Boundary blow-up solutions to fractional elliptic equations in a measure framework

Huyuan Chen[1] **Ying Wang**[2] **Hichem Hajaiej**[3]

Abstract

Let $\alpha \in (0,1)$, Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with C^2 boundary $\partial\Omega$ and ω be the Hausdorff measure on $\partial\Omega$. We denote by $\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}$ a measure

$$\langle \frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}, f\rangle = \int_{\partial\Omega} \frac{\partial^\alpha f(x)}{\partial \vec{n}_x^\alpha} d\omega(x), \quad f \in C^1(\bar{\Omega}),$$

where $\vec{n}_x$ is the unit outward normal vector at point $x \in \partial\Omega$. In this paper, we prove that problem

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= k\tfrac{\partial^\alpha \omega}{\partial \vec{n}^\alpha} \quad \text{in} \quad \bar{\Omega}, \\ u &= 0 \quad \text{in} \quad \Omega^c \end{aligned} \tag{0.1}$$

admits a unique weak solution u_k under the hypotheses that $k > 0$, $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha \in (0,1)$ and g is a nondecreasing function satisfying extra conditions. We prove that the weak solution of (0.1) is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \setminus \bar{\Omega}, \\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) &= +\infty. \end{aligned}$$

Key Words: Fractional Laplacian, Green kernel, Boundary blow-up solution.

MSC2010: 35R11, 35J61, 35R06

1 Introduction

1.1 motivation

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with C^2 boundary $\partial\Omega$. The pioneering works [13, 16] obtained that the nonlinear reaction diffusion equation

$$\begin{aligned} -\Delta u + h(u) &= 0 \quad \text{in} \quad \Omega, \\ u &= +\infty \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1.1}$$

admits a solution if h is a locally Lipschitz continuous function which is increasing and satisfies Keller-Osserman condition

$$\int_1^{+\infty} \left[\int_0^s h(t)dt\right]^{-\frac{1}{2}} ds < +\infty.$$

Great interests in existence, uniqueness and asymptotic behavior of boundary blow-up solution to (1.1) have been taken, see [1, 9, 11, 14, 19, 20, 15]. It is well known that when

[1]hc64@nyu.edu
[2]ywang@dim.uhicle.cl
[3]hh62@nyu.edu

$h(s) = s^p$ with $p > 1$, (1.1) has a unique solution with boundary asymptotic behavior $\rho^{-\frac{2}{p-1}}(x)$, where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$.

Comparing with the Laplacian case, a much richer structure for the solutions set appears for the non-local case. Recently, the authors in [5] obtained very different phenomena of the boundary blow-up solutions to elliptic equations involving the fractional Laplacian, precisely,

$$\begin{aligned} (-\Delta)^\alpha u + |u|^{p-1}u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c, \\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) &= +\infty, \end{aligned} \tag{1.2}$$

where $p > 0$ and the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon\to 0^+} (-\Delta)^\alpha_\epsilon u(x),$$

here for $\epsilon > 0$,

$$(-\Delta)^\alpha_\epsilon u(x) = -\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{u(z) - u(x)}{|z-x|^{N+2\alpha}} dz$$

The existence of boundary blow-up solution of (1.2) is derived by constructing appropriate super and sub-solutions and this construction involves the one dimensional truncated Laplacian of power functions given by

$$C(\tau) = \int_0^{+\infty} \frac{\chi_{(0,1)}(t)|1-t|^\tau + (1+t)^\tau - 2}{t^{1+2\alpha}} dt, \tag{1.3}$$

where $\tau \in (-1, 0)$ and $\chi_{(0,1)}$ is the characteristic function of the interval $(0,1)$. It is known that there exists a unique zero point of (1.3) in $(-1,0)$, denoting $\tau_0(\alpha)$. Then

Proposition 1.1 *[5, Theorem 1.1] Assume that Ω is an open, bounded and connected domain of class C^2 and $\alpha \in (0,1)$. Then we have:*
Existence*: Assume that*

$$1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)},$$

the equation (1.2) possesses at least one solution u satisfying

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} \le \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty. \tag{1.4}$$

Uniqueness*: u is the unique solution of (1.2) satisfying (1.4).*
Nonexistence*: In the following three cases:*

i) For any $\tau \in (-1,0) \setminus \{-\frac{2\alpha}{p-1},\ \tau_0(\alpha)\}$ and

$$1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)} \quad \text{or}$$

ii) For any $\tau \in (-1,0)$ and

$$p \ge 1 - \frac{2\alpha}{\tau_0(\alpha)} \quad \text{or}$$

iii) For any $\tau \in (-1,0) \setminus \{\tau_0(\alpha)\}$ and

$$1 < p \le 1 + 2\alpha,$$

problem (1.2) does not have a solution u satisfying

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} \le \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} < +\infty. \tag{1.5}$$

Special existence for $\tau = \tau_0(\alpha)$. *Assume that*

$$\max\{1 - \frac{2\alpha}{\tau_0(\alpha)} + \frac{\tau_0(\alpha)+1}{\tau_0(\alpha)}, 1\} < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}.$$

Then for any $t > 0$, there is a positive solution u of equation (1.2) satisfying

$$\lim_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau_0(\alpha)} = t.$$

There are some challenging questions to ask:

1. Could $\tau_0(\alpha)$ be expressed explicitly?

2. With what condition of general nonlinearity makes existence hold?

3. The uniqueness and nonexistence restricts in the class functions (1.4) and (1.5), so are there some solutions breaking the assumption (1.4)?

Our interest in this article is to introduce a new method to study the boundary blow-up solutions of semilinear fractional elliptic equations and answer above questions. The main idea is to find suitable type measure concentrated on the whole boundary and then by making basic estimates to prove that the corresponding weak solution solves (1.2). Our first result is stated as follows:

Proposition 1.2 *Let $\alpha \in (0,1)$ and $\tau_0(\alpha)$ is the zero point of $C(\cdot)$ when $C(\cdot)$ given by (1.3), then*

$$\tau_0(\alpha) = \alpha - 1.$$

We observe that the critical value $1 - \frac{2\alpha}{\tau_0(\alpha)}$ in Proposition 1.1 turns out to be $\frac{1+\alpha}{1-\alpha}$. In what follows, we would like to show the details of our new method and answer the second and third questions in the following.

1.2 A new method and main results

Let $\alpha \in (0,1)$ and ω be the Hausdorff measure on $\partial\Omega$. We denote by $\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}$ a measure

$$\langle \frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}, f\rangle = \int_{\partial\Omega} \frac{\partial^\alpha f(x)}{\partial \vec{n}_x^\alpha} d\omega(x), \qquad f \in C^\alpha(\bar{\Omega}),$$

where $\vec{n}_x$ is the unit inward normal vector of $\partial\Omega$ at point x and

$$\frac{\partial^\alpha f(x)}{\partial \vec{n}_x^\alpha} = \lim_{t\to 0^+} \frac{f(x + t\vec{n}_x) - f(x)}{t^\alpha}.$$

In this paper, we are concerned with the existence and uniqueness of weak solution to the semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + g(u) = k\tfrac{\partial^\alpha \omega}{\partial \vec{n}^\alpha} \quad & \text{in} \quad \bar{\Omega}, \\ u = 0 \qquad\quad & \text{in} \quad \bar{\Omega}^c, \end{aligned} \tag{1.6}$$

where $k > 0$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

In [6], the authors studied problem (1.6) replaced $\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}$ by $\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}$ where ν is a Radon measure concentrated on boundary measure. They proved that such a problem has a unique weak solution if g is a continuous nondecreasing function satisfying $g(0) \geq 0$ and

$$\int_1^\infty g(s) s^{-1-\frac{N+\alpha}{N-\alpha}} ds < +\infty. \tag{1.7}$$

Moreover, [6] analyzed the isolated singularity of weak solution of (1.6) in the case that $\nu = \delta_{x_0}$ with $x_0 \in \partial\Omega$. Our aim in this article is to investigate how the Hausdorff measure on $\partial\Omega$ works on the weak solution of (1.6).

Before starting our main theorems we make precise the notion of weak solution used in this note.

Definition 1.1 *We say that u is a weak solution of (1.6), if $u \in L^1(\Omega)$, $g(u) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + g(u)\xi] dx = k \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\omega(x), \qquad \forall \xi \in \mathbb{X}_\alpha.$$

where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$ and $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ denotes the space of functions ξ satisfying:

(i) $\mathrm{supp}(\xi) \subset \bar{\Omega}$*;*

(ii) $(-\Delta)^\alpha \xi(x)$ *exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \leq C$ for some $C > 0$;*

(iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ and $\varepsilon_0 > 0$ such that $|(-\Delta)^\alpha_\varepsilon \xi| \leq \varphi$ a.e. in Ω for all $\varepsilon \in (0, \varepsilon_0]$.

Now we are ready to state our first result for problem (1.6).

Theorem 1.1 *Assume that $k > 0$, $\rho(x) = \mathrm{dist}(x, \partial\Omega)$ and g is a continuous nondecreasing function satisfying $g(0) \geq 0$ and*

$$\int_1^\infty g(s) s^{-1-\frac{1+\alpha}{1-\alpha}} ds < +\infty. \tag{1.8}$$

Then (i) problem (1.6) admits a unique positive weak solution u_k;

(ii) the mapping $k \to u_k$ is increasing and there exists $c_1 \geq 1$ independent of k such that

$$\frac{k}{c_1} \rho(x)^{\alpha-1} \leq u_k(x) \leq c_1 k \rho(x)^{\alpha-1}, \qquad \forall x \in \Omega; \tag{1.9}$$

(iii) if we assume additionally that g is C^β locally in $\mathbb{R}$ with $\beta > 0$, then u_k is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + g(u) = 0 \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \mathbb{R}^N \setminus \bar{\Omega}, \\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) = +\infty. & \end{aligned} \tag{1.10}$$

We remark that in Theorem 1.1 extends the special existence of boundary blow up solutions to fractional elliptic equation (1.10) with general nonlinearity g in integral subcritical case with the critical exponents $\frac{1+\alpha}{1-\alpha}$ with is larger than $\frac{N+\alpha}{N-\alpha}$. Specially, letting $g \equiv 0$, there exists infinitely many boundary blow up $\alpha-$harmonic functions.

