Homework 4 – Calc Emphasizing Proofs

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P1. Let

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0, \end{cases}$$

and

$$f(y) = \begin{cases} 1 & \text{if} \quad y \neq 0, \\ 0 & \text{if} \quad y = 0. \end{cases}$$

(i) Find $f \circ g$.

Answer: When $x = \frac{1}{k\pi}$, $k \in \mathbb{Z} \setminus \{0\}$, *then* $g(x) = 0$, *so* $f(g(x)) = 0$. *When* $x \neq \frac{1}{k\pi}$ *and* $x \neq 0$, *where* $k \in \mathbb{Z} \setminus \{0\}$, *then* $g(x) = x \sin \frac{1}{x} \neq 0$, *then* $f(g(x)) = 1$. *When* $x = 0$, *by the fact of* $g(0) = 1$, *then* $f(g(0)) = 1$. *Therefore,*

$$f \circ g(x) = \begin{cases} 1 & \text{if} \quad x \neq \frac{1}{k\pi} \ (k \in \mathbb{Z} \setminus \{0\}) \quad \text{or} \quad x = 0, \\ 0 & \text{if} \quad x = \frac{1}{k\pi} \ (k \in \mathbb{Z} \setminus \{0\}). \end{cases}$$

(ii) Find $\lim_{x \to 0} g(x)$ and $\lim_{y \to 0} f(y)$.

Answer:

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$
$$\lim_{y \to 0} f(y) = \lim_{y \to 0} 1 = 1.$$

(iii) Conclude.

Answer: $\lim_{x \to 0} g(x)$ *and* $\lim_{y \to 0} f(y)$ *exist, but* $\lim_{x \to 0} f \circ g(x)$ *does not exist.*

Explanation 1: We observe that

$$\lim_{k \to \infty} \frac{1}{k\pi} = \lim_{k \to \infty} \frac{1}{k\pi + \frac{\pi}{2}} = 0,$$

but

$$\lim_{k \to \infty} f \circ g(\frac{1}{k\pi}) = 0 \quad \text{and} \quad \lim_{k \to \infty} f \circ g(\frac{1}{k\pi + \frac{\pi}{2}}) = 1.$$

Explanation 2: Fix $\varepsilon_0 = \frac{1}{2}$ *and for any* $\delta \in (0, 1)$, *let* $K_\delta = \frac{1}{n\pi} + 1$, *then*

$$0 < \frac{1}{k\pi}, \frac{1}{k\pi + \frac{\pi}{2}} < \delta, \quad \forall k \in \mathbb{N}, \ k \geq K_\delta,$$

but

$$|f \circ g(\frac{1}{k\pi}) - f \circ g(\frac{1}{k\pi + \frac{\pi}{2}})| = 1 > \varepsilon_0,$$

therefore, the limit of $f \circ g$ *does not exist as* $x \to 0$.

(iv) *If* $\lim_{x \to x_0} g(x) = y_0$, $\lim_{y \to y_0} f(y) = l$ *and* f *is continuous at* y_0, *then*

$$\lim_{x \to x_0} f \circ g(x) = l.$$

Answer: Since f is continuous at y_0 and $\lim_{y\to y_0} f(y) = l$, *then* $\lim_{y\to y_0} f(y) = f(y_0) = l$. *Combining with* $\lim_{x\to x_0} g(x) = y_0$, *we have that*

$$\lim_{x\to x_0} f\circ g(x) = f(\lim_{x\to x_0} g(x)) = f(y_0) = l.$$