

On the necessity of the assumptions used to prove Hardy-Littlewood and Riesz Rearrangement Inequalities

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Abstract

We prove that supermodularity is a necessary condition for the generalized Hardy-Littlewood and Riesz rearrangement inequalities. We also show the necessity of the monotonicity of the kernels involved in the Riesz-type integral.

Keywords and phrases: rearrangement, inequality, supermodular integrand .

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1 Introduction

Let $F : \mathbb{R}_+^m \rightarrow \mathbb{R}$ be a Borel measurable function. We say that F is supermodular if:

$$F(y + he_i + ke_j) + F(y) \geq F(y + he_i) + F(y + ke_j) \quad (1.1)$$

for every $i, j \in \{1, \dots, m\}$, $i \neq j$, $k, h > 0$,

where $y = (y_1, \dots, y_m)$, and e_i denotes the standard basis vector in \mathbb{R}^m , $i = 1, \dots, m$.

This condition has been first analysed in the context of integral inequalities by G. Lorentz [19].

Note that if F is C^2 , (1.1) is equivalent to

$$\frac{\partial^2 F}{\partial x_i \partial x_j}(x) \geq 0, \quad \text{for every } i, j \in \{1, \dots, m\}, i \neq j. \quad (1.2)$$

The supermodularity condition has been widely used in functional analysis. It was the main assumption to prove the generalized Hardy-Littlewood rearrangement inequality,

$$\int_X F(u_1(x), \dots, u_m(x)) d\mu(x) \leq \int_X F(u_1^*(x), \dots, u_m^*(x)) d\mu(x), \quad (1.3)$$

Here u_i are nonnegative measurable functions that vanish at infinity and u_i^* are their symmetric decreasing rearrangements. The integral is taken over $X = \mathbb{R}^n$, \mathbb{H}^n or \mathcal{S}^n . (1.3)

has been studied by many authors [17, 19, 11, 6, 7]. Its main applications are economics [11], chemistry [24], and nonlinear optics where the profile of stable electromagnetic waves traveling along a planar waveguide are given by the ground states of the energy functional

$$E(u) := \frac{1}{2} \int_{\mathbb{R}} \left(u'^2 - G(|x|, u) \right) dx, \quad (1.4)$$

under the constraint $\|u\|_2 = c > 0$. A crucial step to prove that (1.4) admits an even ground state $u = u(x)$ which is decreasing for $x > 0$, is (1.3), [15].

In [16], we showed that the ground state of the $m \times m$ elliptic system

$$(\mathbf{S}) \begin{cases} \Delta u_1 + \lambda_1 u_1 + g_1(u_1, \dots, u_m) = 0 \\ \dots \\ \Delta u_m + \lambda_m u_m + g_m(u_1, \dots, u_m) = 0 \end{cases},$$

where $g_i = (\partial G / \partial u_i)$, ($i = 1, \dots, m$), exists when G is supermodular, and satisfies some further conditions.

Notice that supermodularity condition is equivalent to the cooperativity of (\mathbf{S}) . As proved by W.C. Troy [24], the latter condition is also necessary for the existence of solutions of (\mathbf{S}) . In [10], G. Carlier viewed the generalized Hardy–Littlewood rearrangement inequality as an optimal transportation problem. He showed that the left-hand side of (1.3) achieves its maximum (i.e. the cost is minimized) under the supermodularity assumption. In this paper, we prove, among other things, that (1.1) is necessary for the inequality (1.3) to hold.

The generalized Riesz rearrangement inequality is

$$\begin{aligned} & \int \cdots \int F(u_1(x_1), \dots, u_m(x_m)) \prod_{i < j} K_{ij}(d(x_i, x_j)) dx_1 \cdots dx_m \\ & \leq \int \cdots \int F(u_1^*(x_1), \dots, u_m^*(x_m)) \prod_{i < j} K_{ij}(d(x_i, x_j)) dx_1 \cdots dx_m, \end{aligned} \quad (1.5)$$

where d denotes distance and the functions K_{ij} are decreasing.

It is closely related to the Brunn–Minkowski inequality of convex geometry. Notice that the integral in (1.5) can represent a physical interaction potential.