Since $\alpha - 1 > -\frac{2\alpha}{p-1}$, so we may call the solutions of (1.10) as the weak boundary blow-up solution from the asymptotic behavior (1.9). Our second interest is to consider the limit of weak boundary blow-up solutions.

Theorem 1.2 *Let $g(s)=s^p$ with $p\in(0,\frac{1+\alpha}{1-\alpha})$ and u_k be the weak solution of (1.6), then*

(i) if $p\in(1+2\alpha,\frac{1+\alpha}{1-\alpha})$, then the limit of $\{u_k\}$ as $k\to\infty$ exists, denoting u_∞, which is a classical solution of (1.2). Moreover, u_∞ satisfies

$$\frac{1}{c_2}\rho(x)^{-\frac{2\alpha}{p-1}}\le u_\infty(x)\le c_2\rho(x)^{-\frac{2\alpha}{p-1}},\qquad \forall x\in\Omega, \tag{1.11}$$

where $c_2\ge 1$.

(ii) if $p\in(0,1+2\alpha]$, then

$$\lim_{k\to\infty}u_k(x)=+\infty,\qquad \forall x\in\Omega.$$

We notice that the limit of weak boundary blow-up solutions is the solution of (1.1) with behavior (1.4) when $p\in(1+2\alpha,\frac{1+\alpha}{1-\alpha})$ stated in Proposition 1.1. As a consequence of Theorem 1.2 *(ii)*, $p\in(0,1+2\alpha]$, there is no solution u of (1.1) such that

$$\lim_{x\in\Omega,x\in\Omega}u(x)\rho^{1-\alpha}(x)\equiv 0\quad\text{or}\quad+\infty.$$

From [6] and Theorem 1.1, the Dirac mass and Hausdorff measure have different contribution to the solution of

$$(-\Delta)^\alpha u+g(u)=0\quad\text{in}\quad\Omega.$$

Our interest is to understand what singularity of the solution to

$$\begin{aligned}(-\Delta)^\alpha u+g(u)&=\tfrac{\partial^\alpha(\omega+\delta_{x_0})}{\partial\vec{n}^\alpha}\qquad &&\text{in}\quad\bar{\Omega},\\ u&=0 &&\text{in}\quad\bar{\Omega}^c,\end{aligned}\tag{1.12}$$

where $x_0\in\partial\Omega$ and δ_{x_0} is the Dirac mass concentrated x_0 on the boundary. Inspired by Definition 1.1, it is natural to give the definition of weak solution of (1.12) as following.

Definition 1.2 *We say that u is a weak solution of (1.12), if $u\in L^1(\Omega)$, $g(u)\in L^1(\Omega,\rho^\alpha dx)$ and*

$$\int_\Omega[u(-\Delta)^\alpha\xi+g(u)\xi]dx=\int_{\partial\Omega}\frac{\partial^\alpha\xi(x)}{\partial\vec{n}_x^\alpha}d\omega(x)+\frac{\partial^\alpha\xi(x_0)}{\partial\vec{n}_{x_0}},\qquad\forall\xi\in\mathbb{X}_\alpha.$$

Theorem 1.3 *Assume that $x_0\in\partial\Omega$, g is a continuous nondecreasing function satisfying $g(0)\ge 0$,*

$$\int_1^\infty g(s)s^{-1-\frac{N+\alpha}{N-\alpha}}ds<+\infty \tag{1.13}$$

and for some $\lambda>0$,

$$g(s+t)\le\lambda[g(s)+g(t)],\quad\forall s,t>0. \tag{1.14}$$

Then problem (1.6) admits a unique positive weak solution v such that

$$\frac{1}{c_3}\left[\rho(x)^{\alpha-1}+\frac{\rho(x)^\alpha}{|x-x_0|^N}\right]\le v(x)\le c_3\left[\rho(x)^{\alpha-1}+\frac{\rho(x)^\alpha}{|x-x_0|^N}\right],\qquad\forall x\in\Omega. \tag{1.15}$$

Moreover, if assume additionally that g is C^β locally in $\mathbb{R}$ with $\beta>0$, then v is a classical solution of (1.10).

From Theorem 1.3, we find out a classical solution of (1.10) with explosive rate $\rho(x)^{\alpha-1}+\frac{\rho(x)^\alpha}{|x-x_0|^N}$, this answers the question 3 in the first part of the introduction.

The boundary blow-up solutions of (1.10) could be searched for by making use of measure type data on boundary and the main difficulty is to do the estimate of $\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}]$ and $g(\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}])$. Especially, it is dedicate to make the estimate of $g(\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}])$ near the boundary when the nonlinearity g is just integral-subcritical, i.e. (1.8).

This article is organized as follows. In Section §2 we present some preliminaries to the Marcinkiewicz type estimate for $\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}]$ and present the existence and uniqueness of weak solution of (1.6) when g is bounded. Section §3, §4 are devoted to prove Theorem 1.1 and Theorem 1.2. Finally, we obtain one typical solution that blows up along the boundary with different power rate.

2 Preliminary

2.1 The Marcinkiewicz type estimate

In order to obtain the weak solution of (1.6) with integral subcritical nonlinearity, we have to introduce the Marcinkiewicz space and recall some related estimate.

Definition 2.1 *Let $\Theta\subset\mathbb{R}^N$ be a domain and ϖ be a positive Borel measure in Θ. For $\kappa>1$, $\kappa'=\kappa/(\kappa-1)$ and $u\in L^1_{loc}(\Theta,d\mu)$, we set*

$$\|u\|_{M^\kappa(\Theta,d\varpi)}=\inf\left\{c\in[0,\infty]:\int_E|u|d\varpi\le c\left(\int_E d\varpi\right)^{\frac{1}{\kappa'}},\ \forall E\subset\Theta,\ E\text{ Borel}\right\}\tag{2.1}$$

and

$$M^\kappa(\Theta,d\varpi)=\{u\in L^1_{loc}(\Theta,d\varpi):\|u\|_{M^\kappa(\Theta,d\varpi)}<+\infty\}.\tag{2.2}$$

The space $M^\kappa(\Theta,d\varpi)$ is called the Marcinkiewicz space of exponent κ, or weak L^κ-space and $\|.\|_{M^\kappa(\Theta,d\varpi)}$ is a quasi-norm.

Proposition 2.1 *[2, 7] Assume that $1\le q<\kappa<\infty$ and $u\in L^1_{loc}(\Theta,d\varpi)$. Then there exists $c_4>0$ dependent of q,κ such that*

$$\int_E|u|^q d\varpi\le c_4\|u\|_{M^\kappa(\Theta,d\varpi)}\left(\int_E d\varpi\right)^{1-q/\kappa}$$

for any Borel set E of Θ.

Denote by G_α the Green kernel of $(-\Delta)^\alpha$ in $\Omega\times\Omega$ and by $\mathbb{G}_\alpha[\cdot]$ the Green operator defined as

$$\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}](x)=\lim_{t\to0^+}\int_{\partial\Omega}G_\alpha(x,y+t\vec{n}_y)t^{-\alpha}d\omega(y).$$

Our purpose in this subsection is to do Marcinkiewicz type estimate for $\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}]$.

Lemma 2.1 *There exists $c_5\ge1$ such that for any $x\in\Omega$,*

$$\frac{1}{c_5}\rho(x)^{\alpha-1}\le\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}](x)\le c_5\rho(x)^{\alpha-1}.\tag{2.3}$$

Proof. Since $\partial\Omega$ is C^2, then there exists $t_0 \in (0, \frac{1}{2})$ such that for any $x \in \Omega_t := \{z \in \Omega, \rho(x) < t\}$ with $t < t_0$, there exists a unique $x_\partial \in \partial\Omega$ such that

$$|x - x_\partial| = \rho(x)$$

and for $t \in (0, t_0)$ letting

$$\mathcal{C}_t = \{x \in \Omega : \ \rho(x) = t\},$$

$\mathcal{C}_t$ is C^2 for $t \in (0, t_0)$ and any Borel set E_t in $\mathcal{C}_t$, there exists unique set $E \subset \partial\Omega$ such that

$$x_t \in E_t \quad \text{for} \quad x \in E.$$

In fact, for $x \in \mathcal{C}_t$ with $t \in (0, t_0)$, there exists a unique $x_\partial \in \partial\Omega$ such that

$$x = t\vec{n}_{x_\partial} + x_\partial \quad \text{and} \quad |x - x_\partial| = t = \rho(x).$$

Denotes by ω_t a measure on $\mathcal{C}_t$ generated by ω such that for $t \in (0, t_0)$,

$$\omega_t(E_t) = \omega(E) \qquad \text{for any Borel set} \ \ E_t \subset \mathcal{C}_t. \tag{2.4}$$

By compactness we only have to prove that (2.3) holds in a neighborhood of any point $\bar{x} \in \partial\Omega$ and without loss of generality, we may assume that

$$x_\partial = 0 \quad \text{and} \quad \vec{n}_{x_\partial} = e_N.$$

From [6, Lemma 2.1], there exists $c_6 > 0$ such that

$$\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x) \le \int_{\partial\Omega} \frac{c_6}{|x - y|^{N-\alpha}} d\omega(y), \qquad \forall x \in \Omega. \tag{2.5}$$

