Homework 6 – Calc Emphasizing Proofs

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Page 181 problem 2. (i)

$$f'(x) = (3x^2 + 8x + 5)\cos(x^3 + 4x^2 + 5x + 2);$$

(ii)

$$f'(x) = 3(2x + \cos x)\sin^2(x^2 + \sin x) \cdot \cos(x^2 + \sin x);$$

(iii)

$$f'(x) = 4(x + \sin x \cos x + \sin x + x \sin x)\sin^2(x^2 + \cos x) \cdot \cos(x^2 + \sin x);$$

(iv)

$$f'(x) = 3(\frac{x}{\cos x})^2 \frac{\cos x + x\sin x}{\cos^2 x} \cos((\frac{x}{\cos x})^3);$$

(v)

$$f'(x) = 3(\frac{x}{\cos x})^2 \frac{\cos x + x\sin x}{\cos^2 x} \cos((\frac{x}{\cos x})^3);$$

(vi)

$$f'(x) = -31^2 \sin x \cdot \cos(x^{31^2-1});$$

(vii)

$$f'(x) = 2\sin x \cdot \cos x \cdot \sin^3 x^2 + 6x\sin^2 x \cdot \cos x^2;$$

(viii)

$$f'(x) = 6\sin^2(\sin^2(\sin x)) \cdot \cos(\sin^2(\sin x)) \cdot \sin(\sin x) \cdot \cos(\sin x) \cdot \cos x;$$

(ix)

$$f'(x) = 6(x + \sin^5 x)^5(1 + 5\sin^4 x \cos x);$$

(x)

$$f'(x) = \cos(\sin(\sin(\sin(\sin(x))))) \cdot \cos(\sin(\sin(\sin(x)))) \cdot \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \cos(x);$$

(xi)

$$f'(x) = 7^3x^6(\sin^7 x^7 + 1)^6 \cos((\sin^7 x^7 + 1)^7) \cdot \sin^6 x^7 \cdot \cos x^7;$$

(xii)

$$f'(x) = 5\{[(x^2 + x)^3 + x]^4 + x\}^4\{4[(x^2 + x)^3 + x]^3[3(x^2 + x)^2(2x + 1) + 1] + 1\};$$

(xiii)

$$f'(x) = 2\cos(x^2 + \sin(x^2 + \sin(x^2))) \cdot [x + (x + x\cos x^2)\cos(x^2 + \sin x^2)];$$

(xiv)

$$f'(x) = 6^4 \cos(6\cos(6\sin(6\cos(6x)))) \cdot \sin(6\sin(6\cos(6x)))\cos(6\cos(6x)) \cdot \sin(6x);$$

(xv)

$$f'(x) = \frac{2[x\cos x^2 \sin^2 x + \sin x^2 \sin x \cos x](1+\sin x) - (\sin x^2 \sin^2 x \cos x)}{(1+\sin x)^2};$$

(xvi)

$$f'(x) = \frac{(x^2 + x\sin x - 2)(1+\cos x) - (x+\sin x)(2x + \sin x + x\cos x)}{(x^2 + x\sin x - 2)^2};$$

(xvii)

$$f'(x) = \cos(\frac{x^3}{\sin(\frac{x^3}{\sin x})})\frac{3x^2\sin(\frac{x^3}{\sin x}) - x^3\cos(\frac{x^3}{\sin x})\frac{3x^2\sin x - x^3\cos x}{\sin^2 x}}{\sin^2(\frac{x^3}{\sin x})};$$

(xviii)

$$f'(x) = \cos(\frac{x}{x - \sin(\frac{x}{x-\sin x})})\frac{x - \sin(\frac{x}{x-\sin x}) - x[1 - \cos(\frac{x}{x-\sin x})\frac{x - \sin x - x(1-\cos x)}{(x-\sin x)^2}]}{(x - \sin(\frac{x}{x-\sin x}))^2};$$

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(i) Since $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ for $x \neq 0$, $h(0) = 3$ and $h'(0) = \sin^2 \sin 1$, then

$$(f\circ h)'(0) = f'(h(0))h'(0) = (6\sin\frac{1}{3} - \cos\frac{1}{3})\sin^2\sin 1.$$

(ii) Since $f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0} x\sin\frac{1}{x} = 0$, then

$$(k\circ f)'(0) = k'(f(0))f'(0) = k'(0)f'(0) = 0.$$

(iii) Since $\alpha'(x) = 2xh'(x^2) = 2x\sin^2(\sin(x^2+1))$, then

$$\alpha'(x^2) = 2x^2\sin^2(\sin(x^4+1)).$$

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Since $g(0) = g'(0) = 0$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\frac{g(x) - g(0)}{x} - g'(0)| \le \epsilon,$$

that is,

$$|g(x)| \le \epsilon|x|.$$

Together with $|\sin\frac{1}{x}| \le 1$,

$$|\frac{g(x)\sin\frac{1}{x}}{x}| \le \epsilon,$$

that is,

$$\lim_{x\to 0}\frac{g(x)\sin\frac{1}{x}}{x} = 0.$$

Then

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{g(x)\sin\frac{1}{x}}{x}.$$

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We prove the result by induction.

Step 1. for $n = 1$, we have that

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a) = \sum_{k=0}^{1} \binom{1}{k} f^{(k)}(a)g^{(1-k)}(a).$$

Step 2. assume that for $n = m$, we have that

$$(fg)^{(m)}(a) = \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(a)g^{(m-k)}(a).$$

Now for $n = m + 1$, we have that

$$\begin{aligned}
(fg)^{(m+1)}(a) &= [f'(a)g(a) + f(a)g'(a)]^{(m)} \\
&= [f'(a)g(a)]^{(m)} + [f(a)g'(a)]^{(m)} \\
&= \sum_{j=0}^{m} \binom{m}{j} f^{(j+1)}(a)g^{(m-j)}(a) + \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(a)g^{(m-k+1)}(a) \\
&= \sum_{k=1}^{m+1} \binom{m}{k-1} f^{(k)}(a)g^{(m-k+1)}(a) + \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(a)g^{(m-k+1)}(a) \\
&= \sum_{k=0}^{m+1} \left[\binom{m}{k-1} + \binom{m}{k}\right] f^{(k)}(a)g^{(m-k+1)}(a) \\
&= \sum_{k=0}^{m+1} (\binom{m+1}{k} f^{(k)}(a)g^{(m-k+1)}(a)
\end{aligned}$$

By induction, we have that for any $n \in \mathbb{N}$,

$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k)}(a).$$