# On a new Class of Variational Problems

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**Abstract**

We prove some fractional rearrangement inequalities on the lattice $h\mathbb{Z}$ with mesh size $h > 0$. We then discuss some applications of our results in some problems related to a class of discrete nonlinear Schrödinger equations.

## 1 Introduction

Fractional differential equations involving the fractional Laplacian $(-\Delta)^s, 0 < s < 1$, arise in many fields; medicine, geology, hydrology, mathematical physics and mathematical biology;[2], [3], [5], [9] and references therein. The model case of nonlinear fractional Schrödinger equations describing the above problems is:

$$\left.\begin{array}{l} i\partial_t\Phi(t,x) + (-\Delta)^s\Phi(t,x) + f(|x|, |\Phi(t,x)|) = 0 \\ \Phi(0,x) = \Phi_0(x). \end{array}\right\} \tag{1.1}$$

There are many interesting underlying problems related to (1.1), especially the study of existence and uniqueness of the solutions. The ones which are of particular interest are the, so called, standing waves, i.e, $\Phi(t,x) = e^{-i\lambda t}u(x)$. Such $\Phi$ solves (1.1) if and only if $u$ is a solution of the following fractional elliptic equation:

$$(-\Delta)^s u + f(|x|, |u|) + \lambda u = 0, \tag{1.2}$$

where $\lambda$ is a Lagrange multiplier.
Ground state solutions of (1.2) are obtained by minimizing the following fractional constrained variational problem:

$$I_c = \inf\{E(u) : u \in S_c\} \tag{1.3}$$

$$E(u) = \frac{1}{2}|\nabla_s u|_2^2 - \int_{\mathbb{R}^N} F(|x|, u)dx,$$

$$|\nabla_s u|_2^2 = C_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy,\ C_{N,s} = (2^{s-1}s/\pi^{N/2}\,\Gamma(\frac{N+s}{2})/\Gamma(1-\frac{s}{2}))$$

$$F(r,t) = \int_0^t f(r,p)dp \quad \text{and} \quad S_c = \left\{u \in H^s(\mathbb{R}^N))\int_{\mathbb{R^N}} u^2 = c^2\right\}$$

Solutions of (1.3) are the best candidates to guarantee the orbital stability of the corresponding standing waves, which is one of the most important properties that one has to investigate for (1.1) due to its tight connections to applications of this fractional nonlinear Schrödinger equation.

(1.3) constitutes in itself a branch of nonlinear analysis as shown by the numerous articles dedicated to this topic during the last decades. In many relevant cases, it is crucial to derive some qualitative and quantitative properties of the minimizers of (1.3) before studying their orbital stability. However in some situations, one has to model (1.3) and then to numerically get the desired informations: This is the general scheme to achieve this goal.

**Step 1**: First functions and functionals are discretized by replacing $\mathbb{R}^N$ by centered balls in zero, by replacing derivatives by finite differences and by restricting the function spaces to finite dimensional spaces.

**Step 2**: The next step consists of analyzing the solutions of (1.3) of finite difference / finite element problems; apriori bounds, asymptotics, symmetry, radiality...

**Step 3**: In the last step, algorithms are designed to compute numerical approximations of solutions of the discretized problems.

In this paper, we consider a one-dimensional lattice $h\mathbb{Z}$ with a mesh size $h > 0$. We denote $x_m = hm$ with $m \in \mathbb{Z}$ and $\Phi_h : \mathbb{R} \times h\mathbb{Z} \to \mathbb{C}$. Then (1.1) becomes in the discrete setting:

$$\begin{cases} i\,\frac{d}{dt}\Phi_h(t,x_m) = h\sum_{n\neq m}\frac{\Phi_h(t,x_m)-\Phi_h(t,x_n)}{|x_m-x_n|^{1+2s}} + \\ f(|x_m|,|\Phi_h(x_m)|) \\ \Phi_h(0,x_m) = \Phi_h^0(x_m). \end{cases} \tag{1.4}$$

Here the fractional power $s$ can also be interpreted as a fixed parameter controlling the decay behavior of the lattice interaction; [2], [9], [10].

In [2], the authors have considered the cubic nonlinear Schrödinger equation (1.4) in the special case $f(r,t) = |t|^2 t; N = 1$. More precisely, they have studied (see 2.1):

$$\begin{cases} i\,\frac{d}{dt}u_h(t,x_m) = \frac{1}{\beta(h)}\sum_{n\neq m} J_{|n-m|}\,[u_h(t,x_m)-u_h(t,x_n)] \\ \pm|u_h(t,x_m)|^2\,u_h(t,x_m) \\ u_h(0,x_m) = u_h(x_m), \end{cases}$$

where $x_m = mh$, $J_n = |n|^{1-2s}$ and $\beta(h) = h^{2s}$.

(1.4) can also be viewed as the model of quantum particles on a lattice with repulsive or self-interactions (depending on the sign of $f$), [2]. When $h \to 0^+$, one does expect that $\Phi_h$ tends in some sense to the solution $\Phi(t,x)$ of (1.1).

When this happens, the justification that (1.4) models perfectly (1.1) is excellent. These kind of simulations are very complicated because of the nonlocal properties of the operator involved. However, some promising progress has been recently made in this direction [3].

Now as indicated in Step 1 above, let $\Omega$ be a centered interval in zero and consider a regular step size ($x_m = hm, x_{m+1} - x_m = h$).
Then discrete standing waves of (1.4) solve the following problem:

$$\sum_{x \in \Omega_h} \frac{u(x+h) + u(x-h) - 2u(x)}{h^{1+2s}} + f(|x|, |u|) + \lambda u = 0. \tag{1.5}$$

The corresponding discretized energy functional is :

$$J_h(u) = \sum_{x \in \Omega_h} \{\frac{1}{2} |\nabla_h^s u|_2 - F(|x|, u)\} h,$$

where $\Omega_h$ consists of the points of regular mesh of steps $h$ that belong to the centered interval $\Omega$ and:

$$|\nabla_h^s u|_2^2 = \sum_{k,\ell} \frac{|u(x_k) - u(x_\ell)|^2}{|k-\ell|^{1+2s} h^{2s}}. \tag{1.6}$$

