Homework 1 – Calc Emphasizing Proofs

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P1. Find all numbers x for which

(i) $|x-3| = 8$.
(ii) $|x-3| < 8$.
(iii) $|x+4| < 2$.
(iv) $|x-1| + |x-2| > 1$.
(v) $|x-1| + |x+1| < 2$.
(vi) $|x-1| + |x+1| < 1$.
(vii) $|x-1| \cdot |x+1| = 0$.
(viii) $|x-1| \cdot |x+2| = 3$.

Answer: (i) *It implies that* $x - 3 = \pm 8$ *and then* $x = 11$ *or* $x = -5$.

(ii) *It implies that* $-8 < x - 3 < 8$ *and then* $-5 < x < 11$.

(iii) *It implies that* $-2 < x + 4 < 2$ *and then* $-6 < x < -2$.

(iv) *For* $x > 2$, *we have that* $x - 1 + x - 2 > 1$, *which implies* $x > 2$;
For $x < 1$, *we have that* $1 - x + 2 - x > 1$, *which implies that* $x < 1$;
For $1 \leq x \leq 2$, *we have that* $x - 1 + 2 - x = 1$, *then it is impossible.*
Therefore, we have that $x > 2$ *or* $x < 1$.

(v) *For* $x \geq 1$, *we have that* $x - 1 + x + 1 = 2x \geq 2$, *so there is no* $x \geq 1$ *such that* $|x-1| + |x+1| < 2$;
For $x \leq -1$, *we have that* $1 - x - x - 1 = -2x \geq 2$, *so there is no* $x \leq -1$ *such that* $|x-1| + |x+1| < 2$;
For $-1 < x < 1$, *we have that* $1 - x + x + 1 = 2$, *so there is no* $x \leq (-1, 1)$ *such that* $|x-1| + |x+1| < 2$.
Therefore, there is no $x \leq \mathbb{R}$ *such that* $|x-1| + |x+1| < 2$.

(vi) *If there is some* x_0 *such that* $|x_0 - 1| + |x_0 + 1| < 1$, *then* $|x_0 - 1| + |x_0 + 1| < 2$, *but from (v) there is no* x *such that* $|x-1| + |x+1| < 2$. *Then there is no* x *such that* $|x_0 - 1| + |x_0 + 1| < 1$.

(vii) $|x-1| \cdot |x+1| = 0$ *implies that* $(x-1)(x+1) = 0$. *Then* $x = 1$ *or* $x = -1$.

(viii) $|x-1| \cdot |x+2| = 3$ *implies that* $(x-1)(x+2) = 3$ *or* $(x-1)(x+2) = -3$, *In the case of* $(x-1)(x+2) = 3$, *we have* $x^2 + x - 5 = 0$ *and then* $x = \frac{-1 \pm \sqrt{21}}{2}$; *In the case of* $(x-1)(x+2) = -3$, *we have* $x^2 + x + 1 = 0$ *and the solution is empty. Therefore, we have that* $x = \frac{-1 \pm \sqrt{21}}{2}$. □

P2. Prove the following:

(i) $|xy| = |x| \cdot |y|$.

(ii) $|\frac{1}{x}| = \frac{1}{|x|}$, if $x \neq 0$. (The best way to do this is to remember what is $|x|^{-1}$ is.)

(iii) $\frac{|x|}{|y|} = |\frac{x}{y}|$, if $y \neq 0$.

(iv) $|x - y| \leq |x| + |y|$.

(v) $|x| - |y| \leq |x - y|$.

(vi) $|(|x| - |y|)| \leq |x - y|$.

(vii) $|x+y+z| \leq |x|+|y|+|z|$. Indicate when equality holds, and prove your statement.

Proof. (i) *Since* $(|xy|)^2 = x^2y^2$, $(|x| \cdot |y|)^2 = x^2y^2$, $|xy| \geq 0$ *and* $|x| \cdot |y| \geq 0$, *then* $|xy| = |x| \cdot |y|$.

