EXPLICIT CONSTRUCTIVE APPROXIMATION TO SYMMETRIZATION VIA ITERATED POLARIZATIONS

HICHEM HAJAIEJ

DEDICATED TO AL BAERNSTEIN II IN THE OCCASION OF HIS 70$^{\mathrm{TH}}$ BIRTHDAY.

Abstract. We construct an explicit approximation to a rearranged function u^* of u via iterated polarizations of u.

1. Introduction

Two-point rearrangement or polarization is a powerful tool to establish numerous functional inequalities. The breakthrough paper of Al Baernstein II, [1], opened the way to many interesting works.

Let us first recall that for any nonnegative real valued function $u : \mathbb{R}^N \to \mathbb{R}_+$ and any half space H of $\mathbb{R}^N$ **containing the origin**, we define the two-point rearrangement (or polarization) of u with respect to H by:

$$u^H(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\} & \text{for } x \in H, \\ \min\{u(x), u(\sigma_H(x))\} & \text{elsewhere;} \end{cases} \tag{1.1}$$

here σ_H is the reflection with respect to H.

We say that a nonnegative function u is a **symmetrizable** function if $\mu\left\{x \in \mathbb{R}^N : u(x) > t\right\} < \infty$ for all $t > 0$, where μ denotes the Lebesgue measure in $\mathbb{R}^N$. $\mathbf{u}^*$ is the unique function equimeasurable with u (see Section 2) such that $u^*(x) = h(|x|)$ with h non-increasing and right-continuous.

Inspired by the work of Al Baernstein II, F. Brock and Y. Solynin proved in [4] that for a nonnegative function $u \in L^p(\mathbb{R}^N)$, its Schwarz symmetrization u^* is the limit of iterated polarizations. Namely

$$u_n = u^{H_1 \ldots H_n} = \left[\left(u^{H_1}\right)^{H_2^{\cdot^{\cdot^{\cdot}}}}\right]^{H_n} \to u^* \in L^p\left(\mathbb{R}^N\right) \text{ provided that } u \in L^p_+\left(\mathbb{R}^N\right). \tag{1.2}$$

It turned out that (1.2) is extremely useful to prove many rearrangement inequalities. Indeed, it reduces complicated symmetrization inequalities to easier combinatorial problems. More precisely, to prove the generalized Hardy-Littlewood inequalities [3, 5], it is sufficient to establish:

$$\int_{\mathbb{R}^N} F\left(u_1(x), \ldots, u_n(x)\right) dx \le \int_{\mathbb{R}^N} F\left(u_1^H(x), \ldots, u_n^H(x)\right) dx \tag{1.3}$$

for any H. Hence using (1.2) and suitable growth conditions, we get

$$\int_{\mathbb{R}^N} F\left(u_1(x), \ldots, u_n(x)\right) dx \le \int_{\mathbb{R}^N} F\left(u_1^*(x), \ldots, u_n^*(x)\right) dx. \tag{1.4}$$

The same approach applies to establish the generalized Riesz inequality. C. Draghici [3] and A. Burchard and the author [5] proved that:

$$\begin{aligned}(1.5)\qquad &\int_{\mathbb{R}^N} \dots \int_{\mathbb{R}^N} F\left(u_1(x_1), \dots, u_n(x_n)\right) \prod_{i<j} K_{i,j}\left(d(x_i, x_j)\right) dx_1 \dots dx_n \\ &\leq \int_{\mathbb{R}^N} \dots \int_{\mathbb{R}^N} F\left(u_1^H(x_1), \dots, u_n^H(x_n)\right) \prod_{i<j} K_{i,j}\left(d(x_i, x_j)\right) dx_1 \dots dx_n\end{aligned}$$

for any $H, u_1, \dots, u_2$ are symmetrizable functions and $K_{i,j}$ are non-increasing kernels. Hence using again the limiting procedure (1.2), we obtain:

$$\begin{aligned}&\int_{\mathbb{R}^N} \dots \int_{\mathbb{R}^N} F\left(u_1(x_1), \dots, u_n(x_n)\right) \prod_{i<j} K_{i,j}\left(d(x_i, x_j)\right) dx_1 \dots dx_n \\ (1.6)\qquad &\leq \int_{\mathbb{R}^N} \dots \int_{\mathbb{R}^N} F\left(u_1^*(x_1), \dots, u_n^*(x_n)\right) \prod_{i<j} K_{i,j}\left(d(x_i, x_j)\right) dx_1 \dots dx_n.\end{aligned}$$