Then an equivalent formulation of the discretized constrained variational problem is:

$$I_h^c = \inf \left\{ E_h(u_h) : \sum_{m \in \mathbb{Z}} u_h^2(x_m) h = c^2 \right\} \tag{1.7}$$

$$E_h(u_h) = \frac{1}{2} |\nabla_h^s u_h|_2^2 h - \sum_{m \in \mathbb{Z}} F(|x_m|, u_h(x_m)) h.$$

for $u_h$ a lattice function defined on $h\mathbb{Z}$. The more one knows about the qualitative properties of solution of (1.7), the more efficient and less difficult is the design of algorithms.
Very recently, the author has established optimal assumptions under which all the minimizers of the continuous constrained variational problem (1.3) are Schwarz symmetric (i.e radial and radially decreasing); [3.4]. His method hinges on the following rearrangement inequalities:

$$|\nabla_s u^*|_2 \leq |\nabla_s u|_2 \tag{1.8}$$

$$|u|_2 = |u^*|_2 \tag{1.9}$$

$$\int_{\mathbb{R}^N} F(|x|, u) dx \leq \int_{\mathbb{R}^N} F(|x|, u^*) dx \tag{1.10}$$

where $u^*$ is the Schwarz rearrangement of $u$, [6].

The main goal of the present paper is to extend the results of [3] to the discrete case. The key step to reach this objective is to prove (1.8) to (1.10) in this setting:

$$\sum_{k,\ell\in\mathbb{Z}} \frac{|u_h^*(x_k)-u_h^*(x_\ell)|^2}{|k-\ell|^{1+2s}} \leq \sum_{k,\ell\in\mathbb{Z}} \frac{|u_h(x_k)-u_h(x_\ell)|^2}{|k-\ell|^{1+2s}} \tag{1.11}$$

$$\sum_{k\in\mathbb{Z}} u_h^2(x_k) = \sum_{k\in\mathbb{Z}} (u_h^*)^2(x_k) \tag{1.12}$$

$$\sum_{k\in\mathbb{Z}} F(|x_k|, u_h(x_k)) \leq \sum_{k\in\mathbb{Z}} F(|x_k|, u_h^*(x_k)) \tag{1.13}$$

$(u_h^*)$ denotes the discrete Schwarz symmetrization of $u_h$, [Definition 2.3]. Amazingly the situation in the discrete setting is very intricating and Challenging. This is due to the appearance of some unexpected phenomena in this kind of problems. In fact Mc Kenna and Reichel have proved in [7] that critical points of a class of discretized variational problems do not generally inherit the same symmetry properties as the critical points of the corresponding continuous problems. More precisely, they were able to show that unlike the continuous case, there are spurious situations in the discrete one, i.e, solutions with no relation to the ones in the continuous setting.
In this work, we will show that such situations cannot occur in our context thanks to the rearrangement inequalities (1.11) to (1.13). Moreover we will establish cases of equality in (1.11) and (1.13). Therefore, we are able to determine hypotheses on $F$ and $s$ for which all the minimizers of (1.7) are Schwarz symmetric (Theorem 4.1). Proving that solutions of the discretized constrained variational problem (1.7) inherit symmetry and monotonocity properties is extremely important for the design of numerical Scheme; [1]. It also implies that we need only to solve numerically these problems on a quarter region instead of the full one. this considerably cuts down the computational cost.
Our paper is organized as follows. In the next section, we give some definitions and preliminary results. In section 3, we will prove (1.11) to (1.13). In the last section, we will show that these inequalities are extremely helpful to prove that all the minimizers of (1.7) are Schwarz symmetric.

From now on $h$ is a fixed stepsize and for $m \in \mathbb{Z}$ $x_m = hm$.

# 2 Notations and Preliminaries

## 2.1 Discrete function spaces

**Definition 2.1**

For sequences $u_h, v_h : h\mathbb{Z} \to \mathbb{R}$, we define:

$$\begin{aligned} (u_h, v_h)_{L_h^2} &= h \sum_{m\in\mathbb{Z}} u_h(x_m) v_h(x_m), \\ \|u_h\|_{L_h^2}^2 &= h \sum_{m\in\mathbb{Z}} u_h^2(x_m). \end{aligned}$$

And more generally for $1 \leq p < \infty$, we define:

$$\|u_h\|_{L_h^p} = (h \sum_{m \in \mathbb{Z}} |u_h(x_m)|^p)^{1/p} \tag{2.2}$$

$$\begin{aligned} L_h^p &= \{u_h \in \mathbb{R}^{h\mathbb{Z}}, \|u\|_{L_h^p} < +\infty\} \text{ is a complete Banach space} \\ \mathbb{R}^{h\mathbb{Z}} &= \{f : h\mathbb{Z} \to \mathbb{R}\}. \end{aligned}$$

For $u_h \in L_h^2$, we define its Fourier transform $\hat{u}_h : [-\pi, \pi] \to \mathbb{C}$ by

$$\hat{u}_h(k) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} u_h(x_m) e^{-imk}.$$

Since $u_h \in L_h^2$, it follows that $\hat{u}_h \in L^2([-\pi, \pi])$ and that

$$u_h(x_m) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{u}_h(k) e^{imk} dk.$$

Using Parseval's identity, we obtain:

$$(u_h, v_h)_{L_h^2} = h \int_{-\pi}^{\pi} \hat{u}_h(k) \hat{v}_h(k) dk.$$

Thanks to this observation, we can introduce the following fractional Sobolev norm for lattice functions $u_h \in L_h^2$: Let $0 \leq s \leq 1$ be given, we define $\|u_h\|_{H_h^s}$ for $u_h \in L_h^2$ by setting

$$\|u_h\|_{H_h^s}^2 = h \int_{-\pi}^{\pi} (1 + h^{-2s} |k|^{2s}) |\hat{u}_h(k)| dk. \tag{2.2}$$

Obviously $\|u_h\|_{H_h^0} = \|u_h\|_{L_h^2}$ and $\|u\|_{H_h^s} < \infty$ for any $u_h \in L_h^2$.