(ii) *Since* $|\frac{1}{x}| \cdot |x| = |\frac{1}{x} \cdot x| = 1$, *then* $|\frac{1}{x}| = \frac{1}{|x|}$. *Or for* $x > 0$, *we deduce that* $|\frac{1}{x}| = \frac{1}{x}$, $\frac{1}{|x|} = \frac{1}{x}$ *then* $|\frac{1}{x}| = \frac{1}{|x|}$. *For* $x < 0$, *we deduce that* $|\frac{1}{x}| = -\frac{1}{x}$, $\frac{1}{|x|} = -\frac{1}{x}$ *then* $|\frac{1}{x}| = \frac{1}{|x|}$.

(iii) *Since* $|\frac{x}{y}| \cdot |y| = |\frac{x}{y} \cdot y| = |x|$, *then* $\frac{|x|}{|y|} = |\frac{x}{y}|$.

(iv) $|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$.

(v) *Since* $|x| = |y + (x - y)| \leq |y| + |x - y|$, *then* $|x| - |y| \leq |x - y|$.

(vi) *Since* $|x||y| \geq xy$ *then*

$$|(|x| - |y|)|^2 = x^2 + y^2 - 2|x||y| \leq x^2 + y^2 - 2xy = |x - y|^2.$$

Together with $|(|x| - |y|)| \geq 0$ *and* $|x - y| \geq 0$, *we have that* $|(|x| - |y|)| \leq |x - y|$.

(vii) $|x + y + z| = |x + (y + z)| \leq |x| + |y + z| \leq |x| + |y| + |z|$. *When the set of the signs of* x, y, z *does not contain* $\{1, -1\}$, *the equality holds.*

P3. If $a_1, \cdots, a_n \geq 0$, then the "arithmetic mean"

$$A_n = \frac{a_1 + \cdots + a_n}{n}$$

and "geometric mean"

$$G_n = \sqrt[n]{a_1 \cdots a_n}$$

satisfy

$$G_n \leq A_n.$$

(a) Suppose that $a_1 < A_n$. Then some a_i satisfies $a_i > A_n$; for convenience, say $a_2 > A_n$. Let $\bar{a}_1 = A_n$ and $\bar{a}_2 = a_1 + a_2 - \bar{a}_1$. Show that

$$\bar{a}_1\bar{a}_2 \geq a_1a_2.$$

Why does repeating this process enough times eventually prove that $G_n \leq A_n$? When does equality hold in the formula $G_n \leq A_n$?

The reasoning in this proof is related to another interesting proof.

(b) Using the fact that $G_n \leq A_n$ when $n = 2$, prove, by induction on k, that $G_n \leq A_n$ for $n = 2^k$.

(c) For a general n, let $2^m > n$. apply part (b) to the 2^m numbers

$$a_1, \cdots, a_n, \underbrace{A_n, \cdots, A_n}_{2^m - n \text{ times}}$$

to prove that $G_n \le A_n$.

Proof. (a) Since

$$\bar{a}_1\bar{a}_2 - a_1a_2 = A_n(a_1 + a_2 - A_n) - a_1a_2 = (A_n - a_1)(a_2 - A_n) \ge 0,$$

then

$$\bar{a}_1\bar{a}_2 \ge a_1a_2.$$

When $n = 2$, *we have* $G_2^2 = a_1a_2$ *and* $A_2^2 = \frac{a_1^2+a_2^2+2a_1a_2}{4} \ge a_1a_2 \ge G_2^2$. *Since* $G_2 \ge 0$ *and* $A_2 \ge 0$, *then* $A_2 \ge G_2$.