Let $\phi : B'_{t_0}(0) \to \mathbb{R}$ such that $(y', \phi(y') \in \partial\Omega$, where $B'_{t_0}(0)$ is the ball centered at origin with radium t_0 in $\mathbb{R}^{N-1}$. We choose some $s_0 \in (0, t_0)$ small enough, there exists $c_7 \ge 1$ such that for any Borel set $E \subset B_{s_0}(0) \cap \partial\Omega$,

$$\frac{1}{c_7}|E'| \le \omega(E) \le c_7|E'|,$$

where

$$E' = \{y' \in \mathbb{R}^{N-1} : \quad (y', \phi(y')) \in E\}.$$

For $s_0 > 0$ small, there exists $c_8 > 0$ such that for $y = (y', y_N) \in B_{s_0}(0) \cap \partial\Omega$

$$|te_N - y| \ge c_8 |te_N - (y', 0)| = c_8 \sqrt{t^2 + |y'|^2}$$

Therefore,

$$\begin{aligned}
\int_{B_{s_0}(0)\cap\partial\Omega} \frac{1}{|te_N - y|^{N-\alpha}} d\omega(y) &\le \int_{B_{s_0}(0)\cap\partial\Omega} \frac{1}{(t^2 + |y'|^2)^{\frac{N-\alpha}{2}}} d\omega(y) \\
&\le c_9 \int_{B'_{s_0}(0)} \frac{1}{(t^2 + |y'|^2)^{\frac{N-\alpha}{2}}} dy' \\
&= c_{10} \int_0^{s_0} \frac{s^{N-2}}{(t^2 + s^2)^{\frac{N-\alpha}{2}}} ds \\
&= c_{10} t^{\alpha-1} \int_0^{\frac{s_0}{t}} \frac{s^{N-2}}{(1 + s^2)^{\frac{N-\alpha}{2}}} ds \\
&\le c_{11} t^{\alpha-1},
\end{aligned}$$

where $c_{11}=\int_0^{+\infty}\frac{c_{10}}{(1+s^2)^{\frac{2-\alpha}{2}}}ds<+\infty$ since $\frac{2-\alpha}{2}>\frac12$. For $y\in\partial\Omega\setminus B_{s_0}(0)$, there exists $c_{12}>0$ such that $|te_N-y|\ge c_{12}s_0$, then

$$\int_{\partial\Omega\setminus B_{s_0}(0)}\frac{1}{|te_N-y|^{N-\alpha}}d\omega(y)\le c_{12}s_0^{\alpha-N}\int_{\partial\Omega\setminus B_{s_0}(0)}d\omega(y)\le c_{12}s_0^{\alpha-N}\omega(\partial\Omega).$$

Therefore, for $t\in(0,t_0)$,

$$\int_{\partial\Omega}\frac{1}{|te_N-y|^{N-\alpha}}d\omega(y)\le c_{12}t^{\alpha-1}.$$

For $x\in\Omega\setminus\Omega_{t_0}$ and $y\in\partial\Omega$, we observe that $|x-y|\ge t_0$, then $\int_{\partial\Omega}\frac{1}{|x-y|^{N-\alpha}}d\omega(y)$ is bounded by some constant dependent of t_0 and the diameter of Ω, thus, (2.3) holds.

We now prove that for $t\in(0,t_0)$,

$$\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}](te_N)\ge\frac{1}{c_6}t^{\alpha-1}. \tag{2.6}$$

For all $s\in(0,\frac t8)$, we have that

$$|te_N-y|>\frac t2\quad\text{for}\quad y\in\mathcal{C}_s\cap B_{\frac t4}(se_N).$$

Therefore,

$$|te_N-y|>\frac t2=\frac12\max\{\rho(y),\rho(te_N)\}$$

and apply [4, Theorem 1.2] to derive that there exists $c_{13}>0$ such that for all $s\in(0,\frac t8)$

$$G_\alpha(te_N,y)\ge c_{13}\frac{\rho^\alpha(y)\rho^\alpha(te_N)}{|te_N-y|^N}=c_{13}\frac{t^\alpha s^\alpha}{|te_N-y|^N},\quad y\in\mathcal{C}_s\cap B_{\frac t4}(se_N). \tag{2.7}$$

Thus,

$$\mathbb{G}_\alpha[s^{-\alpha}\omega_s](te_N)\ge c_{13}\int_{\mathcal{C}_s\cap B_{\frac t4}(se_N)}\frac{t^\alpha}{|te_N-y|^N}d\omega_s(y).$$

Denote

$$D_{t,s}=\mathcal{C}_s\cap B_{\frac t4}(se_N)\quad\text{and}\quad D_t=\partial\Omega\cap B_{\frac t4}(0).$$

We observe that

$$|te_N-y|\le c_{14}t,\quad\forall y\in D_{t,s}$$

and

$$\frac{1}{c_{14}}t^{N-1}\le\omega_s(D_{t,s})\le c_{14}t^{N-1},$$

where $c_{14}>1$, then for any $s\in(0,\frac t8)$

$$\begin{aligned}\int_{\mathcal{C}_s}\frac{t^\alpha}{|te_N-y|^N}d\omega_s(y)\ &\ge\ \int_{D_{t,s}}\frac{t^\alpha}{|x-y|^N}d\omega_{\mathcal{C}_s}(y)\\ &\ge\ c_{15}t^\alpha t^{-N}\omega_s(D_{t,s})\ge c_{16}t^{\alpha-1},\end{aligned}$$

which implies (2.6) by passing the limit of $s\to0^+$. $\square$

Proposition 2.2 *Let* $\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}]$ *given by (2.5) and* $p^* = \frac{1+\alpha}{1-\alpha}$. *Then there exists* $c_{17} > 0$ *such that*

$$\|\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}]\|_{M^{p^*}(\Omega,\rho^\alpha dx)} \le c_{17}. \tag{2.8}$$

Proof. For any Borel set E of Ω satisfying

$$0 < |E| < |\Omega_{t_0}|,$$

where $\Omega_r = \{x \in \Omega,\ \rho(x) < r\}$ for $r > 0$, there exists $t \in (0, t_0)$ such that

$$|E| = |\Omega_t|.$$

Then there exists $c_{18} > 0$ such that

$$|\Omega_t| = \int_0^t \omega_t(\mathcal{C}_t)dt \le c_{18} t.$$

We observe that

$$|E \setminus \Omega_t| = |E| - |E \cap \Omega_t| = |\Omega_t \setminus E|$$

and

$$\rho(y) \le \rho(z), \quad \forall y \in E \setminus \Omega_t,\ \forall z \in \Omega_t \setminus E.$$

Then

$$\int_E \rho^\alpha dx \ \ \ge \ \int_{\Omega_t} \rho^\alpha(x)dx = c_{19} \int_0^t \int_{\mathcal{C}_s} s^\alpha d\omega_s ds \le c_{20} t^{\alpha+1}$$

and together with (2.3), we deduce that

$$\begin{aligned}
\int_E \mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x)\rho^\alpha(x)dx &\le \int_E \int_{\partial\Omega} \frac{c_6}{|x-y|^{N-\alpha}} d\omega(y)\rho^\alpha(x)dx \\
&\le c_{21} \int_E \rho^{2\alpha-1}(x)dx \le c_{21} \int_{\Omega_t} \rho^{2\alpha-1}(x)dx \\
&= c_{22} \int_0^t \int_{\mathcal{C}_s} s^{2\alpha-1} d\omega_s ds = c_{23} t^{2\alpha} \\
&\le c_{23} (\int_E \rho^\alpha dx)^{\frac{2\alpha}{1+\alpha}},
\end{aligned}$$

where $c_{22}, c_{23} > 0$. Therefore,

$$\int_E \mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x)\rho^\alpha(x)dx \le c_{23}|E|^{\frac{2\alpha}{1+\alpha}}.$$

Together with

$$\frac{2\alpha}{1+\alpha} = \frac{p^*-1}{p^*},$$

we derive (2.8). This completes the proof. $\square$

2.2 Existence for bounded nonlinearity

We extend Hausdorff measure ω to $\bar{\Omega}$ by zero inside Ω, still denoting ω. For bounded C^2 domain, it follows [18, p 57] that ω is a Radon measure in $\bar{\Omega}$. In the approximating to weak solution of (1.6), we consider a sequence $\{g_n\}$ of C^1 nonnegative functions defined on $\mathbb{R}_+$ such that $g_n(0) = g(0)$,

$$g_n \le g_{n+1} \le g, \quad \sup_{s\in\mathbb{R}_+} g_n(s) = n \quad \text{and} \quad \lim_{n\to\infty} \|g_n - g\|_{L^\infty_{loc}(\mathbb{R}_+)} = 0. \tag{2.9}$$

Proposition 2.3 *Assume that $\{g_n\}_n$ is given by (2.9). Then*

$$\begin{aligned} (-\Delta)^\alpha u + g_n(u) &= k\tfrac{\partial^\alpha \omega}{\partial \vec{n}^\alpha} \quad && \text{in} \quad \Omega, \\ u &= 0 && \text{in} \quad \Omega^c \end{aligned} \tag{2.10}$$

admits a unique positive weak solution $u_{k,n}$ satisfying

(i) the mapping $k \to u_{k,n}$ is increasing, the mapping $n \to u_{k,n}$ is decreasing

$$k\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x) - k\mathbb{G}_\alpha[g_n(k\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}])](x) \le u_{k,n}(x) \le k\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x), \quad \forall x \in \Omega; \tag{2.11}$$

(ii) $u_{k,n}$ is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + g_n(u) &= 0 \quad && \text{in} \quad \Omega, \\ u &= 0 && \text{in} \quad \mathbb{R}^N \setminus \bar{\Omega}, \\ \lim\nolimits_{x\in\Omega, x\to\partial\Omega} u(x) &= +\infty. \end{aligned} \tag{2.12}$$

Proof. Since ω is a Radon measure in $\bar{\Omega}$, we could apply [6, Theorem 1.1] to obtain that problem (5.6) admits a unique weak solution $u_{k,n}$ satisfying that (i) and $u_{k,n}$ is a classical solution of

$$(-\Delta)^\alpha u + g_n(u) = 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \bar{\Omega}.$$

From Lemma 2.1 and (2.11), there exists $c_{24} \ge 1$ such that

$$\frac{1}{c_{24}} \rho(x)^{\alpha-1} \le u_{k,n}(x) \le c_{24} \rho(x)^{\alpha-1}, \quad x \in \Omega. \tag{2.13}$$

Therefore, $u_{k,n}$ is a classical solution of (2.12). □

In particular, let $g_0 \equiv 0$, we have that

Corollary 2.1 $\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}]$ *is a classical solution of*

$$\begin{aligned} (-\Delta)^\alpha u &= 0 \quad && \text{in} \quad \Omega, \\ u &= 0 && \text{in} \quad \mathbb{R}^N \setminus \bar{\Omega}, \\ \lim\nolimits_{x\in\Omega, x\to\partial\Omega} u(x) &= +\infty. \end{aligned} \tag{2.14}$$

With the help of Corollary 2.1, we are in the position to prove Proposition 1.2.

