}{u^2 + v^2} = [\partial_i(\frac{u}{v})]^2 \frac{v^4}{u^2 + v^2} \qquad \text{for } 1 \leq i \leq 3.$$

Using (3.9), it follows that

$$\int_B |\nabla(\frac{u}{v})|^2 \frac{v^4}{u^2 + v^2} dx = 0. \tag{3.12}$$

Hence $\nabla(\frac{u}{v}) = 0$ on B. This implies that there exists a constant K such that $\frac{u}{v} = K$ on B. But $x_0 \in B$, we then necessarily have that $K \equiv 0$. This shows that Ω is also an open subset of $\mathbb{R}^3$.
The proofs of a) and b) identical.
Now we turn to the characterization of the orbit Z_c :
Let $z = e^{i\sigma} w, \sigma \in \mathbb{R}$, $w \in W_c$. Then $z \in \hat{S}_c$ and $\hat{E}(z) = E(w) = I_c = \hat{I}_c$. Then $\{e^{i\sigma} w : \sigma \in \mathbb{R}, w \in W_c\} \in Z_c$.
Conversely for $z = (u, v) \in Z_c$, set $w = |z|$, then we have $\hat{E}(z) = E(w) = \hat{I}_c = I_c$ and $w \in W_c$.
If $v \equiv 0$, $w = |w| > 0$ on $\mathbb{R}^3$ and so $z = e^{i\sigma} w \in W_c$, where $\sigma = 0$ if $u > 0$ and $\sigma = \pi$ if $u < 0$ on $\mathbb{R}^3$. Otherwise by part b), we certainly have that $v(x) \neq 0$ for all $x \in \mathbb{R}^3$.
In this case, it follows by (3.10) that $\nabla(\frac{u}{v}) = 0$ on $\mathbb{R}^3$.
This implies that we can find a constant $\alpha \in \mathbb{R}$ such that $u \equiv \alpha v$ on $\mathbb{R}^3$. Hence $z = (\alpha + i)v$ and $w = |\alpha + i||v|$.
Let $\theta \in \mathbb{R}$ be such $(\alpha + i) = |\alpha + i| e^{i\theta}$ and let $\varphi = 0$ if $v > 0$ and $\varphi = \pi$ if $v < 0$ on $\mathbb{R}^3$. Setting $\sigma = \theta + \varphi$, we have that $z = (\alpha + i)v = |\alpha + i| e^{i\theta} |v| e^{i\varphi} = w e^{i\varphi}$, where $w \in W_c$.

Acknowledgment The author is very grateful to Stefan Krömer for some illuminating discussions.

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