**Definition 2.2** The $L_h^2$ norm of the fractional gradient of a lattice function $u_h$ is defined by:

$$\|\nabla_s u_h\|_{L_h^2}^2 = \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \frac{|u_h(x_k) - u_h(x_\ell)|^2}{|x_k - x_\ell|^{1+2s}} \frac{1}{h^{2s+1}} \tag{2.3}$$

## 2.2 Discrete functional inequalities

First let us recall, for the convenience of the reader, that Sobolev embeddings are still valid in the discrete setting. In our context, we will need the following ones: For $s < \frac{1}{2}$, we have:

$H_h^s$ is continuously embedded in $L_h^{p+2}$ for $p < 4s$, (2.4)

$H_h^s(\widetilde{\mathbb{Z}})$ is compactly embedded in $L_h^{\ell+2}(\widetilde{\mathbb{Z}})$ for $p < 4s$ and $\widetilde{\mathbb{Z}}$ is a bounded lattice of $\mathbb{Z}$. (2.5)

**Discrete fractional Gagliardo-Nirenberg Inequality**:

Following the proof of [Lemma 3.2, 2], we can easily prove that:

$$\|u_h\|_{L_h^{\ell+2}} \leq K \|\nabla_s u_h\|_{L_h^2}^{\theta} \|u\|_{L_h^2}^{1-\theta} \tag{2.6}$$

for any $\ell < 4s, \theta = \dfrac{\ell}{2s(\ell+2)}$

## 2.3 Schwarz symmetrization in $h\mathbb{Z}$

**Definition 2.3**: If $u_h : h\mathbb{Z} \to \mathbb{R}_+$ is bounded from above, the discrete Schwarz symmetrization of $u_h$ is the unique function $u_h^* : h\mathbb{Z} \to \mathbb{R}_+$ such that:

1. For all $k \geq 0 : u_h^*(x_k) \geq u_h^*(x_{-k}) \geq u_h^*(x_{k+1})$.

2. For all $t \in \mathbb{R} : \#\{k \in \mathbb{Z} : u_h^*(x_k) > t\} = \#\{k \in \mathbb{Z} : u_h(x_k) > t\}$.

We can define explicity $u_h^*$ by the following formula:

$$u_h^*(x_k) = \begin{cases} \sup\{t \in \mathbb{R} : \#\{\ell \in \mathbb{Z} : u_h(x_\ell) > t\} \leq 2|k| + 1 & \text{if } k \leq 0 \\ \sup\{t \in \mathbb{R} : \#\{\ell \in \mathbb{Z} : u_h(x_\ell) > t\} \leq 2k & \text{if } k \geq 0 \end{cases}$$

The construction of $u_h^*$ goes thus by taking for $u_h^*(0)$ the maximum value of $u_h$, for $u_h^*(h)$ the second largest value of $u_h$, $u_h^*(-h)$ the third one and so on.
**Definition 2.4** A function $u_h : h\mathbb{Z} \to \mathbb{R}_+$ is admissible if $\#\{\ell \in \mathbb{Z} : u_h(x_\ell) \geq t\} < \infty$ for all $t > 0$.
**Remark**: If $u_h \in L_h^p$ for $1 < p < \infty$ and u is non$-$negative, then u is admissible.
**Lemma 2.5** For every $t > 0$ then:

$$\#\{k \in \mathbb{Z}, u_h(x_k) = t\} = \#\{k \in \mathbb{Z} : u_h^*(x_k) = t\}.$$

**Proposition 2.6** (Cavalieri's principle in the discrete setting)
Let $f : \mathbb{R}_+ \to \mathbb{R}_+$, $u_h : h\mathbb{Z} \to \mathbb{R}_+$ be admissible.
If $f(0) = 0$, then:

$$\sum_{k\in\mathbb{Z}} f(u_h(x_k)) = \sum_{k\in\mathbb{Z}} f(u_h^*(x_k)). \tag{2.7}$$

**Proof** We have:

$$\sum_{k\in\mathbb{Z}} f(u_h(x_k)) = \sum_t f(t)\#\{k \in \mathbb{Z} : u_h(x_k) = t\} + f(0)\,\#\,\{k \in \mathbb{Z} : u_h(x_k) = 0\}.$$

The analogous happens for $u_h^*$ and we can conclude using Lemma 2.5 and the fact that $f(0) = 0$.
**Corollary 2.7** If $u_h \in L_h^p$ and $u_h$ is non-negative, then $u_h^* \in L_h^p$ and

$$\|u_h\|_{L_h^p} = \|u_h^*\|_{L_h^p}. \tag{2.8}$$

Now we need to prove some preliminary results about approximation of a Schwarz rearrangement $u_h^*$ of $u_h$ by repeated polarizations. This will be crucial to establish the discrete symmetrization inequalities (1.11) and (1.13). We will use some ideas and techniques developped by the author in [6] in the continuous setting. Let us first define the polarization in the discrete setting.
**Definition 2.8** The set of semi finite open intervals whose boundary is contained in $h\mathbb{Z}/2$ is denoted by $\mathcal{H}^h = [ah/2, +\infty[a \in \mathbb{Z}$. For $H \in \mathcal{H}^h$, the reflexion with respect to $\partial H$ is

denoted by $\sigma_H$. Note that if $H \in \mathcal{H}^h, \sigma_H(h\mathbb{Z}) = h\mathbb{Z}$.

**Definition 2.9** The polarization of $u_h : h\mathbb{Z} \to \mathbb{R}_+$ with respect to $H \in \mathcal{H}^h$ is the function $u_h^H : h\mathbb{Z} \to \mathbb{R}_+$ defined by:

$$u_h^H(x_k) = \begin{cases} \max\{u_h(x_k), u_h(\sigma_H(x_k)) & \text{if } x_k \in h\mathbb{Z} \cap H \\ \min\{u_h(x_k), u_h(\sigma_H(x_k)) & \text{if } x_k \in h\mathbb{Z} \backslash H. \end{cases}$$

**Proposition 2.10**:

Let $u_h : h\mathbb{Z} \to \mathbb{R}_+$ and $v_h : h\mathbb{Z} \to \mathbb{R}_+$ be admissible.
If $u_h v_h \in L_h^1$ and $u_h^H v_h^H \in L_h^1$, then

$$\sum_{k\in\mathbb{Z}} u_h(x_k) v_h(x_k) \leq \sum_{k\in\mathbb{Z}} u_h^H(x_k) v_h^H(x_k). \tag{2.9}$$

Moreover if $v_h = v_h^H$ and there is equality in (2.9), then

$$u_h^H(x_k) = u_h(x_k) \text{ and } u_h^H(\sigma_H(x_k)) = u_h(\sigma_H(x_k))$$

for any $x_k \in h\mathbb{Z} \cap H$ such that $v_h(x_k) > v_h(\sigma_H(x_k))$.