Proof of Proposition 1.2. We first prove that $\tau_0(\alpha) \le \alpha - 1$. Inversely, if $\tau_0(\alpha) > \alpha - 1$, then we have that

$$1 - \frac{2\alpha}{\tau_0(\alpha)} > \frac{1+\alpha}{1-\alpha} > 1 + 2\alpha.$$

On the one hand, it follows by [5, Theorem 1.1] that for $p = \frac{1+\alpha}{1-\alpha}$, problem

$$\begin{aligned} (-\Delta)^\alpha u + |u|^{p-1}u = 0 \qquad & \text{in} \quad \Omega, \\ u = 0 \qquad & \text{in} \quad \mathbb{R}^N \setminus \bar{\Omega}, \\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) = +\infty & \end{aligned} \tag{2.15}$$

admits a solution w such that

$$\frac{1}{c_{25}} \le w(x)\rho^{1-\alpha}(x) \le c_{25}, \quad x \in \Omega,$$

where $c_{25} > 1$ and

$$-\frac{2\alpha}{\frac{1+\alpha}{1-\alpha} - 1} = \alpha - 1.$$

On the other hand, form Corollary 2.1 we know that for any $\mu > 0$, $\mu\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}]$ is a super solution of problem (2.15). Furthermore, from Lemma 2.1,

$$\frac{1}{c_5} \le \mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x)\rho^{1-\alpha} \le c_5.$$

Now choosing $\mu_1 = \frac{1}{2c_{25}c_5}$ and $\mu_2 = 2c_{25}c_5$, we derive that

$$\limsup_{x\in\Omega,\ x\to\partial\Omega} \mu_1 \mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x)\rho(x)^{1-\alpha} < \frac{1}{c_{25}}, \quad \liminf_{x\in\Omega,\ x\to\partial\Omega} \mu_2 \mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x)\rho(x)^{1-\alpha} > c_{25}.$$

thus, from [5, Proposition 6.1], there is no solution u such that

$$\frac{1}{c_{25}} \le \liminf_{x\in\Omega,\ x\to\partial\Omega} u(x)\rho(x)^{1-\alpha} \le \limsup_{x\in\Omega,\ x\to\partial\Omega} u(x)\rho(x)^{1-\alpha} \le c_{25}.$$

The contradiction is obvious.

We finally prove that $\tau_0(\alpha) \ge \alpha - 1$. Inversely, if $\tau_0(\alpha) < \alpha - 1$, then we have that

$$1 - \frac{2\alpha}{\tau_0(\alpha)} < \frac{1+\alpha}{1-\alpha}.$$

From [5, Theorem 1.1] nonexistence (ii), there is no solution u of problem (2.15) with

$$\max\{1 - \frac{2\alpha}{\tau_0(\alpha)}, \frac{2\alpha}{1-\alpha}\} < p < \frac{1+\alpha}{1-\alpha} \tag{2.16}$$

such that

$$0 < \liminf_{x\in\Omega,\ x\to\partial\Omega} u(x)\rho(x)^{1-\alpha} \le \limsup_{x\in\Omega,\ x\to\partial\Omega} u(x)\rho(x)^{1-\alpha} < +\infty. \tag{2.17}$$

Let $\bar{\tau} = 2\alpha - (1-\alpha)p$, then

$$\bar{\tau} > 2\alpha - (1-\alpha)\frac{1+\alpha}{1-\alpha} = \alpha - 1$$

and

$$\bar{\tau} < 2\alpha - (1-\alpha)\frac{2\alpha}{1-\alpha} = 0.$$

For $t_0>0$ small, $\Omega_{t_0}=\{x\in\Omega,\rho(x)<t_0\}$ is C^2 and define

$$V_1(x)=\begin{cases} d(x)^{\bar\tau}, & x\in\Omega_{t_0},\\ l(x), & x\in\Omega\setminus\Omega_{t_0},\\ 0, & x\in\Omega^c,\end{cases} \tag{2.18}$$

where the function l is positive such that V_1 is C^2 in Ω. From [5, Proposition 3.2 (ii)], there exists $\delta_1\in(0,t_0]$ and $c_{26}>1$ such that

$$\frac{1}{c_{26}}\rho(x)^{\bar\tau-2\alpha}\le(-\Delta)^\alpha V_1(x)\le c_{26}\rho(x)^{\bar\tau-2\alpha},\quad \forall x\in\Omega_{\delta_1}.$$

We observe that $\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}]$ is a super solution of (2.15) with p in (2.16). Now we define

$$W_\mu(x)=\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}]-\mu V_{\bar\tau}(x)-\mu^2\mathbb{G}_\alpha[1],$$

where $\mathbb{G}_\alpha[1]$ is the solution of

$$\begin{aligned}(-\Delta)^\alpha u=1 &\quad \text{in}\quad \Omega,\\ u=0 &\quad \text{in}\quad \mathbb{R}^N\setminus\Omega.\end{aligned}$$

We see that $\bar\tau-2\alpha=(\alpha-1)p$, there exists $\mu_1>0$ such that for $\mu\ge\mu_1$ and $x\in\Omega_{\delta_1}$,

$$(-\Delta)^\alpha W_\mu(x)+|W_\mu|^{p-1}W_\mu(x)\ \le\ -c_{26}\mu\rho(x)^{\bar\tau-2\alpha}+c_5^p\rho(x)^{(\alpha-1)p}\le 0$$

and there exists $\mu_2>0$ such that for $\mu\ge\mu_2$ and $x\in\Omega\setminus\Omega_{\delta_1}$,

$$(-\Delta)^\alpha W_\mu(x)+|W_\mu|^{p-1}W_\mu(x)\ \le\ c_{26}\mu\max_{\Omega\setminus\Omega_{\delta_1}}|(-\Delta)^\alpha V_1|-\mu^2+\Big(\min_{\Omega\setminus\Omega_{\delta_1}}\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}]\Big)^p\le 0,$$

Therefore, for $\mu=\max\{\mu_1,\mu_2\}$, W_μ is a sub solution of (2.15) with p in (2.16) and

$$c_5\rho^{\alpha-1}\ge\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}]\ge W_\mu\quad\text{and}\quad \liminf_{x\in\Omega,\ x\to\partial\Omega}W_\mu(x)\rho^{1-\alpha}(x)\ge\frac{1}{c_5}.$$

By [5, Theorem 2.6], there exists a solution u of (2.15) with p in (2.16) satisfying (2.17). A contradiction is obtained and the proof is complete. □

3 Proof of Theorem 1.1

Lemma 3.1 (i) *Assume that g is a continuous nondecreasing function satisfying $g(0)\ge 0$ and (1.8). Then*

$$\lim_{\rho(x)\to 0^+}\mathbb{G}_\alpha[g(\mathbb{G}_\alpha[k\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}])](x)\rho(x)^{1-\alpha}=0. \tag{3.1}$$

(ii) *Assume that $p\in(0,\frac{1+\alpha}{1-\alpha})$, then there exists $c_{27}>0$ such that for any $x\in\Omega_t$ with $t\in(0,t_0)$,*

$$\mathbb{G}_\alpha[(\mathbb{G}_\alpha[\frac{\partial^\alpha\omega}{\partial\vec{n}^\alpha}])^p](x)\le c_{27}\rho(x)^{2\alpha-(1-\alpha)p}+c_{33} \tag{3.2}$$

Proof. (i) Without loss of generality, we may assume that

$$0 \in \partial\Omega, \quad \vec{n}_0 = e_N, \quad x_s = se_N$$

and we just need prove (3.2) and (3.1) for x_s with $s \in (0, t_0)$. It follows by Lemma 2.1 that

$$\frac{1}{c_5}\rho(x)^{\alpha-1} \le \mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x) \le c_5\rho(x)^{\alpha-1}, \qquad \forall x \in \Omega. \tag{3.3}$$

Combining with monotonicity of g, we have that

$$\begin{aligned}
\mathbb{G}_\alpha[g(\mathbb{G}_\alpha[k\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}])](x_s)s^{1-\alpha} &\le \int_\Omega G_\alpha(x_s,z)g(c_5k\rho(z)^{\alpha-1})dzs^{1-\alpha} \\
&\le \int_\Omega \frac{c_5\rho^\alpha(z)}{|x_s-z|^{N-\alpha}}g(c_5k\rho(z)^{\alpha-1})dzs^{1-\alpha} \\
&= c_5\left[\int_{B_{t_0}(0)\cap\Omega} \frac{s^{1-\alpha}\rho^\alpha(z)}{|x_s-z|^{N-\alpha}}g(c_5k\rho(z)^{\alpha-1})dz\right. \\
&\quad \left. + \int_{\Omega\setminus B_{t_0}(0)} \frac{s^{1-\alpha}\rho^\alpha(z)}{|x_s-z|^{N-\alpha}}g(c_5k\rho(z)^{\alpha-1})dz\right] \\
&:= A_1(s) + A_2(s).
\end{aligned}$$

Let $B'_{t_0}(0)$ be the ball with radium t_0 and centered at the origin in $\mathbb{R}^{N-1}$ and since $\partial\Omega$ is C^2, there exists a C^2 function $\psi : B'_\eta(0) \to \mathbb{R}$ such that

$$(z', \psi(z') \in \partial\Omega \quad \text{for any } z' \in B'_\eta(0),$$

where $\eta > 0$. Denote

$$\Psi(y) = (y', \psi(y')) + y_N\vec{n}_{(y',\psi(y'))}, \quad \forall y = (y', y_N) \in Q_\eta,$$

where

$$Q_\eta = \{z = (z', z_N) \in \mathbb{R}^{N-1} \times \mathbb{R},\ |z'| < \eta,\ 0 < z_N < \eta\}.$$