Moreover two-point rearrangement enables us to study equality cases in (1.4) and (1.6). Once again it reduces a very hard functional analysis problem to a much less difficult combinatorial one. Indeed, to study equality cases in the generalized Hardy-Littlewood and Riesz inequalities it is sufficient to determine equality cases in (1.3) and (1.5) (respectively). This was done by A. Burchard and the author in [3, Theorem 2]. The other key tool was the following result

$$(1.7)\qquad v = v^* \Leftrightarrow v = v^H \qquad \forall H.$$

Polarization is an efficient tool to establish Pólya-Szegö inequality. Indeed, if u is a nonnegative function in the Sobolev space $W^{1,p}(\mathbb{R}^N)$, then u^H is also in the same space, [8], and $|\nabla u|_p = |\nabla u^H|_p$, which implies that $|\nabla u|_p = |\nabla u^{H_1 \dots H_n}|_p$, the lower semi-continuity of $|\cdot|_p$ and (1.2) enables us to conclude that

$$(1.8)\qquad |\nabla u|_p \geq |\nabla u^*|_p.$$

Recently, M. Squassina and the author, [8], have extended this inequality to integrands depending on u and its gradient. Our main ingredient was polarization. Generalized Pólya-Szegö inequality has numerous applications in quasilinear equations.

However the study of equality cases in the Generalized Pólya-Szegö inequality needs subtle informations about the iterated polarizations. The method of Brock and Solynin to establish (1.2) is based on maximization techniques in which one cannot have concrete informations about the maximum and consequenlty the sequence constructed in [4, Lemma 6.1] is very abstracted. In [9], Van Schaftingen has slightly improved the construction of [4] but his proof is not direct either and the underlying ideas he used are also based on implicit maximization problem.

The goal of this paper is to give an explicit construction of a sequence (u_n) obtained by iterated polarizations of u with respect to some half spaces H. This construction is a key ingredient in establishing equality cases in the generalized Pólya-Szegö inequality.

2. **Notation and Definitions**

- All statements about measurability refer to the Lebesgue measure μ in $\mathbb{R}^N$ unless it is indicated ($N \in \mathbb{N}^*$).
- In an integral where no domain of integration (variable of integration) is indicated, it is to be understood that the integration extends over all $\mathbb{R}^N$ (respectively the variable of integration is $d\mu$)

- $M(\mathbb{R}^N)$ is the set of real valued measurable functions, for $p > 1$; $L^p(\mathbb{R}^N) = \left\{u \in M(\mathbb{R}^N) : |u|_p < \infty\right\}$ where $|u|_p = \left(\int |u|^p\right)^{\frac{1}{p}}$, $|\cdot|$ is the euclidean norm in $\mathbb{R}^N$. $L_p^+(\mathbb{R}^N)$ is the cone of non-negative functions of $L_p(\mathbb{R}^N)$.
- The set of symmetrizable functions $F_N = \left\{u \in M_+(\mathbb{R}^N) : \mu\left\{x \in \mathbb{R}^N : u(x) > t\right\} < \infty \quad \forall t > 0\right\}$; $M_+(\mathbb{R}^N)$ is the set of nonnegative measurable functions.
- If u and v are in F_N, we say that u is **equimeasurable with** v if $\mu\{x \in \mathbb{R}^N : u(x) > t\} = \mu\{x \in \mathbb{R}^N : v(x) > t\} \quad \forall t > 0$; we write $u \sim v$.
- For $u \in F_N$, its **Schwarz symmetrization $\mathbf{u}^*$** is the unique function such that $u \sim u^*$ with $u^*(x) = h(|x|); h : (0, \infty) \to \mathbb{R}_+$ is nonincreasing and right continuous. When $u = u^*$, we say that u is **Schwarz symmetric**. When h is strictly decreasing, we say that u^* is strictly decreasing too (for a more detailed account, see [7]).
- It is well-known that for any half space H containing $O_{\mathbb{R}^N}$, we have that: u, u^H and u^* are equimeasurable [1], therefore:

$$|u|_p = |u^H|_p = |u^*|_p \text{ for any } u \in L_p^+(\mathbb{R}^N). \tag{2.1}$$

Topology of Half spaces: $\mathcal{H}$ denotes the set of closed half spaces H of $\mathbb{R}^N$ containing $\mathbf{O}_{\mathbb{R}^N}$.