**Proof**: For any $x_k \in h\mathbb{Z} \cap H$:

$$\begin{array}{c} u_h(x_k)v_h(x_k) + u_h(\sigma_H(x_k))v_h(\sigma_H(x_k)) \leq \\ u_h^H(x_k)v_h^H(x_k) + u_h^H(\sigma_H(x_k))v_h^H(\sigma_H(x_k)) \end{array} \tag{2.10}$$

Summing these inequalities and noticing that $u_h^H(x_k)v_h^H(x_k) = u_h(x_k)v_h(x_k)$ for $x_k \in h\mathbb{Z} \cap /H$, we obtain (2.9).
In case, we have equality in (2.9), we have also equality in (2.10) for $x_k \in h\mathbb{Z} \cap H$. But by our assumption on $v$, this means that $u_h(\sigma_H(x_k)) \leq u_h(x_k)$ for every $x_k \in h\mathbb{Z} \cap H \Rightarrow u_h = u_h^H$.

For the latter, we will need two particular type of polarizations .

**Definition 2.11**:

$H_+ = ]0, +\infty[, H_- = ]-\infty, \frac{h}{2}[$ so that:

$$u_h^{H_+}(x_k) = \begin{cases} \max(u_h(x_k), u_h(x_k)) & \text{if } k \geq 0 \\ \min(u_h(x_k), u_h(x_k)) & \text{if } k \leq 0 \end{cases} \tag{2.11}$$

$$u_h^{H_-}(x_k) = \begin{cases} \max(u_h(x_k), u_h(x_{1-k})) & \text{if } k \leq 0 \\ \min(u_h(x_k), u_h(x_{1-k})) & \text{if } k \geq 1 \end{cases} \tag{2.12}$$

The aim of the following pragraph is to show that $u_h^*$ is a limit of iterated polarization. For $u_h : h\mathbb{Z} \to \mathbb{R}_+$, define $T_h u_h = (u_h^{H_-})^{H_+}$. Iterating $T_h$, one gets $u, u^{H_- H_+}, u^{H_- H_+ H_- H_+}$ we shall prove that $T_h^n u_h$ goes to $u_h^*$ as $n \to \infty$.

**Proposition 2.12** The sequence $(T_h^n u_h)_{n\geq 0}$ is precompact in $(X_h, d)$, where the metric $d$ is defined by

$$d(u_h, v_h) = \sum_{k\in\mathbb{Z}} \frac{|u_h(x_k) - v_h(x_k)|}{1 + 2^{|k|}|u_h(x_k) - v_h(x_k)|}.$$

Moreover, for any cluster point $v_h$:

$$\#\{k \in \mathbb{Z} : v_h(x_k) > t\} = \#\{k \in \mathbb{Z} : u_h(x_k) > t\}$$

for any $u_h$ admissible and any $t > 0$.
**Proof**: $X_h = \{u_h : h\mathbb{Z} \to \mathbb{R}_+\}$ endowed with the metric $d$ which is a complete metric space:

$$u_{h,n}(x_k) \to x_h(x_k) \; \forall \; k \in \mathbb{Z} \Leftrightarrow d(u_{h,n}, u_h) \to 0.$$

Now first observe that by induction on $n \geq 0$, we certainly have that

$$\inf_{|\ell| \leq |k|} u_h(x_\ell) \leq (T_h^n u_h)(x_k) \leq \sup_{|\ell| \geq |k|} u_h(x_k).$$

The precompactness then follows by a standard diagonal argument.
Let $v_h$ be a cluster point of the sequence $(T_h^n u_h)$. Assume that $T_h^{n_j} u_h(x_k) \to v_h(x_k) \; \forall \; k \in \mathbb{Z}$. Then:

$$\#\{k \in \mathbb{Z} : u_h^H(x_k) > t\} = \#\{k \in \mathbb{Z} : u_h(x_k) > t\}; \, \forall t > 0.$$

Therefore for any $n \geq 0$:

$$\{k \in \mathbb{Z} : (T_h^n u_h)(x_k) > t\} = \#\{k \in \mathbb{Z} : u_h(x_k) > t\}; \; \forall t > 0.$$

This implies that:

$$\#\{k \in \mathbb{Z} : v_h(x_k) > t\} \leq$$

$$\leq \underline{\lim}_{\ell \to \infty} \#\{k \in \mathbb{Z} : u_h(x_k) > t\}; \, \forall t > 0.$$

For the converse inequality, let $A > 0$ and note that:

$$\#\{k \in \mathbb{Z} : |k| \leq A \text{ and } u_h^H(x_k) \geq t + \frac{1}{A}\} \geq$$

$$\geq \#\{k \in \mathbb{Z} : |k| \leq A \text{ and } u_h(x_k) \geq t + \frac{1}{A}\},$$

hence

$$\#\{k \in \mathbb{Z} : |k| \leq A \text{ and } (T_h^n u_h)(x_k) \geq t + \frac{1}{A}\} \geq$$

$$\#\{k \in \mathbb{Z} : |k| \leq A \text{ and } u_h(x_k) \geq t + \frac{1}{A}\}; \, \forall t > 0.$$

And we can conclude by letting $A$ tend to infinity that

$$\#\{k \in \mathbb{Z} : v_h(x_k) \geq t\} = \#\{k \in \mathbb{Z} : u_h(x_k) \geq t\}; \; \forall t > 0.$$

**Proposition 2.13**:
If $u_h : h\mathbb{Z} \to \mathbb{R}_+$ is admissible, then:

$$T_h^n u_h(x_k) \to u_h^*(x_k) \text{ for every } k \in \mathbb{Z}.$$

**Proof**: We know by Proposition 2.12 that $(T_h^n u_h)_{n\geq 0}$ is precompact in $(X_h, d)$. Assume that $T_h^{n_j} u_h(x_k) \to v_h(x_k)$ for every $k \in \mathbb{Z}$. We need to prove that $v_h = u_h^*$.
First not that for any $k \geq 0 : T_h^n u_h(x_k) \geq T_h^n u_h(x_{-k})$ so $v(x_k) \geq v(x_{-k})$.
For $k \geq 0, \ell \in \mathbb{Z}$ set $w_\ell(x_k) = \begin{cases} 1 & \text{if } \ell \leq k \\ 0 & \text{if } \ell > k \end{cases}$
$w^{H-} = w$, and by the previous proposition

$$\sum_{\ell\in\mathbb{Z}} (T_h^{n_j} u_h)^{H-}(x_\ell) w_k(x_\ell) \leq \sum_{\ell\in\mathbb{Z}} v_h(x_\ell) w_k(x_\ell).$$

Letting $j \to \infty$, one gets

$$\sum_{\ell\in\mathbb{Z}} v_h^{H-}(x_\ell) w_k(x_\ell) \leq \sum_{\ell\in\mathbb{Z}} v_h(x_\ell) w_h(x_\ell)$$

since $\sigma_H(x_{-k}) = x_{k+1}$ one has $w_k(x_{-\ell}) = w_k(\sigma_H(x_{-\ell}))$.
Thus using cases of equality established in Proposition 2.10, we can conclude.