Thus, Ψ is a C^2 diffeomorphism mapping such that

$$\Psi(y) = y, \quad \forall y = te_N,\ t \in (0, \eta).$$

Therefore, if $t_0 > 0$ is chosen small enough, we have that $\Omega \cap B_{t_0}(0) \subset \Psi(Q_\eta)$ and there exists $c_{28} > 1$ such that for $z = \Psi(y) \in \Psi(Q_\eta)$,

$$\rho(z) = y_N \quad \text{and} \quad \frac{1}{c_{28}}|se_N - y| \le |se_N - z| \le c_{28}|se_N - y|. \tag{3.4}$$

Then we have that

$$\begin{aligned}
A_1(s) &= \int_{B_{t_0}(0)\cap\Omega} \frac{s^{1-\alpha}\rho^\alpha(z)}{|x_s-z|^{N-\alpha}}g(c_5k\rho(z)^{\alpha-1})dz \\
&\le c_{28}\int_{B_{t_0}(0)\cap\Omega} \frac{s^{1-\alpha}\rho^\alpha(z)}{|x_s-\Psi^{-1}(z)|^{N-\alpha}}g(c_5k\rho(z)^{\alpha-1})dz \\
&\le c_{28}\int_{Q_\eta} \frac{s^{1-\alpha}y_N^\alpha}{|x_s-y|^{N-\alpha}}g(c_5ky_N^{\alpha-1})dy.
\end{aligned}$$

For $s\in(0,\frac{1}{8}\eta)$, we decompose Q_η as following

$$Q_{i,0}=\{z=(z',z_N)\in Q_{t_0}:\ \frac{is}{2}\le|z'|<\frac{(i+1)s}{2},\ 0<z_N<\frac{s}{2}\}$$

and

$$Q_{i,j}=\{z=(z',z_N)\in Q_{t_0}:\ \frac{is}{2}\le|z'|<\frac{(i+1)s}{2},\ (j+\frac{1}{2})s\le z_N<(j+\frac{3}{2})s\},$$

where $i=0,1,\cdots N_s$, $j=1,\cdots N_s$ and N_s is the largest integer number such that $N_s\le\frac{\eta}{8s}$. Then we derive that

For $y\in Q_{0,1}$, we have that $\frac{s}{2}\le y_N\le\frac{3s}{2}$ and

$$\begin{aligned}\int_{Q_{0,1}}\frac{s^{1-\alpha}y_N^\alpha}{|x_s-y|^{N-\alpha}}g(c_5ky_N^{\alpha-1})dy &\le c_5s^{1+\alpha}g\left(c_{29}ks^{\alpha-1}\right)\int_{B_{2/3}(e_N)}\frac{1}{|e_N-z|^{N-\alpha}}dz\\ &= c_5r^{-\frac{1+\alpha}{1-\alpha}}g(c_{29}kr)\int_{B_{2/3}(e_N)}\frac{1}{|\vec{n}_0-z|^{N-\alpha}}dz\\ &\le c_{30}r^{-\frac{1+\alpha}{1-\alpha}}g(c_{29}kr),\end{aligned}\tag{3.5}$$

where $r=s^{\alpha-1}$ and $c_{29},c_{30}>0$.

For $y\in Q_{i,0}$ with $i=0,\cdots,N_s$, we obtain that $|x_s-y|\ge\frac{i+1}{4}s$ and

$$\begin{aligned}\int_{Q_{i,0}}\frac{s^{1-\alpha}y_N^\alpha}{|x_s-y|^{N-\alpha}}g(c_5ky_N^{\alpha-1})dy &\le \frac{c_{31}s^{1-N}}{(1+i)^{N-\alpha}}\int_{Q_{i,0}}y_N^\alpha g(c_5ky_N^{\alpha-1})dy\\ &= \frac{c_{32}}{(1+i)^{N-\alpha}}\int_0^{\frac{s}{2}}t^\alpha g(c_5kt^{\alpha-1})dt\\ &= \frac{c_{32}(1-\alpha)^{-1}}{(1+i)^{N-\alpha}}\int_{s^{-\frac{1}{N-\alpha}}}^{\infty}\tau^{-1-\frac{1+\alpha}{1-\alpha}}g\left(c_5k\tau\right)d\tau,\end{aligned}\tag{3.6}$$

where $r=s^{\alpha-1}$ and $c_{31},c_{32}>0$.

For $y\in Q_{i,j}$ with $i=0,\cdots,N_s$, $j=1,\cdots,N_s$ and $(i,j)\ne(0,1)$, we derive that $|x_s-y|\ge\frac{i+j}{4}s$ and $(j+\frac{1}{2})s\le y_N<(j+\frac{3}{2})s$

$$\begin{aligned}\int_{Q_{i,j}}\frac{s^{1-\alpha}y_N^\alpha}{|x_s-y|^{N-\alpha}}g(c_5ky_N^{\alpha-1})dy &\le \frac{c_{33}s^{1-N}}{(i+j)^{N-\alpha}}(js)^\alpha g\left(c_5k(js)^{\alpha-1}\right)|Q_{i,j}|\\ &= \frac{c_{34}j^\alpha}{(i+j)^{N-\alpha}}r^{-\frac{1+\alpha}{1-\alpha}}g(c_5kj^{\alpha-1}r),\end{aligned}\tag{3.7}$$

where $r=s^{\alpha-1}$ and $c_{33},c_{34}>0$.

Therefore, there exist $c_{35},c_{36}>0$ such that

$$A_1(s)\le\sum_{i=0,j=0}^{N_s}\frac{c_{35}j^\alpha r^{-\frac{1+\alpha}{1-\alpha}}}{(i+j+1)^{N-\alpha}}g(c_{29}kj^{\alpha-1}r)+\sum_{i=0}^{N_s}\frac{c_{36}}{(1+i)^{N-\alpha}}\int_{s^{-\frac{1}{N-\alpha}}}^{\infty}\tau^{-1-\frac{1+\alpha}{1-\alpha}}g\left(c_5k\tau\right)d\tau.$$

Since

$$\sum_{i=0}^{N_s}\frac{c_{36}}{(1+i)^{N-\alpha}}\int_{s^{-\frac{1}{N-\alpha}}}^{\infty}\tau^{-1-\frac{1+\alpha}{1-\alpha}}g\left(c_5k\tau\right)d\tau\le c_{37}\int_{s^{-\frac{1}{N-\alpha}}}^{\infty}\tau^{-1-\frac{1+\alpha}{1-\alpha}}g\left(c_5k\tau\right)d\tau$$

which tends to 0 as $s \to 0^+$ by hypothesis (1.8).

For any $\epsilon > 0$, there exists $n_\epsilon > 1$ such that

$$\sum_{i,j=n_\epsilon}^{N_s} \frac{c_{35}}{(i+j+1)^{N-\alpha}} \le \epsilon,$$

and since $\{(j^{\alpha-1}r)^{-\frac{1+\alpha}{1-\alpha}} g(c_{29}kj^{\alpha-1}r)\}$ is uniformly bounded, we imply that

$$\begin{aligned}\sum_{i,j=n_\epsilon}^{N_s} \frac{c_{35}j^\alpha}{(i+j+1)^{N-\alpha}} r^{-\frac{1+\alpha}{1-\alpha}} g(c_{29}kj^{\alpha-1}r) &= \sum_{i,j=n_\epsilon}^{N_s} \frac{c_{35}j^{-1}}{(i+j+1)^{N-\alpha}} (j^{\alpha-1}r)^{-\frac{1+\alpha}{1-\alpha}} g(c_{29}kj^{\alpha-1}r) \\ &\le c_{36}\epsilon.\end{aligned}$$

For $i, j < n_\epsilon$, there exists $s_\epsilon \in (0, \eta)$ such that

$$j^\alpha r^{-\frac{1+\alpha}{1-\alpha}} g(c_{29}kj^{1-\alpha}r) \le \epsilon, \quad \text{for } r \ge s_\epsilon^{\alpha-1},$$

thus, for any $\epsilon > 0$, there exists s_ϵ such that for $s \in (0, s_\epsilon)$

$$\begin{aligned}\sum_{i=0,j=0}^{n_\epsilon} \frac{c_{35}j^\alpha}{(i+j+1)^{N-\alpha}} r^{-\frac{1+\alpha}{1-\alpha}} g(c_{29}kj^{\alpha-1}r) &\le \epsilon \sum_{i=0,j=0}^{N_s} \frac{c_{35}j^{-1}}{(i+j+1)^{N-\alpha}} \\ &\le c_{37}\epsilon.\end{aligned}$$

Then we deduce that

$$\lim_{s\to 0^+} A_1(s) = 0.$$

Therefore,

$$\lim_{s\to 0^+} \mathbb{G}_\alpha[g(\mathbb{G}_\alpha[k\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}])](x_s)s^{1-\alpha} = 0.$$

Since $|x_s - z| \ge c_{38}t_0$ for $\Omega \setminus B_{t_0}(0)$, therefore,

$$\begin{aligned}A_2 &\le c_{39}s^{1-\alpha} \int_{\Omega\setminus B_{t_0}(0)} \rho^\alpha(z) g(c_5k\rho(z)^{\alpha-1})dz \\ &\le c_{39}s^{1-\alpha}\omega(\partial\Omega) \int_0^{d_0} t^\alpha g(c_5kt^{\alpha-1})dt \\ &\le c_{40}s^{1-\alpha},\end{aligned} \tag{3.8}$$

where $c_{39}, c_{40} > 0$ and $d_0 = \max_{x\in\Omega} \rho(x)$. Thus, (3.1) holds.