We assume that $A_{n-1} \ge G_{n-1}$. *Then we prove* $A_n \ge G_n$. *In fact, assume that* $a_1 < A_n$ *and* $a_2 > A_n$, *then choose* $\bar{a}_1 = A_n$ *and* $\bar{a}_2 = a_1 + a_2 - \bar{a}_1$. *We see that*

$$\bar{a}_2 + a_3 + \cdots + a_n = (n-1)A_n,$$

therefore, by assumption,

$$A_n^{n-1} = \left[\frac{\bar{a}_2 + a_3 + \cdots + a_n}{n-1}\right]^{n-1} \ge \bar{a}_2a_3\cdots a_n,$$

which implies that

$$\begin{aligned} A_n^n &= A_nA_n^{n-1} \\ &\ge A_n\bar{a}_2a_3\cdots a_n \\ &\ge a_1a_2a_3\cdots a_n = G_n^n. \end{aligned}$$

By induction, we have that

$$A_n \ge G_n, \quad n \in \mathbb{N}.$$

When $a_1 = a_2 = \cdots = a_n$, *we have* $G_n = A_n$.

(b) When $n = 2$, *we have* $G_2^2 = a_1a_2$ *and* $A_2^2 = \frac{a_1^2+a_2^2+2a_1a_2}{4} \ge a_1a_2 \ge G_2^2$. *Since* $G_2 \ge 0$ *and* $A_2 \ge 0$, *then* $A_2 \ge G_2$.

Assume that $A_{2^k} \ge G_{2^k}$ *holds when* $n = 2^k$.

For $n = 2^{k+1}$, *we see that*

$$\begin{aligned} A_{2^{k+1}} &= \frac{a_1 + \cdots + a_{2^{k+1}}}{2^{k+1}} \\ &= \frac{\frac{a_1+\cdots+a_{2^k}}{2^k} + \frac{a_{2^k+1}+\cdots+a_{2^{k+1}}}{2^{k+1}}}{2} \\ &\ge \frac{\sqrt[2^k]{a_1\cdots a_{2^k}} + \sqrt[2^k]{a_{2^k+1}\cdots a_{2^k}}}{2} \\ &\ge \sqrt{\sqrt[2^k]{a_1\cdots a_{2^k}}\,\sqrt[2^k]{a_{2^k+1}\cdots a_{2^k}}} \\ &= G_{2^{k+1}}, \end{aligned}$$

where the first inequality used the $A_{2^k} \ge G_{2^k}$.

By induction, we prove that $G_n \leq A_n$.

(c) For $n \leq 2^m$, we have that

$$A_{2^m} = \frac{a_1 + \cdots + a_n + \underbrace{A_n + \cdots A_n}_{2^m - n}}{2^m}$$

By applying (b), we have

$$\begin{aligned} A_n = \frac{nA_n + \underbrace{A_n + \cdots A_n}_{2^m - n}}{2^m} &= \frac{a_1 + \cdots + a_n + \underbrace{A_n + \cdots A_n}_{2^m - n}}{2^m} \\ &\geq \sqrt[2^m]{a_1 \cdots a_n A_n^{2^m - n}} \\ &= \sqrt[2^m]{G_n^n A_n^{2^m - n}}, \end{aligned}$$

then we have $A_n \geq G_n$. □

P4. The following is recursive definition of a^n:

$$a^1 = a,$$

$$a^{n+1} = a^n \cdot a.$$

Prove, by induction, that

$$(i) \qquad a^{n+m} = a^n \cdot a^m,$$

$$(ii) \qquad (a^n)^m = a^{nm}.$$

(Don't try to be fancy: use either induction on n or induction on m, not both at once.)

Proof. (i) It is obvious that $a^{n+1} = a^n \cdot a$. Assume that $a^{n+m-1} = a^n \cdot a^{m-1}$ holds, then

$$a^{n+m} = a^{n+m-1} \cdot a = a^n \cdot a^{m-1} \cdot a = a^n \cdot a^m.$$

(ii) It is obvious that $(a^n)^1 = a^n$. Assume that $(a^n)^{m-1} = a^{n(m-1)}$ holds, by (i), we have that

$$(a^n)^m = (a^n)^{m-1} \cdot a^n = a^{n(m-1)} \cdot a^n = a^{nm-n+n} = a^{nm}.$$