# 3 Discrete symmetrization inequalities

**Definition 3.1**

- A function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is supermodular if:

$$G(x + x_0, y + y_0) + G(x, y) \geq G(x, y + y_0) + G(x + x_0, y) \tag{3.1}$$

for any $x, y \in \mathbb{R}, x_0, y_0 > 0$.

- A function $K : |h\mathbb{Z}| \times \mathbb{R}_+ \to \mathbb{R}$ is $|h\mathbb{Z}|$ supermodular ($|h\mathbb{Z}| = h\mathbb{N}$) if

$$K(|(m + m_0)h|, y + y_0) + K(|mh|, y) \geq K(|mh|, y + y_0) + K(|(m + m_0)h|, y) \tag{3.2}$$

for any $m \in \mathbb{Z}, y \in \mathbb{R}_+$, $m_0 \in \mathbb{N}, y_0 > 0$.
We say that $G$ is strictly supermodular if (3.1) holds true with a strict sign. The function $K$ is $h\mathbb{Z}$ strictly supermodular when (3.2) holds with a strict sign.

In the sequel, we will make a frequent use of the following property : If $u_h$ is admissible:

$$u_h = u_h^* \Leftrightarrow u_h = u_h^H \ \ \forall\, \sigma_H \in H. \tag{3.3}$$

**Theorem 3.1.**

i) Let $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a supermodular function then:

$$\sum_{k\in\mathbb{Z}} G(u_h(x_k), v_h(x_k)) \leq \sum_{k\in\mathbb{Z}} G(u_h^H(x_k), v_h^H(x_k)) \tag{3.4}$$

for any admissible functions $u_h$ and $v_h$.

If $v_h = v_h^H$ and $G$ is strictly supermodular, then equality holds true in (3.4) if and only if $u_h = u_h^H$.

ii) If in addition $G(.,.)$ is continuous, non-decreasing with respect to each variable and $\sum_{k\in\mathbb{Z}} G(u_h^*(x_k), v_h^*(x_k)) < \infty$ then we have

$$\sum_{k\in\mathbb{Z}} G(u_h(x_k), v_h(x_k)) \leq \sum_{k\in\mathbb{Z}} G(u_h^*(x_k), v_h^*(x_k)) \tag{3.5}$$

for any admissible function $u_h$.

If $v_h = v_h^*$ and $G$ is strictly supermodular, then equality holds in (3.5) if and only if $u_h = u_h^*$.

**Proof**

i) By the supermodularity of $G$, we certainly have for any $x_k \in h\mathbb{Z} \cap H$ that:

$$\begin{aligned} G(u_h(x_k), v_h(x_k)) \quad &+ G(u_h(\sigma_H(x_k)), v_h(\sigma_H(x_k)) \\ &\leq G(u_h^H(x_k), v_h^H(x_k)) + G(u_h^H(\sigma_H(x_k)), v_h^H(\sigma_H(x_k)) \end{aligned} \tag{3.6}$$

Summing up this inequality and noticing that $u_h^H(x_k)v_h^H(x_k) = u_h(x_k)v_h(x_k)$ for any $x_k \in h\mathbb{Z} \cap \partial H$, we obtain (3.4).
Now in case we have equality in (3.3), we will also have equality in (3.6) for any $x_k \in h\mathbb{Z} \cap \partial H$ by the strict supermodularity of $G$. Now since we are also assuming that $v_h = v_h^H$, it follows that

$$u_h(\sigma_H(x_k)) \leq u_h(x_k) \ \ \forall \ x_k \in h\mathbb{Z} \cap H; \text{ i.e }, u_h = u_h^H. \tag{3.7}$$

ii) By the continuity and the monotonicity of $G$, (3.5) follows immediately from (3.4) by applying the Theorem of monotone convergence. More precisely if $(T_{u_h}^n)$ is the sequence of iterated polarizations constructed in section 2, we obviously have:

$$\sum_{k\in\mathbb{Z}} G(u_h(x_k), v_h(x_k)) \leq \sum_{k\in\mathbb{Z}} G(T_{u_h}(x_k), T_{v_h}(x_k)) \leq ... \leq \sum_{k\in\mathbb{Z}} G(T_{u_h}^n(x_k), T_{v_h}^n(x_k)) \tag{3.8}$$

Thus letting $n$ go to infinity, the result follows and we certainly have

$$\sum_{k\in\mathbb{Z}} G(u_h(x_k), v_h(x_k)) \leq \sum_{k\in\mathbb{Z}} G(T_{u_h}(x_k), T_{v_h}(x_k)) \leq ... \leq \sum_{h\in\mathbb{Z}} G(u_h^*(x_k), v_h(x_k)). \tag{3.9}$$

Now if we have equality in (3.5), we will certainly have equality in (3.9):

$$\sum_{k\in\mathbb{Z}} G(u_h(x_k), v_h(x_k)) = \sum_{k\in\mathbb{Z}} G(T_{u_h}(x_k), T_{v_h}(x_k)) = ... = \sum_{k\in\mathbb{Z}} G(u_h^*(x_k), v_h^*(x_k)).$$

But we are supposing that $v_h = v_h^* \Rightarrow v_h^H = v_h \ \forall \ H$. Thus using cases of equality of part i), it follows that $u_h = u_h^H \ \forall \ H$, which is equivalent to say that $u_h = u_h^*$ by (3.3).