(ii) When $g(s) = s^p$ with $p \in (0, \frac{1+\alpha}{1-\alpha})$, we observe that (3.5) becomes that

$$\int_{Q_{0,1}} \frac{s^{1-\alpha}y_N^\alpha}{|x_s - y|^{N-\alpha}} (c_5y_N^{\alpha-1})^p dy \le c_5^p s^{1+\alpha+(\alpha-1)p},$$

(3.6) turns out to

$$\begin{aligned}\int_{Q_{i,0}} \frac{s^{1-\alpha}y_N^\alpha}{|x_s - y|^{N-\alpha}} (c_5y_N^{\alpha-1})^p dy &= \frac{c_{41}}{(1+i)^{N-\alpha}} \int_0^{\frac{s}{2}} t^{\alpha+(\alpha-1)p}dt \\ &\le \frac{c_{42}}{(1+i)^{N-\alpha}} s^{1+\alpha+(\alpha-1)p}\end{aligned}$$

and (3.7) becomes that

$$\int_{Q_{i,j}} \frac{s^{1-\alpha} y_N^\alpha}{|x_s - y|^{N-\alpha}} (c_5 y_N^{\alpha-1})^p dy \quad \le \quad \frac{c_{43}}{(1+i+j)^{N-\alpha}} s^{1+\alpha+(\alpha-1)p}.$$

Therefore, we have that

$$\int_{B_{t_0}(0)\cap\Omega} \frac{\rho^\alpha(y)}{|x_s - y|^{N-\alpha}} \rho(y)^{(\alpha-1)p} dy \le c_{44} s^{2\alpha+(\alpha-1)p}.$$

which, combining (3.8), implies (3.2). $\square$

Proof of Theorem 1.1. *To prove the existence of weak solution.* Take $\{g_n\}$ a sequence of C^1 nondecreasing functions defined on $\mathbb{R}$ satisfying $g_n(0) = g(0)$ and (2.9). By Proposition 2.3, problem (5.6) admits a unique weak solution $u_{k,n}$ such that

$$0 < u_{k,n} \le \mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}] \quad \text{in} \quad \Omega$$

and

$$\int_\Omega [u_{k,n}(-\Delta)^\alpha \xi + g_n(u_{k,n})\xi] dx = k \int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\omega(x), \quad \forall \xi \in \mathbb{X}_\alpha. \tag{3.9}$$

For any compact set $\mathcal{K} \subset \Omega$, we observe from [6, Lemma 3.2] that for some $\beta \in (0, \alpha)$,

$$\|u_{k,n}\|_{C^\beta(\mathcal{K})} \le c_{45} k.$$

Therefore, up to some subsequence, there exists u_k such that

$$\lim_{n\to\infty} u_{k,n} = u_k \quad \text{in } \Omega.$$

Then $g_n(u_{k,n})$ converge to $g(u_k)$ in Ω as $n \to \infty$. By Proposition 2.2 and (3.19) in [6], we have that

$$u_{k,n} \to u_k \text{ in } L^1(\Omega), \quad \|g_n(u_n)\|_{L^1(\Omega, \rho^\alpha dx)} \le c_{46} \|\mathbb{G}_\alpha[\frac{\partial^\alpha |\nu|}{\partial \vec{n}^\alpha}]\|_{L^1(\Omega)}$$

and

$$m(\lambda) \le c_{47} \lambda^{-\frac{1+\alpha}{1-\alpha}} \quad \text{for} \quad \lambda > \lambda_0,$$

where

$$m(\lambda) = \int_{S_\lambda} \rho_{\partial\Omega}^\alpha(x) dx \quad \text{with} \quad S_\lambda = \{x \in \Omega : \mathbb{G}_\alpha[\frac{\partial^\alpha |\nu|}{\partial \vec{n}^\alpha}] > \lambda\}.$$

For any Borel set $E \subset \Omega$, we have that

$$\begin{aligned}
\int_E |g_n(u_n)| \rho_{\partial\Omega}^\alpha(x) dx &\le \int_{E\cap \tilde{S}^c_{\frac{\lambda}{k}}} g\left(k\mathbb{G}_\alpha[\frac{\partial^\alpha |\nu|}{\partial \vec{n}^\alpha}]\right) \rho_{\partial\Omega}^\alpha(x) dx + \int_{E\cap \tilde{S}_{\frac{\lambda}{k}}} g\left(k\mathbb{G}_\alpha[\frac{\partial^\alpha |\nu|}{\partial \vec{n}^\alpha}]\right) \rho_{\partial\Omega}^\alpha(x) dx \\
&\le \tilde{g}\left(\frac{\lambda}{k}\right) \int_E \rho_{\partial\Omega}^\alpha(x) dx + \int_{\tilde{S}_{\frac{\lambda}{k}}} \tilde{g}\left(k\mathbb{G}_\alpha[\frac{\partial^\alpha |\nu|}{\partial \vec{n}^\alpha}]\right) \rho_{\partial\Omega}^\alpha(x) dx \\
&\le \tilde{g}\left(\frac{\lambda}{k}\right) \int_E \rho_{\partial\Omega}^\alpha(x) dx + \tilde{m}\left(\frac{\lambda}{k}\right) \tilde{g}\left(\frac{\lambda}{k}\right) + \int_{\frac{\lambda}{k}}^\infty \tilde{m}(s) d\tilde{g}(s),
\end{aligned}$$

where $\tilde{g}(r) = g(|r|) - g(-|r|)$.

On the other hand,

$$\int_{\frac{\lambda}{k}}^{\infty} \tilde{g}(s)d\tilde{m}(s) = \lim_{T\to\infty} \int_{\frac{\lambda}{k}}^{T} \tilde{g}(s)d\tilde{m}(s).$$

Thus,

$$\begin{aligned}\tilde{m}\left(\frac{\lambda}{k}\right)\tilde{g}\left(\frac{\lambda}{k}\right) + \int_{\frac{\lambda}{k}}^{T} \tilde{m}(s)d\tilde{g}(s) &\leq c_{47}\tilde{g}\left(\frac{\lambda}{k}\right)\left(\frac{\lambda}{k}\right)^{-\frac{1+\alpha}{1-\alpha}} + c_{47}\int_{\frac{\lambda}{k}}^{T} s^{-\frac{1+\alpha}{1-\alpha}} d\tilde{g}(s) \\ &\leq c_{48}T^{-\frac{1+\alpha}{1-\alpha}}\tilde{g}(T) + \frac{c_{49}}{\frac{1+\alpha}{1-\alpha}+1}\int_{\frac{\lambda}{k}}^{T} s^{-1-\frac{1+\alpha}{1-\alpha}}\tilde{g}(s)ds.\end{aligned}$$

By assumption (1.8) and [6, Lemma 3.4] with $p = \frac{1+\alpha}{1-\alpha}$, $T^{-\frac{N+\alpha}{N-\alpha}}\tilde{g}(T) \to 0$ when $T \to \infty$, therefore,

$$\tilde{m}\left(\frac{\lambda}{k}\right)\tilde{g}\left(\frac{\lambda}{k}\right) + \int_{\frac{\lambda}{k}}^{\infty} \tilde{m}(s)\ d\tilde{g}(s) \leq \frac{c_{49}}{\frac{1+\alpha}{1-\alpha}+1}\int_{\frac{\lambda}{k}}^{\infty} s^{-1-\frac{1+\alpha}{1-\alpha}}\tilde{g}(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when $\lambda \to \infty$. The conclusion follows: for any $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\frac{c_{49}}{\frac{1+\alpha}{1-\alpha}+1}\int_{\frac{\lambda}{k}}^{\infty} s^{-1-\frac{1+\alpha}{1-\alpha}}\tilde{g}(s)ds \leq \frac{\epsilon}{2}.$$

For λ fixed, there exists $\delta > 0$ such that

$$\int_E \rho_{\partial\Omega}^{\alpha}(x)dx \leq \delta \Longrightarrow \tilde{g}\left(\frac{\lambda}{k}\right)\int_E \rho_{\partial\Omega}^{\alpha}(x)dx \leq \frac{\epsilon}{2},$$

which implies that $\{g_n \circ u_n\}$ is uniformly integrable in $L^1(\Omega, \rho_{\partial\Omega}^{\alpha}dx)$. Then $g_n \circ u_n \to g \circ u_\nu$ in $L^1(\Omega, \rho_{\partial\Omega}^{\alpha}dx)$ by Vitali convergence theorem.

Passing to the limit as $n \to +\infty$ in the identity (3.9), it implies that

$$\int_\Omega [u_k(-\Delta)^\alpha \xi + g(u_k)\xi]dx = k\int_{\partial\Omega} \frac{\partial^\alpha \xi(x)}{\partial \vec{n}_x^\alpha} d\omega(x), \quad \forall \xi \in \mathbb{X}_\alpha.$$

Then u_k is a weak solution of (1.6). Moreover, it follows by the fact

$$k\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}] - g(k\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}]) \leq u_k \leq k\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}] \quad \text{in } \Omega. \tag{3.10}$$

Uniqueness of weak solution. Let u_1, u_2 be two weak solutions of (1.6) and $w = u_1 - u_2$. Then $(-\Delta)^\alpha w = g_n(u_2) - g_n(u_1)$ and $g_n(u_2) - g_n(u_1) \in L^1(\Omega, \rho^\alpha dx)$. By Kato's inequatlity, see [8, Proposition 2.4], for $\xi \in \mathbb{X}_\alpha$, $\xi \geq 0$, we have that

$$\int_\Omega |w|(-\Delta)^\alpha \xi dx + \int_\Omega [g_n(u_1) - g_n(u_2)]\text{sign}(w)\xi dx \leq 0.$$

Combining with $\int_\Omega [g_n(u_1) - g_n(u_2)]\text{sign}(w)\xi dx \geq 0$, then we have

$$w = 0 \quad \text{a.e. in } \Omega.$$

Regularity of $u_{k,n}$ and u_k. Since g_n is C^1 in $\mathbb{R}$, then by [6, Lemma 3.2], we have

$$\|u_{k,n}\|_{C^{2\alpha+\beta}(\mathcal{K})} \leq c_{50}k, \tag{3.11}$$

for any compact set $\mathcal{K}$ and some $\beta \in (0, \alpha)$. Then $u_{k,n}$ is $C^{2\alpha+\beta}$ locally in Ω. Together with the fact that $u_{n,k}$ is classical solution of (2.12), we derive by Theorem 2.2 in [5] that u_k is a classical solution of (2.12).

