Homework 2 – Calc Emphasizing Proofs

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P1. Describe the graph of g in terms of the graph of f if

(i) $g(x) = f(x) + c$.
(ii) $g(x) = f(x + c)$. (It is easy to make a mistake here.)
(iii) $g(x) = cf(x)$. (Distinguish the cases $c = 0$, $c > 0$, $c < 0$.)
(iv) $g(x) = f(cx)$.
(v) $g(x) = f(\frac{1}{x})$.
(vi) $g(x) = f(|x|)$.
(vii) $g(x) = |f(x)|$.
(viii) $g(x) = \max(f, 0)$.
(ix) $g(x) = \min(f, 0)$.
(x) $g(x) = \max(f, 1)$.

Answer: (i) The graph of f is moved c units up if $c \geq 0$ or moves $-c$ units down if $c < 0$.
(ii) The graph of f is moved c units left if $c \geq 0$ or moves $-c$ units right if $c < 0$.
(iii) When $c = 0$, the graph of g is the horizontal axis. When $c > 0$, the graph of f expands c times along the vertical axis. When $c < 0$, the graph of f first expands $-c$ times along the vertical axis and then turns as the symmetry about the horizontal axis.
(iv) When $c = 0$, the graph of g is the line $y = f(0)$. When $c > 0$, the graph of f expands $\frac{1}{c}$ times along the horizontal axis. When $c < 0$, the graph of f first expands $-c$ times along the horizontal axis and then turns as the symmetry about the vertical axis.
(v) Replace every point on the graph $(x, f(x))$ by $(x, f(\frac{1}{x}))$.
(vi) Keep the part of the graph of f lying on the right hind of y-axis and then reflected it about the y-axis.
(vii) Keep the positive part of the graph of f and reflected the negative part of the graph of f about the x-axis.
(viii) Cut off the negative part of f into 0.
(ix) Cut off the positive part of f into 0.
(x) Cut off the part of the graph f lying below the line $y = 1$ and replace this part into $y = 1$.

P2. The symbol $[x]$ denotes the largest integer which is $\leq x$. Thus, $[2.1] = [2] = 2$ and $[-0.9] = [-1] = -1$. Draw the graph of the following functions (they are all quite interesting, and several will reappear frequently in other problems).

(i) $f(x) = [x]$.
(ii) $f(x) = x - [x]$.
(iii) $f(x) = \sqrt{x - [x]}$.
(iv) $f(x) = [x] + \sqrt{x - [x]}$.
(v) $f(x) = [\frac{1}{x}]$.
(vi) $f(x) = \frac{1}{[\frac{1}{x}]}$.

P3. (a) Show that the square of the distance from (c, d) to (x, mx) is

$$x^2(m^2 + 1) + x(-2md - 2c) + d^2 + c^2.$$

Using Problem 1-18 to find the minimum of these numbers, show that the distance from (c, d) to the graph of $f(x) = mx$ is

$$|cm - d|/\sqrt{m^2 + 1}.$$

(b) Find the distance from (c, d) to the graph of $f(x) = mx + b$. (Reduce this case to part (a).)

Answer: (a) We denote D by the distance from (c, d) to (x, mx).

$$\begin{aligned} D^2 &= (c-x)^2 + (d-mx)^2 \\ &= c^2 - 2cx + x^2 + d^2 - 2dmx + m^2x^2 \\ &= x^2(m^2+1) + x(-2md - 2c) + d^2 + c^2 \\ &= (m^2+1)\left[\left(x - \frac{dm+c}{m^2+1}\right)^2 - \left(\frac{dm+c}{m^2+1}\right)^2 + \frac{d^2+c^2}{m^2+1}\right]. \end{aligned}$$

For $x = \frac{dm+c}{m^2+1}$, we have that

$$\begin{aligned} D^2_{min} &= (m^2+1)\left[-\left(\frac{dm+c}{m^2+1}\right)^2 + \frac{d^2+c^2}{m^2+1}\right] \\ &= \frac{(d-mc)^2}{m^2+1}, \end{aligned}$$

then

$$D_{min} = \frac{|d-mc|}{\sqrt{m^2+1}}.$$

(b) We denote L by the distance from (c, d) to the graph of $f(x) = mx + b$.

$$\begin{aligned} L^2 &= (c-x)^2 + (d-mx-b)^2 \\ &= c^2 - 2cx + x^2 + (d-b)^2 - 2(d-b)mx + m^2x^2 \\ &= x^2(m^2+1) + x(-2md - 2c + 2mb) + (d-b)^2 + c^2 \\ &= (m^2+1)\left[\left(x - \frac{dm+c-mb}{m^2+1}\right)^2 - \left(\frac{dm+c-mb}{m^2+1}\right)^2 + \frac{(d-b)^2+c^2}{m^2+1}\right]. \end{aligned}$$

For $x = \frac{dm+c-mb}{m^2+1}$, we have that

$$\begin{aligned} L^2_{min} &= (m^2+1)\left[-\left(\frac{dm+c-mb}{m^2+1}\right)^2 + \frac{(d-b)^2+c^2}{m^2+1}\right] \\ &= \frac{(d-mc-b)^2}{m^2+1}, \end{aligned}$$

then

$$L_{min} = \frac{|d-mc-b|}{\sqrt{m^2+1}}.$$

P4. (a) Using Problem 22, show that the number x' and y' indicated in Figure 31 are given by

$$x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y,$$

$$y' = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y.$$

(b) Show that the set of all (x, y) with $(\frac{x'}{\sqrt{2}})^2 - (\frac{y'}{\sqrt{2}})^2 = 1$ is the same as the set of all (x, y) with $xy = 1$.

Answer: (a) We know that x' is the distance from (x, y) to $y = -x$, y' is the distance from (x, y) to $y = x$. Using Problem 22,

$$x' = \frac{|-x-y|}{\sqrt{2}}, \qquad y' = \frac{|x-y|}{\sqrt{2}}.$$

Since x, y are both positive and $x < y$, then

$$x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y, \qquad y' = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y.$$

(b) From (a), we have that

$$1 = \left(\frac{x'}{\sqrt{2}}\right)^2 - \left(\frac{y'}{\sqrt{2}}\right)^2 = \left(\frac{1}{2}x + \frac{1}{2}y\right)^2 - \left(-\frac{1}{2}x + \frac{1}{2}y\right)^2 = xy.$$