**Remark** Hypotheses on $G$ used in part ii) can be relaxed.
In fact, it is sufficient to suppose that $G$ is supermodular and that $\sum_{k\in\mathbb{Z}} G(u_h^*(x_k),0)$ and $\sum_{k\in\mathbb{Z}} G(0,v_h^*(x_k)) < \infty$, since $\tilde{G}(s_1,s_2) = G(s_1,s_2) - G(s_1,0) - G(0,s_2)$ satisfies all the assumptions of Theorem 3.1. Therefore the monotonicity of the function with respect to each variable can be removed.
**Theorem 3.2**: If $F$ is a function: $h\mathbb{N}\times\mathbb{R}_+ \to \mathbb{R}$ satisfying

1. $F(|x_m|,.)$ is continuous for any $m\in\mathbb{Z}$,
2. $-F$ is $|h\mathbb{Z}|$ supermodular,
3. $\sum_{k\in\mathbb{Z}} F(|x_k|,0) < \infty$, then

$$\sum_{k\in\mathbb{Z}} F(|x_k|,u_h(x_k)) \le \sum_{k\in\mathbb{Z}} F(|x_k|,u_h^*(x_k)) \tag{3.10}$$

for any admissible $u_h$.

Moreover if $-F$ is strictly $|h\mathbb{Z}|$ supermodular and we have equality in (3.10), then $u_h = u_h^*$.

**Proof** the proof is identical to the one of the previous result.

**Theorem 3.3** (Discrete fractional Polya Szegö inequality).
Let $u_h : h\mathbb{Z}\to\mathbb{R}_+$ be admissible, then :

$$\begin{aligned} |\nabla_s u_h|^2_{L^2_h} &= \sum_{\ell,k\in\mathbb{Z}} \frac{|u_h(x_k)-u_h(x_\ell)|^2 h}{|kh-\ell h|^{1+2s}} = \frac{1}{h^{2s}} \sum_{\ell,k\in\mathbb{Z}} \frac{|u_h(x_k)-u_h(x_\ell)|^2}{|k-\ell|^{1+2s}} \\ &\ge \frac{1}{h^{2s}} \sum_{\ell,k\in\mathbb{Z}} \frac{|u_h^*(x_k)-u_h^*(x_\ell)|^2}{|k-\ell|^{1+2s}} = |\nabla_s u_h^*|^2_{L^2_h}. \end{aligned} \tag{3.11}$$

If one has equality in (3.9), then $u_h(x_k) = u_h^H(x_k)$ or $u_h^H(x_k) = u_h(\sigma_H(x_k))$ (i.e. $u_h$ and $u_h^*$ are equal up to a translation).
**Proof** Let $k,\ell\in\mathbb{Z}$ be such that $\partial H \not\subset (k,\ell)$, set $k' = \sigma_H(k)$ and $\ell' = \sigma_H(\ell)$, then $|k-\ell| = |k'-\ell'| \le |k-\ell'| = |k'-\ell|$.
Hence

$$\frac{|u_h(x_k)-u_h(x_\ell)|^2}{|k-\ell|^{1+2s}} + \frac{|u_h(x_k)-u_h(x_{\ell'})|^2}{|k-\ell'|^{1+2s}} + \frac{|u_h(x_{k'})-u_h(x_\ell)|^2}{|k'-\ell|^{1+2s}} +$$

$$\frac{|u_h(x_{k'})-u_h(x_{\ell'})|^2}{|k'-\ell'|^{1+2s}} = \frac{1}{|k'-\ell|^{1+2s}} P(u_h) + \left(\frac{1}{|k-\ell|^{1+2s}} - \frac{1}{|k'-\ell|^{1+2s}}\right) Q(u_h)$$

where

$$P(u_h) = |u_h(x_k) - u_h(x_\ell)|^2 + |u_h(x_k) - u_h(x_{\ell'})|^2 + |u_h(x_{k'}) - u_h(x_\ell)|^2 +$$

$$|u_h(x_{k'}) - u_h(x_{\ell'})|^2 \text{ and } Q(u_h) = |u_h(x_k) - u_h(x_\ell)|^2 + |u_h(x_{k'}) - u_h(x_{\ell'})|^2.$$

Noticing that $P(u_h) = P(u_h^H)\ \forall\ H \in \mathcal{H}^h$ and that

$$\left(\frac{1}{|k-\ell|^{1+2s}} - \frac{1}{|k'-\ell|^{1+2s}}\right) Q(u_h) \geq \left(\frac{1}{|k-\ell|^{1+2s}} - \frac{1}{|k'-\ell|^{1+2s}}\right) Q(u_h^H)$$

and summing over $k$ and $\ell$, enables us to conclude that

$$|\nabla_s u_h|^2_{L^2_h} \geq |\nabla_s u_h^H|^2_{L^2_h}\ \forall\ H \in \mathcal{H}^h.$$

Finally thanks to Proposition 2.12, we certainly have that $|\nabla_s u_h|^2_{L^2_h} \geq |\nabla_s u_h^*|^2_{L^2_h}$ for any admissible $u_h$.
Cases of equality are obtained in the same way as part ii) of Theorem 3.1.

# 4 Discrete Fractional constained variational problem

In this section we will study the following constrained variational problem:

$$I_c^h = \inf\{E_h(u_h) : u_h \in S_c^h\}. \tag{4.1}$$

$$E_h(u_h) = \left\{\frac{1}{2}|\nabla_s u_h|^2_{L^2_h} - \int F(|x_k|, u_h(x_k))\right\} h$$

$$S_c^h = \left\{u_h \in H_h^s : \sum_{k\in\mathbb{Z}} u_h^2(x_k)h = c^2\right\},$$

where $c$ is a prescribed real number.
Our main result in this section is:
**Theorem 4.1**: Let $F : |h\mathbb{Z}| \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the following assumptions:
(F0) $F(|x_m|, t) \leq F(|x_m|, |t|)\ \forall\ t \in \mathbb{R}$.
(F1) $F(|x_m|, .)$ is continuous $\forall\ m \in \mathbb{Z}$.
(F2) $\exists\ K > 0$ and $0 < \ell < 4s$ such that $\forall\ m \in \mathbb{Z}, t \geq 0,\ 0 \leq F(|x_m|, t) \leq K(t^2 + t^{\ell+2})$.
(F3) $\forall\ \varepsilon > 0, \exists\ m_0 \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$ such that

$$F(|x_m|, t) \leq \varepsilon t^2\ \ \forall\ m > m_0 \text{ and } |t| \leq |t_0|.$$

(F4) $-F$ is $h\mathbb{Z}$ supermodular
(F5) $F(|x_m|, \theta t) \geq \theta^2 F(|x_m|, t)\ \forall\ m \in \mathbb{Z}, \theta > 1, t \in \mathbb{R}$.
(F6) $\exists\ \delta > 0, t_1 > 0, m_1 \in \mathbb{Z}, \alpha > 0$ such that $F(|x_m|, t) > \delta t^\alpha$ for any $m > m_1$ and $|t| < |t_1|$, where $1 + 2s > \frac{\alpha}{2}$.