To prove (1.9.) Plugging (3.1) and (3.3) into (3.10), we obtain that (1.9). □

4 Proof of Theorem 1.2

4.1 Strong singularity for $p \in (1+2\alpha, \frac{1+\alpha}{1-\alpha})$

In this subsection, we consider the limit of $\{u_k\}$ as $k \to \infty$, where u_k is the weak solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p = k\tfrac{\partial^\alpha \omega}{\partial \vec{n}^\alpha} \quad &\text{in} \quad \bar{\Omega}, \\ u = 0 \quad &\text{in} \quad \bar{\Omega}^c, \end{aligned}$$

here $p \in (1+2\alpha, \frac{1+\alpha}{1-\alpha})$. From Theorem 1.1, we know that $k \mapsto u_k$ is increasing and u_k is a classical solution of (1.2).

In order to control the limit of $\{u_k\}$ as $k \to \infty$, we have to obtain barrier function, i.e. a suitable super solution of (1.2). To this end, we consider C^2 function w_p satisfying

$$w_p(x) = \begin{cases} \rho(x)^{-\frac{2\alpha}{p-1}}, & \text{for } x \in \Omega_{t_0}, \\ 0, & \text{for } x \in \Omega^c. \end{cases} \tag{4.1}$$

We see that $w_p \in L^1(\Omega)$ if $\frac{2\alpha}{p-1} < 1$, i.e. $p > 1+2\alpha$.

Lemma 4.1 *Assume that $p \in (1+2\alpha, \frac{1+\alpha}{1-\alpha})$ and w_p is defined in (4.1). Then there exists $\lambda_0 > 0$ such that $\lambda_0 w_p$ is a super solution of (1.2).*

Proof. For $p \in (1+2\alpha, \frac{1+\alpha}{1-\alpha})$, we have that $-\frac{2\alpha}{p-1} \in (-1, 0)$ and from [5, Proposition 3.2], it shows that there exists $c(p) < 0$ such that

$$(-\Delta)^\alpha w_p(x) \geq c(p)\rho(x)^{-\frac{2\alpha}{p-1}-2\alpha}, \quad x \in \Omega.$$

Thus, taking $\lambda_0 = |c(p)|^{\frac{1}{p-1}}$, we derive that

$$(-\Delta)^\alpha(\lambda_0 w_p) + (\lambda_0 w_p)^p \geq 0 \quad \text{in} \quad \Omega.$$

Together with $\lambda_0 w_p > 0$ in Ω^c, $\lambda_0 w_p$ is a super solution of (1.2). The proof ends. □

We observe that the super solution $\lambda_0 w_p$ constructed in Lemma 4.1 provide a upper bound for u_∞.

Proof of Theorem 1.2 (i)**.** For $p \in (1+2\alpha, \frac{1+\alpha}{1-\alpha})$, we have that

$$-\frac{2\alpha}{p-1} \in (-1, -1+\alpha)$$

and it follows by (3.3) that

$$u_k(x) \leq k\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x) \leq \frac{c_5 k}{|x|^{1-\alpha}}, \quad x \in \Omega.$$

Then $\lim_{x\in\Omega, \rho(x)\to 0} \frac{u_k(x)}{w_p(x)} = 0$ and we claim that

$$u_k \leq \lambda_0 w_p \quad \text{in} \quad \Omega.$$

In fact, if it fails, then there exists $z_0 \in \Omega$ such that

$$(u_k - \lambda_0 w_p)(z_0) = \inf_{\Omega}(u_k - \lambda_0 w_p) = \operatorname{ess\,inf}_{\mathbb{R}^N}(u_k - \lambda_0 w_p) < 0.$$

Then we have $(-\Delta)^\alpha(u_k - \lambda_0 w_p)(z_0) < 0$, which contradicts the fact that

$$(-\Delta)^\alpha(u_k - \lambda_0 w_p)(z_0) = \lambda_0 w_p^p(z_0) - u_k^p(z_0) > 0.$$

By monotonicity of the mapping $k \to u_k$, there holds

$$u_\infty(x) := \lim_{k\to\infty} u_k(x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

which is a classical solution of (1.2) and

$$u_\infty(x) \le \lambda_0 w_p(x) = \lambda_0 |x|^{-\frac{2\alpha}{p-1}}, \quad \forall x \in \Omega.$$

By applying Stability Theorem [5, Theorem 2.4], we obtain that u_∞ is a classical solution of (1.2).

Finally, we claim that there exists $c_{51} > 0$ such that for $t \in (0, t_0)$,

$$u_\infty(x) \ge c_{51} t^{-\frac{2\alpha}{p-1}}, \quad \forall x \in \mathcal{C}_t. \tag{4.2}$$

Indeed, let $t_k = (\sigma^{-1}k)^{\frac{p-1}{(1-\alpha)p-1-\alpha}}$, where $\sigma > 0$ will be chosen later, then $k = \sigma t_k^{\frac{(1-\alpha)p-1-\alpha}{p-1}}$ and for $x \in \mathcal{C}_t$, we apply Lemma 3.1 with $p \in (1+2\alpha, \frac{1+\alpha}{1-\alpha})$ that

$$\begin{aligned} u_k(x) &\ge k\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x) - k^p \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}])^p](x) \\ &\ge c_5 k t^{\alpha-N}[1 - c_{52} k^{p-1} t^{(\alpha-1)p+\alpha+1}] \\ &\ge c_5 \sigma t_k^{-\frac{2\alpha}{p-1}}[1 - c_{52}\sigma^{p-1} t_k^{p-1}(t_k/2)^{(\alpha-1)p+\alpha+1}] \\ &\ge c_5 \sigma t_k^{-\frac{2\alpha}{p-1}}[1 - c_{52}\sigma^{p-1} 2^{(1-\alpha)p-\alpha-1}] \\ &\ge \frac{c_5\sigma}{2}\rho(x)^{-\frac{2\alpha}{p-1}}, \end{aligned}$$

where we choose σ such that $c_{52}\sigma^{p-1}2^{(N-\alpha)p-\alpha-N} = \frac{1}{2}$. Then for any $x \in \Omega$, there exists $k > 0$ such that $x \in \Omega$ and then

$$u_\infty(x) \ge u_k(x) \ge \frac{c_5\sigma}{2}\rho(x)^{-\frac{2\alpha}{p-1}}, \qquad \forall x \in \Omega.$$

This ends the proof. $\square$

4.2 The limit of $\{u_k\}$ blows up when $p \in (0, 1+2\alpha]$

In this subsection, we derive the blow-up behavior of the limit of $\{u_k\}$ when $p \in (0, 1+2\alpha]$. To this end, we have to do more estimate for u_k.

Lemma 4.2 *Assume that $g(s) = s^p$ with $p \in (1, 1+2\alpha]$ and u_k is the solution of (1.6) obtained by Theorem 1.1. Then there exist $c_{52} > 0$, $r_0 \in (0, \frac{1}{4})$ and $\{r_k\}_k \subset (0, r_0)$ satisfying $r_k \to 0$ as $k \to \infty$ such that*

$$u_k(x) \ge \frac{c_{52}}{\rho(x)}, \qquad \forall x \in \Omega_{t_0}. \tag{4.3}$$

Proof. We divide α, p into 3 cases:

Case I: $1 < \frac{2\alpha}{1-\alpha} < 1+2\alpha$ and $p \in [\frac{2\alpha}{1-\alpha}, 1+2\alpha]$; Case II: $1 < \frac{2\alpha}{1-\alpha} < 1+2\alpha$ and $p \in [1, \frac{2\alpha}{1-\alpha})$;
Case III: $\frac{2\alpha}{1-\alpha} \le 1$ and $p \in (1, 1+2\alpha]$; Case IV: $\frac{2\alpha}{1-\alpha} \ge 1+2\alpha$ and $p \in (1, 1+2\alpha]$.

To prove (4.3) in Case I and Case III. Let $t_j = j^{-\frac{1}{\alpha}}$ with $j \in (k_0, k)$, then $j = t_j^{-\alpha}$. Applying Lemma 3.1 (ii), for $p \ge \frac{2\alpha}{1-\alpha}$, we have that for $x \in \Omega_{t_j} \setminus \Omega_{\frac{t_j}{2}}$,

$$\begin{aligned} u_j(x) &\ge j\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x) - j^p \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}])^p](x) \\ &\ge c_5^{-1} j t_j^{\alpha-1} - c_{27} j^p \rho(x)^{(\alpha-1)p+2\alpha} \\ &\ge c_5^{-1} t_j^{-1} - c_{27} t_j^{-\alpha p-(1-\alpha)p+2\alpha} \\ &\ge \frac{1}{2c_5}\rho(x)^{-1}, \end{aligned}$$

where the last inequality holds since $-\alpha p - (1-\alpha)p + 2\alpha > -1$ and $r_j \to 0$ as $j \to \infty$. Then for any $x \in \Omega_{t_0}$, there exists $j \in (k_0, k)$ such that $x \in \Omega_{t_j} \setminus \Omega_{\frac{t_j}{2}}$ and then

$$u_k(x) \ge u_j(x) \ge \frac{1}{2c_5}\rho^{-1}(x), \qquad \forall x \in \Omega_{t_0}.$$

