Homework 3 – Calc Emphasizing Proofs

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P1. Find the following limits.

(*i*) $\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}$.

(*ii*) $\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x}$.

(*iii*) $\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2}$.

Answer: (*i*) $\frac{1}{2}$; (*ii*) 0; (*iii*) $\frac{1}{2}$.

P2. (a) Prove that if $\lim_{x\to a} f(x) = l$, then $\lim_{x\to a} |f|(x) = |l|$.

(b) Prove that if $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then $\lim_{x\to a} \max(f,g)(x) = \max(l,m)$ and similarly for min.

Answer: (a) Since $\lim_{x\to a} f(x) = l$, *then* $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $\forall x \in (a-\delta, a+\delta)\setminus\{a\}$, *we have that* $|f(x) - l| < \varepsilon$. *Then* $||f|(x) - |l|| \le |f(x) - l| < \varepsilon$, *i.e.* $\lim_{x\to a} |f|(x) = |l|$.

(b)

$$\begin{aligned}\lim_{x\to a}\max(f,g)(x) &= \frac{1}{2}\left(\lim_{x\to a} f(x) + \lim_{x\to a} g(x) + |\lim_{x\to a} f(x) - \lim_{x\to a} g(x)|\right)\\ &= \frac{1}{2}(l + m + |l - m|)\\ &= \max(l,m).\end{aligned}$$

P3. (a) Prove that $\lim_{x\to 0} 1/x$ does not exist, i.e., show that $\lim_{x\to 0} 1/x = l$ is false for every number l.

(b) Prove that $\lim_{x\to 1} 1/(x-1)$ does not exist.

Answer: (a) If there exist some number l *such that* $\lim_{x\to 0} 1/x = l$, *then* $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $\forall x \in (-\delta,\delta)\setminus\{0\}$, *we have that* $|1/x - l| < \varepsilon$. *Then* $|\frac{1}{x}| - |l| \le |1/x - l| < \varepsilon$, *so* $|x| > \frac{1}{|l|+\varepsilon}$, *which contradict with* $\forall x \in (-\delta,\delta)\setminus\{0\}$.

(b) Suppose that $x - 1 = y$, *then* $\lim_{x\to 1} 1/(x-1) = \lim_{y\to 0} 1/y$, *does not exist by (a).*

P4. Find the following limits.

(*i*) $\lim_{x\to\infty}\frac{x+\sin^3 x}{5x+6}$.

(*ii*) $\lim_{x\to\infty}\frac{x\sin x}{x^2+5}$.

(*iii*) $\lim_{x\to\infty}\sqrt{x^2+x} - x$.

(*iv*) $\lim_{x\to\infty}\frac{x^2(1+\sin^2 x)}{(x+\sin x)^2}$.

Answer: (*i*) $\frac{1}{5}$; (*ii*) 0; (*iii*) $\frac{1}{2}$; (*iv*) *Non-existence.*