We equip it with the endowed norm ensuring that $H_n \to H$ if and only if there is a sequence of isometries $i_n : \mathbb{R}^N \to \mathbb{R}^N$ such that $H_n = i_n(H)$ and i_n converges to identity when n goes to infinity.

In the sequel, we only consider closed half spaces that contain $\mathbf{O}_{\mathbb{R}^N}$.

3. Preliminaries and Main Result

Before we state our main Theorem, let us prove some intermediate results. We start by giving Lemma 3.1 which can be very useful for many other purposes.

3.1. Polarization Inequalities and strict Inequalities:

Lemma 3.1. *Let u, v be two elements of F_N, $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a Borel measurable function such that:*
i) $F(w_1, w_2) + F(z_1, z_2) \leq F(\max(z_1, w_1), \max(z_2, w_2)) + F(\min(z_1, w_1), \min(z_2, w_2))$ for any $z_1 \neq w_1$ and $z_2 \neq w_2$. Then:

$$\int F(u, v) \leq \int F(u^H, v^H) \text{ for any } H \in \mathcal{H}$$

provided that both integrals are finite.

If in addition $v = v^*$ is strictly decreasing and i) holds with strict inequality then: If $\int F(u, v) = \int F(u^H, v^H) \quad \forall H \in \mathcal{H}$ then $u = u^H$ for any $H \in \mathcal{H}$, and therefore $u = u^*$.

Proof: Using the integrability assumptions, we can write:

$$\int F(u(x),v(x))\,dx = \int_H F(u(x),v(x)) + \int_H F(u(\sigma_H(x)),v(\sigma_H(x)))$$

$$\int F\left(u^H(x),v^H(x)\right)\,dx = \int_H F\left(u^H(x),v^H(x)\right) + \int_H F\left(u^H(\sigma_H(x)),v^H(\sigma_H(x))\right)$$

Therefore $\int F\left(u^H(x),v^H(x)\right) - \int F(u(x),v(x)) =$
$\int_H F\left(u^H(x),v^H(x)\right) - F(u(x),v(x)) - F(u(\sigma_H(x)),v(\sigma_H(x))) + F(u(\sigma_H(x)),v(\sigma_H(x)))$ (*)

For $x \in H$, set $z_1 = u(x), z_2 = v(x), w_1 = u(\sigma_H(x)), w_2 = v(\sigma_H(x))$. Then $\max(z_1,w_1) = u^H(x)$ and $\min(z_1,w_1) = u^H(\sigma_H(x))$. The result follows thanks to i).

Now if $v = v^*$ is strictly decreasing and (*) equals zero then $(u(x) - u(\sigma_H(x)))$ and $(v(x) - v(\sigma_H(x)))$ have the same sign (up to a set of measure zero).

Since $v = v^H \quad \forall H \in \mathcal{H}$, we certainly have that $v(x) \geq v(\sigma_H(x))$ for almost every $x \in H$ (here we used that $|\sigma_H(x)| \geq |x|$ for $x \in H$).

Hence $u(x) \geq u(\sigma_H(x))$ for a.e. $x \in H$. Using (1.7), the result follows. □

Corollary 3.2. *(Hardy-Littlewood inequality for polarization)*
For any $u \in L^p_+(\mathbb{R}^N)$ and $v \in L^q_+(\mathbb{R}^N)$ with $\frac{1}{p} + \frac{1}{q} = 1$:

(1) $\int uv \leq \int u^H v^H \qquad \forall H \in \mathcal{H}$.
If additionally $v = v^*$ is strictly decreasing then:
(2) *If $\int uv = \int u^H v \quad \forall H \in \mathcal{H}$, then $u = u^*$.*

Proof: set $F(r,s) = rs$ and apply Lemma 3.1. □

Remark 1: A similar result was proved in [7] with u^H replaced by u^* and v^H replaced by v^*.