Then (4.1) admits a Schwarz symmetric minimizer $u_c^h = (u_c^h)^*$.
If in addition (F4) holds with a strict sign, then all minimizers of (4.1) are Schwarz symmetric.
Before proving this result, we need the following lemma:
**Lemma 4.2**: Under (F6) $I_c^h < 0 \; \forall \, c \in \mathbb{R}$.
**Proof**: let $0 < p < 1, u_h \in S_c^h$, then $u_h^p(x_m) = p^{\frac{1}{2}} u_{ph}(x_m)$ is also in $S_c^h$.

$$E_h(u_h^p) = \sum_{k,\ell\in\mathbb{Z}} \frac{|u_h^p(x_k) - u_h^p(x_\ell)|^2}{h^{2s}|k-\ell|^{1+2s}} - \sum_{m\in\mathbb{Z}} F(|x_m|, u_h^p(x_m))$$

$$\begin{aligned}
&\sum_{k,\ell\in\mathbb{Z}} \frac{|p^{1/2}up_h(x_k) - p^{1/2}up_h(x_\ell)|^2}{|k-\ell|^{1+2s}} - \delta \sum_{|m|\geq|m_2|p^{\alpha/2}u_h^{p\alpha}} u_h^p(x_m) \\
\leq\; & \frac{p^{2s}}{h^{2s}} \sum_{k,\ell\in\mathbb{Z}} \frac{|u_h(x_k) - u_h(x_\ell)|^2}{|k-\ell|^{1+2s}} - \delta p^{-1} p^{\frac{\alpha}{2}} \sum_{|m|\geq|m_3|} u_h^\alpha(x_m) \\
\leq\; & \frac{p^{2s}}{h^{2s}} \sum_{k,\ell\in\mathbb{Z}} \frac{|u_h(x_k) - u_h(x_\ell)|^2}{|k-\ell|^{1+2s}} - p^{\frac{\alpha}{2}-1}\delta \sum_{|m|\geq|m_3|} u_h^\alpha(x_m)
\end{aligned}$$

the choice of $p$ and $\alpha$ enables us to conclude.
**Proof of Theorem 4.1**:
**Step 1**: **(4.1) is well posed ($I_c^h > -\infty$ and all minimizing sequences are bounded in $H_h^s$).**
By (F2), we can write:

$$\sum_{m\in\mathbb{Z}} (F(|x_m|, u_h(x_m))h \leq K(\sum_{m\in\mathbb{Z}} u_h^2(x_m)h + \sum_{m\in\mathbb{Z}} u_h^{\ell+2}(x_m)h) \tag{4.2}$$

Now using the fractional discrete Gagliardo-Nirenberg inequality, (2.6), it follows:

$$\|u_h\|_{L_h^{\ell+2}} \leq K' \|u_h\|_{L_h^2}^{1-\theta} \|\nabla_s u_h\|_{L_h^2}^{\theta} \quad \text{where } \theta = \frac{\ell}{2s(\ell+2)}$$

which implies that

$$\|u_h\|_{L_h^{\ell+2}}^{\ell+2} \leq K'' \{\|u_h\|_{L_h^2}^{(1-\theta)(\ell+2)} \|\nabla_s u_h\|_{L_h^2}^{\theta(\ell+2)}\}. \tag{4.3}$$

Now using Young inequality, we have:

$$\|u_h\|_{L_h^2}^{(1-\theta)(\ell+2)} \|\nabla_s u\|_{L_h^2}^{\theta(\ell+2)} \leq \frac{1}{p}\varepsilon^p \|\nabla_s u_h\|_{L_h^2}^{p\theta(\ell+2)} + \frac{1}{q\varepsilon^q}\|u_h\|_{L_h^2}^{q(1-\theta)(\ell+2)} \tag{4.4}$$

for any $\varepsilon > 0$, $p > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, thus choosing $p = \frac{2}{\theta}(\ell+2) = \frac{4s}{\ell}$, we get

$$\begin{aligned}
\|u_h\|_{L_h^{\ell+2}}^{\ell+2} &\leq \tfrac{K''}{p}\varepsilon^p \|\nabla_s u_h\|_{L_h^2}^2 + \tfrac{K''}{q\varepsilon^q}\|u_h\|_{L_h^2}^{q(1-\theta)(\ell+2)} \\
&= \tfrac{K''}{p}\varepsilon^p \|\nabla_s u_h\|_{L_h^2}^2 + \tfrac{K''}{q\varepsilon^q} c^{q(1-\theta)(\ell+2)}
\end{aligned} \tag{4.5}$$

for any $u_h^c \in S_c^h$.
Therefore

$$\begin{aligned} E_h(u_h) &\geq \frac{1}{2}\|\nabla_s u_h\|_{L_h^2}^2 - Kc^2 - K''K\varepsilon^p\|\nabla_s u_h\|_{L_h^2}^2 \\ &- \frac{KK''}{q\varepsilon^q}c^{q(1-\theta)(\ell+2)} \\ &\geq (\frac{1}{2} - \frac{KK''}{p}\varepsilon^p)\|\nabla_s u_h\|_{L_h^2}^2 - Kc^2 - \frac{KK''}{q\varepsilon^q}c^{q(1-\theta)(\ell+2)}. \end{aligned}$$