The most relevant case is $K_{ij}(d(x_i, x_j)) = \omega_{ij}(|x_i - x_j|)$ where ω_{ij} are nonnegative nonincreasing functions. (1.5) is then equivalent to

$$\begin{aligned} & \int \cdots \int F(u_1(x_1), \dots, u_m(x_m)) \omega_{ij}(|x_i - x_j|) dx_1 \cdots dx_m \\ & \leq \int \cdots \int F(u_1^*(x_1), \dots, u_m^*(x_m)) \omega_{ij}(|x_i - x_j|) dx_1 \cdots dx_m. \end{aligned} \quad (1.6)$$

(1.5) and (1.6) have been studied in [22, 23, 3, 4, 5, 1, 20, 13, 21, 12, 25, 7]. All these results use the supermodularity of F . Among the numerous applications of (1.6), let us mention that it was extremely useful to prove the existence and uniqueness of the minimizing solution

of Choquard's equation [18]. In the special case that F is a product and $m = 2$, (1.6) hold on the standard spheres and hyperbolic spaces [2, 3] and it still contains the isoperimetric inequality as a limit. In this paper we will show (see Proposition 3.2 below) that (1.1) is necessary for (1.6) to hold. Notice that, in [7], cases of equality of (1.3) and (1.6) were established under the strict supermodularity condition, that is to say (1.1) with the strict inequality sign. We will also prove that such a condition is inescapable to establish cases of equality in (1.3) and (1.6) (see Remark 1 below).

2. Notation and preliminaries

We fix $n \in \mathbb{N}$. Let μ the Lebesgue measure on \mathbb{R}^n . If $x \in \mathbb{R}^n$ and $r > 0$, let $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$. For any set $M \subset \mathbb{R}^n$, let $\mathbf{1}_M$ denote its characteristic function. By \mathcal{M} we denote the set of all measurable functions on \mathbb{R}^n . For a Borel measurable function $F : \mathbb{R}_+^m \rightarrow \mathbb{R}$, nonincreasing functions $w_{ij} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and nonnegative $u_i \in \mathcal{M}$, we study the following generalized Hardy–Littlewood type functional,

$$I(u_1, \dots, u_m) = \int_{\mathbb{R}^n} F(u_1(x), \dots, u_m(x)) dx, \quad (2.1)$$

and the generalized Riesz type functional.

$$J(u_1, \dots, u_m) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} F(u_1(x_1), \dots, u_m(x_m)) \omega_{ij}(|x_i - x_j|) dx_1 \cdots dx_m. \quad (2.2)$$

If $f \in \mathcal{F}$, let f^* denote its *Schwarz symmetrization* which is the unique lower continuous function which is radially symmetric, radially nonincreasing and such that $\mu\{a < f \leq b\} = \mu\{a < f^* \leq b\}$ for all numbers $\inf f < a < b$ (see [1]). Notice that if $f = \mathbf{1}_M$, where M is Lebesgue measurable with finite measure, then $f^* = \mathbf{1}_{B_R(0)}$ where R is chosen such that $\mu(B_R(0)) = \mu(M)$. Accordingly, a function $u \in \mathcal{F}$ is called *Schwarz symmetric* if it is radial and radially decreasing. We say that it is strictly Schwarz symmetric if it is radial and strictly radially decreasing.

3. Results

Proposition 3.1 (*Necessity of the supermodularity condition in the generalized Hardy–Littlewood inequality*)

Let $F : (\mathbb{R}_+)^m \rightarrow \mathbb{R}$ be a Borel measurable function which vanishes on hyperplanes, that is, $F(y_1, \dots, y_m) = 0$ if $y_k = 0$ for some $k \in \{1, \dots, m\}$. If

$$\int_{\mathbb{R}^n} F(u_1(x), \dots, u_m(x)) dx \leq \int_{\mathbb{R}^n} F(u_1^*(x), \dots, u_m^*(x)) dx,$$

for any $(u_1, \dots, u_m) \in \mathcal{F}^m$ then F satisfies (1.1).

Proof: Suppose that (1.1) is not true. Then there exist $i, j \in \{1, \dots, m\}$, $i \neq j$, $0 \leq a < b$, $0 \leq c < d$ and $m - 2$ nonnegative numbers α_l , ($l \in \{1, \dots, m\} \setminus \{i, j\}$), such that:

$$\begin{aligned} & F(\alpha_1, \dots, a, \dots, d, \dots, \alpha_{m-2}) + F(\alpha_1, \dots, b, \dots, c, \dots, \alpha_{m-2}) \\ & > F(\alpha_1, \dots, b, \dots, d, \dots, \alpha_{m-2}) + F(\alpha_1, \dots, a, \dots, c, \dots, \alpha_{m-2}). \end{aligned} \quad (3.1)$$