Corollary 3.3. *(non-expansivity of two-point rearrangement)*
For any $u, v \in L^p_+(\mathbb{R}^N)$, $|u^H - v^H|_p \leq |u - v|_p \qquad \forall H \in \mathcal{H}$.

Proof: Set $F(r,s) = -|r-s|^p$ and apply Lemma 3.1. □

Remark 2.1: Corollary 3.3 tells us that if u_n is a sequence in $L^p_+(\mathbb{R}^N)$ converging to u in $L^p(\mathbb{R}^N)$, then $u_n^H \to u^H$ in $L^p(\mathbb{R}^N)$ for any $H \in \mathcal{H}$.

Remark 2.2: A similar result [3, Therorem 1] is obtained for Schwarz symmetrization. It implies that if $u_n \to u$ in $L^p(\mathbb{R}^N)$ then $u_n^* \to u^*$.

3.2. **Density Result.**

Lemma 3.4. *(Brock, Solynin [4, Lemma 6.1])*
Let $u \in L^p_+(\mathbb{R}^N)$ and $(H_n)_{n\geq 1}$ be a sequence of closed half spaces of $\mathbb{R}^N$ containing $0_{\mathbb{R}^N}$. Then:

$$u_n = u^{H_1 \ldots H_n} = \left[\left(u^{H_1}\right)^{H_2^{\cdot^{\cdot^{\cdot}}}}\right]^{H_n} \text{ is relatively compact in } L^p(\mathbb{R}^N).$$

Proof: Our method is more direct than that by Brock and Solynin. "Note that if u is Lipschitzean with compact support then the result is evident by Arzela's theorem since $|u_n|^p = \ldots = |u|^p$, (u_n) is equibounded and $w_{u_n} \leq \ldots \leq w_u$, asserting that u_n is equicontinuous (here w_u denotes the modulus of continuity of u)."

For the more general context, we will use Riesz-Fréchet-Kolmogorov results. First using (2.1), it follows that $\int |u^{H_1\ldots H_n}| = \int |u^{H_1\ldots H_{n-1}}|^p = \ldots \int |u|^p$.

Therefore (u_n) is bounded in $L^p(\mathbb{R}^N)$.

On the other hand, given $\varepsilon > 0$, we can find $R > 0$ such that $\int\limits_{|x|>R} |u|^p < \varepsilon$.

Since $\int\limits_{|x|>R} |u_{n+1}|^p \leq \int\limits_{|x|>R} |u_n|^p \leq \ldots \leq \int\limits_{|x|>R} |u|^p$, we can deduce that for any $n \in \mathbb{N}$: $\int\limits_{|x|>R} |u_n|^p \leq \varepsilon$.

Now for any τ_δ a family of strictly decreasing Schwarz symmetric functions we can catch a positive δ satisfying:

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} |u(x) - u(y)|^p \tau_\delta(x-y)\, dx\, dy < \varepsilon.$$

By [3, Theorem 2], we have:

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} |u_n(x) - u_n(y)|^p \tau_\delta(x-y)\, dx\, dy < \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} |u(x) - u(y)|^p \tau_\delta(x-y)\, dx\, dy < \varepsilon.$$

Finally applying the Riesz-Fréchet-Kolmogorov theorem [2], the conclusion follows. □

Theorem:
Let $u \in L^p_+(\mathbb{R}^N)$, $(H_n)_{n\geq 1}$ be a dense sequence in the set of closed half spaces containing $0_{\mathbb{R}^N}$.

Define $(u_n)_{n\geq 0}$ by $\begin{cases} u_0 &= u \\ u_{n+1} &= u_n^{H_1\ldots H_{n+1}} \end{cases}$

Then $u_n \to u^*$ in $L^p(\mathbb{R}^N)$.

