# Homework 5 – Calc Emphasizing Proofs

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P1. Suppose that $A$ and $B$ are two nonempty sets of numbers such that $x \leq y$ for all $x$ in $A$ and all $y$ in $B$.

($i$) Prove that $\sup A \leq y$ for all $y$ in $B$.

($ii$) Prove that $\sup A \leq \inf B$.

*Answer:* ($i$) *Since* $x \leq y$ *for all* $x$ *in* $A$ *and all* $y$ *in* $B$, *then any* $y$ *in* $B$ *is an upper bound of* $A$. *We know that* $\sup A$ *is the least upper bound of* $A$. *Then we have that* $\sup A \leq y$ *for all* $y$ *in* $B$.

($ii$) *By* ($i$), *we have that* $\sup A \leq y$ *for all* $y$ *in* $B$, *then* $\sup A$ *is a lower bound of* $B$. *We know that* $\inf B$ *is the least upper bound of* $B$. *Then we have that* $\sup A \leq \inf B$.

P2. Let $A$ and $B$ be two nonempty sets of numbers which are bounded above, and let $A+B$ denote the set of all numbers $x+y$ with $x$ in $A$ and $y$ in $B$. Prove that $\sup(A+B) = \sup A + \sup B$.

*Answer: One side, we have that* $x \leq \sup A$, $\forall x \in A$ *and* $y \leq \sup B$, $\forall y \in B$. *Then* $x + y \leq \sup A + \sup B$, $\forall x \in A, y \in B$, *that is,* $\sup A + \sup B$ *is an upper bound of* $A + B$. *Since* $\sup(A+B)$ *is the least upper bound of* $A+B$. *Therefore,* $\sup(A+B) \leq \sup A+\sup B$.

*The other side, for any* $\varepsilon > 0$, *there exist* $x \in A$ *and* $y \in B$ *such that* $\sup A \leq x + \frac{\varepsilon}{2}$ *and* $\sup B \leq y + \frac{\varepsilon}{2}$. *Then* $\sup A + \sup B \geq x + y \geq \sup A + \sup B - \varepsilon$. *By the arbitrary of* $\varepsilon$, *we have that* $\sup A + \sup B \geq x + y \geq \sup A + \sup B$.

*As a consequence, we have that* $\sup(A + B) = \sup A + \sup B$.

P3. ($i$) Consider a sequence of closed intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2], \cdots$ Suppose that $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all $n$. Prove that there is a point $x$ which is in every $I_n$.

($ii$) Show that this conclusion is false if we consider open intervals instead of closed intervals.

*Answer:* ($i$) *Let* $A = \{a_1, a_2, \cdots\}$ *and* $B = \{b_1, b_2, \cdots\}$. *Since* $I_n = [a_n, b_n]$, $a_n \leq a_{n+1}$ *and* $b_{n+1} \leq b_n$ *for all* $n$, *then* $I_1 \supset I_2 \supset \cdots \supset I_n$ *and* $\sup A = a_n \leq b_n = \inf B$. *Take* $x = \frac{a_n+b_n}{2}$, *we know that* $x \in I_n$ *for every* $n$.

($ii$) *Let* $a_1 = a_2 = \cdots = a_n = 0$ *and* $b_n = \frac{1}{2^n}$. *If there exists* $a$ *such that*

$$a \in J_n := (0, b_n), \quad n \in \mathbb{N},$$

*then*

$$a > 0 \quad \text{and} \quad a < \frac{1}{2^n}, \quad n \in \mathbb{N}$$

*From* $a < \frac{1}{2^n}$ *for any* $n \in \mathbb{N}$, *we have that* $a \leq 0$, *which is impossible.*