Thus $I_c^h > -\infty$ and all minimizing sequences are bounded in $H_h^s$.
**Step 2**: **Existence of Schwarz symmetric minimizing sequence**.
By symmetrization inequalities proved in section 3, we certainly have thanks to the assumption made on $F$, that

$$E_h(|u_h|) \leq E(u_h).$$

so we can suppose without loss of generality that $u_h$ is non-negative:

$$E_h(u_h^*) \leq E(u_h).$$

**Step 3**: Let $u_{h,n} = u_{h,n}^*$ be a Schwarz symmetric minimizing sequence of (4.1), then we can find $m_0 \in \mathbb{Z}$ such that

$$u_{n,h}^*(x_m) \leq \frac{c}{\sqrt{h}} \quad \forall\, n \in \mathbb{N} \tag{4.6}$$

On the other hand by the weak lower semi-continuity of $\|\ \|_{L_h^2}$ we have that $\|\nabla_s u_h\|_{L_h^2} \leq \liminf \|\nabla_s u_h\|_{L_h^2}$.
Now fix $m_4 \in \mathbb{N}$, since $u_{h,n}$ converges weakly to $u_h$ (up to a subsequence since it is bounded in $H_h^s$), it follows that it converges strongly to $u_h$ in $L_h^{\ell+2}(|m| \leq m_4)$.
This implies that

$$\lim \sum_{|m|\leq m_4} F(|x_m|, u_{h,n}(x_m)) = \sum_{|k|\leq m_4} F(|x_m|, u_h(x_m))$$

(4.6) together with (F3) imply that

$$\sum_{|m|\geq p_0} F(|x_m|, u_{h,n}(x_m)) \text{ and } \sum_{|m|\geq p_0} F(|x_m|, u_h(x_m)) < \epsilon,\ \forall \epsilon > 0$$

for $p_0 \in \mathbb{N}$ big enough.
In conclusion

$$\lim_{n\to\infty} \sum_{m\in\mathbb{Z}} F(|x_m|, u_{h,n}(x_m)) = \sum_{m\in\mathbb{Z}} F(|x_m|, u_h(x_m)).$$

**Step 4**: $I_c^h$ is achieved:
By the weak lower semi-continuity of the norm $L_h^2$, we know that

$$\sum_{m\in\mathbb{Z}} u_h^2(x_m)h \leq c^2.$$

Then observe that $u_h \neq 0$ since we know by Lemma 4.2 $I_c^h < 0$ and $F(.,0) = 0$ by (F2). Now set $t^h = \frac{c^2}{\|u_h\|_{L_h^2}}$ then $t^h \geq 1$.
On the other hand

$$I_c^h \leq E_h(t^h u_h) \leq (t^h)^2 E(u_h) \leq (t^h)^2 I_c \Rightarrow t^h \leq 1$$

by the strict negativity of $I_c^h$.

If $-F$ is strictly $h\mathbb{Z}$ supermodular, then it follows from Theorem 3.2 that all minimizers are Schwarz symmetric.

**Remarks** If $\ell = 4s$, it is easy to reproduce all the steps provided that $c$ is small enough $(0 < c < (\frac{1}{2KK'')^4})$.

If $\lim\limits_{t\to\infty} \dfrac{F(|x_m|,t)}{t^{\ell+2}} \geq A > 0$, then $I_c^h = -\infty$.

# 5 Some Applications

In the very special case $F(x,u) = \frac{1}{p+2}u^{p+2}$ and the fractional Laplacian is replaced by the classical one, the nonlinear Schrödinger equation (1.1) becomes:

$$i\partial_t \varphi = -\Delta\varphi - |\varphi|^p \varphi,$$

and its non-local versions arise in many domains, where one has to consider a lattice with a quantum particle sitting at each side, interacting with the others. Such lattice systems are used to understand electron transport in biopolymers like organic semiconductors, molecular crystals, and DNA.

As a first approximation, it is worth to consider a discrete model of quantum particles at lattice points with two kinds of interactions: nearest-neighbor interactions appearing as a discrete Laplacian term, representing interactions between base pairs in DNA; and self-interactions appearing as a p-nonlinear term, representing interactions within a base pair.

In a much better approximation, one has to take into account the long-range interactions (which need not be of fixed range, because DNA is constantly in flux). So we consider the same p self-interaction term as before, and inverse power-law long-range interactions for s parameters.

Our main result is another continuum limit: for certain values of s, solutions of the discrete model converge weakly to a solution of the NLS with fractional-order Laplacian:

$$i\partial_t \varphi = (-\Delta)^s \varphi - |\varphi|^p \varphi.$$

The dynamics are governed by the discrete NLS on the lattice of mesh size h, and we prove in the last section of this paper that taking the mesh size of the lattice to zero (the continuum limit), gives macroscopic behavior described by the focusing p-NLS:

It is important to have the same qualitative and quantititative properties of the discretized problems of the fractional NLS. The main difficulties are that there is no canonical discretization of the fractional derivative and that the most physical one doesn't obviously play well with the fractional derivative. We constructed a discrete fractional calculus and an interpolation of the discrete functions based on a special mollification (see section 2). This framework is compatible with the fractional derivative. We also have developed some ingenious harmonic analysis techniques to conduct the key steps of the proof in Fourier space.

For other nonlinearities covered by our integrand F, the lattice models Ferro-magnets and spin glasses, Theorem 4.1 would clarify our understanding of glass and phenomena like neural networks and other models approximating Bose-Einstein condensation. A dynamic quantum icing model is the next step. Let us point out that Bose-Einstein condensates are unusual states of matter near absolute zero that can be used to slow and briefly stop light, as well as convert light to matter and back. There are excellent applications, such as quantum information processing and increased accuracy in measurements by inter-ferrometry with atom lasers instead of traditional photon lasers. But the Bose Einstein Condensate is fragile and difficult to work with, so it is vital to work out the theory.

More precisely, the particles are so super-cooled (to a few billionths of a degree Kelvin) that they all fall into the **ground state** and exhibit quantum mechanical behavior macroscopically, in effect they condense into a quantum super-particle. Only two years ago, the Bose Einstein condensates macroscopic behavior was explained mathematically: the microscopic repulsive interactions between quantum particles give rise, in the scaling limit, to quantum macro-behavior governed by the p- nonlinear Schrödinger equation (p-NLS):

$$i\partial_t\varphi = -\Delta\varphi + b|\varphi|^{p+1}\varphi.$$

Again usually the latter equation does not take into account natural anomalous diffusion phenomena, and in the most realistic model, one has to replace the classical Laplacian by the fractional Laplacian. As stated above the ground state solutions play a key role here. This special non-linearity is also covered by our main result ( see Theorem 4.1).

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