Proof: Using Lemma 3.4, (up to a subsequence) u_n converges to v in $L^p(\mathbb{R}^N)$. Let f be a strictly decreasing Schwarz symmetric function in $L^q_+(\mathbb{R}^N)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

It follows, using m times [(1), Corollary 3.2], that:

$$\int u_n^{H_1\ldots H_m} f \leq \int u_{n+1} f; \quad \text{for any } m \in \mathbb{N} \text{ with } m \leq n. \tag{3.1}$$

Using Remark 2.1, we obtain by letting n go to infinity:

$$\int v^{H_1\ldots H_m} f \leq \int v f. \tag{3.2}$$

On the other hand, using once again [(1), Corollary 3.2], we know that:

$$\int v f \leq \int v^{H_1} f \leq \int v^{H_1 H_2} f \leq \ldots \int v^{H_1\ldots H_m} f. \tag{3.3}$$

(3.2) together with (3.3) imply that:

$$\int v f = \int v^{H_1} f = \int v^{H_1 H_2} f = \ldots \int v^{H_1\ldots H_m} f. \tag{3.4}$$

Hence it follows by [(2), Corollary 3.2] that $v = v^{H_1}$, $v^{H_1} = v^{H_1 H_2}$, ..., $v^{H_1\ldots H_{m-1}} = v^{H_1\ldots H_m}$.

Therefore $v = v^{H_1}$, $v^{H_1} = \left(v^{H_1}\right)^{H_2} = v^{H_2} = v$. It follows that $v = v^{H_k}$ for $1 \leq k \leq m$. But this is true for any $m \leq n$, from which we deduce that:

$$v = v^{H_k} \quad \forall k \in \mathbb{N}. \tag{3.5}$$

Now since $(H_n)_{n\geq 1}$ is dense in $\mathcal{H}$, for any $H \in \mathcal{H}$ we can find a subsequence (we will also denote it (H_n)) such that there exists $(i_n)_{n\geq 1}$ a sequence of isometries such that i_n converges to the identity with $H_n = i_n(H)$. Hence:

$$v^{H_n} \to v^H. \tag{3.6}$$

(3.5) together with (3.6) imply that $v = v^H \quad \forall H \in \mathcal{H}$ and therefore using (1.7):

$$v = v^*. \tag{3.7}$$

To conclude, we need to prove that $u^* = v^*$. Remark 2.2 tells us that:

$$u_n^* \to v^* \text{ in } L^p(\mathbb{R}^N). \tag{3.8}$$

On the other hand, we know that:

$$u^* \sim u \sim u_n \sim u_n^*. \tag{3.9}$$

$u_n^* \to v^*$ and $u_n^* \sim u^* \Rightarrow v^* \sim u^*$, thus $u^* = v^*$. □

Acknowledgement

The author is very grateful to F. Brock who brought to his attention the problem and for many fruitful discussions with him during his visit to AUB in June 2008. He also would like to thank A. Burchard and T. Weth for several discussions in connection with some material discussed in this paper.

Happy birthday Al, you showed us the way!

References

[1] A. Baernstein II. A unified approach to symmetrization. Partial differential equations of elliptic type. Cambridge Univ. Press, Cambridge, (1994), 47-91.
[2] H. Brezis. Analyse fonctionnelle, Théorie et applications. Editions Masson, 1984.
[3] A. Burchard, H. Hajaiej. Rearrangement inequalities for functional with monotone integrands. Journal of Functional Analysis, 233, (2006), 561–582.
[4] F.Brock, Y. Solynin. An approach to symmetrization via polarization. Trans. Amer. Math. Soc., 352, (2000), 1759-1796.
[5] F. Brock. Rearrangement inequalities à la Hardy-littlewood. J. Ineq. Appl., (2000), 309–320.
[6] C. Draghici. Rearrangement inequalities with applications to ratio of heat kernels. Potential Analysis 22, (2005), 351–374.
[7] H. Hajaiej. Cases of equality and strict inequality in the extended Hardy-Littlewood inequalities. Proc. Roy. Soc. Edinburgh, 135 A, (2005), 643–661.
[8] H. Hajaiej, M. Squassina. Generalized Pólya-Szegö inequality and applications to some quasilinear problems. Under review.
[9] J. Van Schaftingen. Approximation of symmetrization and symmetry of critical points. Top Methods Nonlinear Anal., (2006), 61–85.

Institut Préparatoire aux Etudes d'Ingénieurs de Tunis
E-mail address: hichem.hajaiej@gmail.com