Now let E and F be two measurable sets such that $E \cap F = \emptyset$, $\mu(F) = \mu(E) < \infty$, and $u_i = a\mathbf{1}_E + b\mathbf{1}_F$, $u_j = c\mathbf{1}_F + d\mathbf{1}_E$, and $u_l = \alpha_l(\mathbf{1}_F + \mathbf{1}_E)$ for $l \in \{1, \dots, m\} \setminus \{i, j\}$. Then $u_i^* = b\mathbf{1}_{B_r(0)} + a\mathbf{1}_A$, where r is chosen such that $\mu(B_r(0)) = \mu(F) = \mu(E)$, and A is the annulus $B_{r'} \setminus \overline{B_r}$ and $\mu(A) = \mu(F) = \mu(E)$. It follows that $u_j^* = d\mathbf{1}_{B_r(0)} + c\mathbf{1}_A$, $u_l^* = \alpha_l\mathbf{1}_{B_{r'}(0)}$, and

$$\begin{aligned} & \int F(u_1(x), \dots, u_n(x)) dx \\ & = [F(\alpha_1, \dots, a, \dots, d, \dots, \alpha_{m-2}) + F(\alpha_1, \dots, b, \dots, c, \dots, \alpha_{m-2})]\mu(E), \\ & \int F(u_1^*(x), \dots, u_n^*(x)) dx \\ & = [F(\alpha_1, \dots, b, \dots, d, \dots, \alpha_{m-2}) + F(\alpha_1, \dots, a, \dots, c, \dots, \alpha_{m-2})]\mu(E). \end{aligned}$$

Hence we have in view of (3.1),

$$\int F(u_1(x), \dots, u_n(x)) dx > \int F(u_1^*(x), \dots, u_n^*(x)) dx,$$

a contradiction. ■

Remark 1: Following the same approach we can easily conclude that the strict supermodularity assumption, that is, (1.1) with strict inequality sign, is necessary to establish cases of equality. In [7], (1.3) was proven under the supermodularity assumption (1.1) and an integrability condition, and cases of equality were established assuming (1.1) with strict inequality sign.

Proposition 3.2: (*Necessity of supermodularity assumption in Riesz-type integral*)

Let $j : (0, +\infty) \rightarrow \mathbb{R}$ be a function which is nonincreasing and not identically equal to zero, and such that

$$\lim_{r \rightarrow +\infty} r^{n-1}j(r) = 0. \quad (3.2)$$

Suppose that $\Psi : [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$ is a Borel measurable function satisfying

$$\Psi(u, 0) = \Psi(0, v) = 0 \quad \forall u \geq 0, v \geq 0. \quad (3.3)$$

Finally assume that for all nonnegative functions $f, g \in L^\infty(\mathbb{R}^n)$ with compact support there holds

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(f(x), g(y))j(|x-y|) dx dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(f^*(x), g^*(y))j(|x-y|) dx dy. \quad (3.4)$$

Then Ψ is supermodular, that is,

$$\Psi(a + a', b + b') - \Psi(a + a', b) - \Psi(a, b + b') + \Psi(a, b) \geq 0 \quad \forall a, a', b, b' \in [0, +\infty). \quad (3.5)$$

Proof: In view of (3.2), and since j is non-increasing and nontrivial, j is non-negative, too, and there exist positive numbers ε and t_0 with $t_0 > 2\varepsilon$ and such that

$$j(t) < j(s) \quad \text{whenever } 0 \leq s \leq \varepsilon \text{ and } |t - t_0| \leq \varepsilon. \quad (3.6)$$

We fix $z \in \mathbb{R}^n$ with $|z| = t_0$, and $\rho > t_0 + \varepsilon$.

Now let $a, a', b, b' \in [0, +\infty)$. Then, if $f = a\mathbf{1}_{B_R(0)} + a'\mathbf{1}_{B_\varepsilon(0)}$, and $g = b\mathbf{1}_{B_\rho(0)} + b'\mathbf{1}_{B_\varepsilon(z)}$, where $R > \varepsilon$, we have that $f = f^*$, and $g^* = b\mathbf{1}_{B_\rho(0)} + b'\mathbf{1}_{B_\varepsilon(0)}$. Using (3.3), a short computation shows that (3.4) is equivalent to

$$\begin{aligned} 0 &\leq \{\Psi(a + a', b + b') - \Psi(a + a', b) - \Psi(a, b + b') + \Psi(a, b)\} \cdot \\ &\quad \cdot \int_{B_\varepsilon(0)} \left[\int_{B_\varepsilon(0)} j(|x - y|) dy - \int_{B_\varepsilon(z)} j(|x - y|) dy \right] dx \\ &+ (\Psi(a, b + b') - \Psi(a, b)) \int_{B_R(0)} \left[\int_{B_\varepsilon(0)} j(|x - y|) dy - \int_{B_\varepsilon(z)} j(|x - y|) dy \right] dx. \end{aligned} \quad (3.7)$$

Set

$$H(x) := \int_{B_\varepsilon(0)} j(|x - y|) dy, \quad x \in \mathbb{R}^n.$$

Since j is nonnegative and nonincreasing, H is nonnegative, radial and radially nonincreasing, that is we can write $H(x) =: h(|x|)$, with h nonincreasing. Since j satisfies (3.2), we also have $\lim_{t \rightarrow \infty} t^{n-1}h(t) = 0$. Setting

$$I(t) := \int_{B_t(0)} H(x) dx - \int_{B_t(z)} H(x) dx, \quad (t > 0),$$

it follows that I is nonnegative, and moreover, (3.6) implies that

$$I(\varepsilon) > 0. \quad (3.8)$$

Finally, if $t > t_0$, then

$$I(t) \leq \int_{B_{t+t_0}(0) \setminus B_{t-t_0}(0)} H(x) dx = n\omega_n \int_{t-t_0}^{t+t_0} s^{n-1}h(s) ds,$$

where ω_n denotes the volume of the unit ball in \mathbb{R}^n . Hence

$$\lim_{t \rightarrow \infty} I(t) = 0. \quad (3.9)$$

Now inequality (3.7) becomes

$$\begin{aligned} 0 &\leq \{\Psi(a + a', b + b') - \Psi(a + a', b) - \Psi(a, b + b') + \Psi(a, b)\} I(\varepsilon) \\ &+ (\Psi(a, b + b') - \Psi(a, b)) I(R). \end{aligned}$$

Sending $R \rightarrow +\infty$, we obtain

$$0 \leq \{\Psi(a + a', b + b') - \Psi(a + a', b) - \Psi(a, b + b') + \Psi(a, b)\} I(\varepsilon), \quad (3.10)$$

and (3.5) follows from (3.8). ■

Remark 2: Proposition 3.2 tells us in particular that Ψ is supermodular if (3.4) holds for $j = \mathbf{1}_{B_R(0)}$, with some $R > 0$.

Proposition 3.3. *Let $\Psi : [0, +\infty) \times [0, +\infty)$ be nontrivial, and satisfies (3.5) and (3.3), and let $h \in C(\mathbb{R}^n)$. Assume that there holds*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(f(x), g(y)) h(x - y) dx dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(f^*(x), g^*(y)) h(x - y) dx dy \quad (3.11)$$

for all bounded nonnegative functions f and g with compact support. Then h is radial and radially nonincreasing, that is, it can be written as $j(|x|) = h(x)$, ($x \in \mathbb{R}^n \setminus \{0\}$), where $j : (0, +\infty) \rightarrow \mathbb{R}$ is a nonincreasing function.

Proof : By choosing $a = b = 0$ in (3.5) and taking into account (3.3), we have that Ψ is nonnegative, and since Ψ is nontrivial, there are numbers $a > 0$, $b > 0$ such that $\Psi(a, b) > 0$. Let $z_1, z_2 \in \mathbb{R}^n$ with $0 < |z_1| < |z_2|$, and set $R := (|z_1| + |z_2|)/2$. Let $f = a(\mathbf{1}_{B_R(0)} + \mathbf{1}_{B_\varepsilon(z_2)} - \mathbf{1}_{B_\varepsilon(z_1)})$, and $g = b\mathbf{1}_{B_\varepsilon(0)}$, where $0 < \varepsilon < (|z_2| - |z_1|)/2$. Then $g = g^*$ and $f^* = a\mathbf{1}_{B_R(0)}$. Hence (3.11) gives

$$0 \leq \Psi(a, b) \left\{ \int_{B_\varepsilon(z_1)} \int_{B_\varepsilon(0)} h(x - y) dx dy - \int_{B_\varepsilon(z_2)} \int_{B_\varepsilon(0)} h(x - y) dx dy \right\}.$$

Sending $\varepsilon \rightarrow 0$ and taking into account $\Psi(a, b) > 0$ we find that $h(z_1) \geq h(z_2)$. Since h is continuous, this implies $h(x) \geq h(y)$ iff $|x| \leq |y|$. Hence h is radial and radially nonincreasing as claimed. ■

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