To prove (4.3) in Case II and Case IV. Let $r_j = j^{-\frac{1}{\alpha}}$ with $j \in (k_0, k)$, then $j = r_j^{-\alpha}$ and for $x \in \Omega_{t_j} \setminus \Omega_{\frac{t_j}{2}}$, we have that

$$\begin{aligned} u_j(x) &\ge j\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}](x) - j^p \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}])^p](x) \\ &\ge c_5^{-1} j \rho(x)^{\alpha-1} - c_{27} j^p \\ &\ge c_5^{-1} t_j^{-1} - c_{27} t_j^{-\alpha p} \\ &\ge \frac{1}{2c_5}\rho(x)^{-1}, \end{aligned}$$

where the last inequality holds since $-\alpha p > -1$ and $r_j \to 0$ as $j \to \infty$. For any $x \in \Omega_{t_0}$, there exists $j \in (k_0, k)$ such that $x \in \Omega_{t_j} \setminus \Omega_{\frac{t_j}{2}}$ and then

$$u_k(x) \ge u_j(x) \ge \frac{1}{2c_5}\rho(x)^{-1}, \qquad \forall x \in \Omega_{t_0}.$$

The proof ends. $\square$

Proof of Theorem 1.2 (ii)**.** It derives by Lemma 4.2 that

$$\pi_k := \int_{\Omega_{t_0} \setminus \Omega_{\frac{t_k}{2}}} u_k(x) \ge c_{52} \int_{\Omega_{t_0} \setminus \Omega_{\frac{t_k}{2}}} \rho^{-1}(x)dx \to \infty \quad \text{as } k \to \infty. \tag{4.4}$$

Then

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 \quad \text{in} \quad \Omega \setminus \Omega_{t_0}, \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \\ u &= u_k \quad \text{in} \quad \Omega_{t_0} \end{aligned} \tag{4.5}$$

admits a unique solution w_k. By Comparison Principle,

$$u_k \ge w_k \quad \text{in} \quad B_{\varrho_0}(y_0). \tag{4.6}$$

Let $\tilde w_k = w_k - u_k\chi_{\Omega\setminus\Omega_{t_0}}$, then $\tilde w_k = w_k$ in $\Omega\setminus\Omega_{t_0}$ and for $x\in\Omega\setminus\Omega_{t_0}$,

$$\begin{aligned}
(-\Delta)^\alpha \tilde w_k(x) &= -\lim_{\epsilon\to0^+}\int_{B_{\varrho_0}(y_0)\setminus B_\epsilon(x)}\tfrac{w_k(z)-w_k(x)}{|z-x|^{N+2\alpha}}dz\\
&\quad +\lim_{\epsilon\to0^+}\int_{B^c_{\varrho_0}(y_0)\setminus B_\epsilon(x)}\tfrac{w_k(x)}{|z-x|^{N+2\alpha}}dz\\
&= -\lim_{\epsilon\to0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(x)}\tfrac{w_k(z)-w_k(x)}{|z-x|^{N+2\alpha}}dz+\int_{B_{r_0}(0)}\tfrac{u_k(z)}{|z-x|^{N+2\alpha}}dz\\
&\ge (-\Delta)^\alpha w_k(x)+c_{53}\pi_k,
\end{aligned}$$

where $c_{53}=(|y_0|+r_0)^{-N-2\alpha}$ and the last inequality follows by the fact of

$$|z-x|\le|x|+|z|\le|y_0|+r_0\quad \text{for } \forall z\in B_{\frac14}(0),\quad \forall x\in\Omega\setminus\Omega_{t_0}.$$

Therefore,

$$\begin{aligned}
(-\Delta)^\alpha\tilde w_k(x)+\tilde w_k^p(x) &\ge (-\Delta)^\alpha w_k(x)+w_k^p(x)+c_{53}\pi_k\\
&= c_{53}\pi_k,\qquad \forall x\in\Omega\setminus\Omega_{t_0},
\end{aligned}$$

that is, $\tilde w_k$ is a super solution of

$$\begin{aligned}
(-\Delta)^\alpha u+u^p = c_{53}\pi_k \quad &\text{in}\quad \Omega\setminus\Omega_{t_0},\\
u=0\qquad &\text{in}\quad (\Omega\setminus\Omega_{t_0})^c.
\end{aligned}\tag{4.7}$$

Let η_1 be the solution of

$$\begin{aligned}
(-\Delta)^\alpha u=1\quad &\text{in}\quad \Omega\setminus\Omega_{t_0},\\
u=0\quad &\text{in}\quad (\Omega\setminus\Omega_{t_0})^c.
\end{aligned}$$

Then $(c_{53}\pi_k)^{\frac1p}\frac{\eta_1}{2\max_{\mathbb{R}^N}\eta_1}$ is sub solution of (4.7) for k large enough. By Comparison Principle, we have that

$$\tilde w_k(x)\ge (c_{53}\pi_k)^{\frac1p}\frac{\eta_1(x)}{2\max_{\mathbb{R}^N}\eta_1},\quad \forall x\in B_{\varrho_0}(y_0),$$

which implies that

$$w_k(y)\ge c_{54}(c_{53}\pi_k)^{\frac1p},\qquad \forall y\in\Omega\setminus\Omega_{t_0},$$

where $c_{54}=\min_{x\in B_{\varrho_0}(y_0)}\frac{\eta_1(x)}{2\max_{\mathbb{R}^N}\eta_1}$. Therefore, (4.6) and (4.4) imply that

$$\lim_{k\to\infty}u_k(y)\ge\lim_{k\to\infty}w_k(y)=\infty,\qquad \forall\Omega\setminus\Omega_{t_0}.$$

Similarly, we can prove

$$\lim_{k\to\infty}u_k(y)\ge\lim_{k\to\infty}w_k(y)=\infty,\qquad \forall y\in\Omega.$$

The proof ends. $\square$

5 Proof of Theorem 1.3

In this section, we are devoted to consider the solution of (1.10) with different blow-up speeds. Without loss of generality, we assume that

$$x_0=0\in\partial\Omega.$$

For $k, j \in \mathbb{N}$, donte
$$\nu = \omega + \delta_0, \tag{5.1}$$
and then
$$\mathbb{G}_\alpha[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}] = \mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}] + \mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}],$$
which is a $\alpha-$harmonic function of (2.14), since $\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}]$ is a $\alpha-$harmonic function of (2.14) and $\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}]$ is $\alpha-$ harmonic.

Proposition 5.1 *Let $\mathbb{G}_\alpha[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}]$ given by (5.1) and $p_N^* = \frac{N+\alpha}{N-\alpha}$. Then there exists $c_{55} > 0$ such that*
$$\|\mathbb{G}_\alpha[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}]\|_{M^{p_N^*}(\Omega,\rho^\alpha dx)} \le c_{55}. \tag{5.2}$$

Proof. Since
$$\|\mathbb{G}_\alpha[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}]\|_{M^{p_N^*}(\Omega,\rho^\alpha dx)} \le k\|\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}]\|_{M^{p_N^*}(\Omega,\rho^\alpha dx)} + j\|\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}]\|_{M^{p_N^*}(\Omega,\rho^\alpha dx)}. \tag{5.3}$$
From Proposition 2.2 and $p* = \frac{1+\alpha}{1-\alpha} > \frac{N+\alpha}{N-\alpha}$, on the one hand, we have that
$$\begin{aligned} \|\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}]\|_{M^{p_N^*}(\Omega,\rho^\alpha dx)} &\le c_{56}\|\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}]\|_{M^{p^*}(\Omega,\rho^\alpha dx)} \\ &\le c_{56}c_{55}, \end{aligned} \tag{5.4}$$
On the other hand, for $\lambda > 0$, denote
$$S_\lambda = \{x \in \Omega : \mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}](x) > \lambda\} \quad \text{and} \quad m(\lambda) = \int_{S_\lambda} \rho^\alpha_{\partial\Omega}(x)dx$$
and from [6, Lemma 3.3] with $\nu = \delta_0$, there exist $\lambda_0 > 1$ and $c_{57} > 0$ such that for any $\lambda \ge \lambda_0$,
$$\tilde{m}(\lambda) \le c_{57}\lambda^{-\frac{N+\alpha}{N-\alpha}}.$$
which implies that
$$\|\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}]\|_{M^{p_N^*}(\Omega,\rho^\alpha dx)} \le c_{55}, \tag{5.5}$$
Thus, (5.2) follows by (5.4) and (5.5). Combining [6, Theorem 1.1] and the proof of Proposition 2.3, we have following result.

Proposition 5.2 *Assume that $\{g_n\}_n$ is given by (2.9). Then*
$$\begin{aligned} (-\Delta)^\alpha u + g_n(u) &= k\tfrac{\partial^\alpha \nu}{\partial \vec{n}^\alpha} \quad &\text{in} \quad \Omega, \\ u &= 0 \quad &\text{in} \quad \Omega^c \end{aligned} \tag{5.6}$$
admits a unique positive weak solution $u_{k,j,n}$ satisfying

(i) the mappings $k \to u_{k,j,n}$, $j \to u_{k,j,n}$ are increasing, the mapping $n \to u_{k,j,n}$ is decreasing
$$\mathbb{G}_\alpha[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}](x) - \mathbb{G}_\alpha[g_n(\mathbb{G}_\alpha[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}])](x) \le u_{k,n}(x) \le \mathbb{G}_\alpha[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}](x), \quad \forall x \in \Omega; \tag{5.7}$$
(ii) $u_{k,n}$ is a classical solution of (2.12).

Lemma 5.1 *There exists $c_{58} > 1$ such that*

$$\frac{1}{c_{58}}\rho^\alpha(x)|x|^{-N} \le \mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}] \le c_{58}\rho^\alpha(x)|x|^{-N}, \qquad \forall x \in \Omega. \tag{5.8}$$

Proof. For any $x \in \Omega$, there exists $s > 0$ such that $x \in \mathcal{C}_s$, and then letting $y_t = t\vec{n}_0$ with $t \in (0, s/2)$, we have that

$$|y_t - x| = s - t > \frac{s}{2} = \frac{1}{2}\max\{\rho_{\partial\Omega}(y_t), \rho_{\partial\Omega}(x)\}.$$

Thus, it follows by apply [4, Theorem 1.1,Theorem 1.2] that

$$\frac{1}{c_{59}}\frac{\rho^\alpha_{\partial\Omega}(y_t)\rho^\alpha_{\partial\Omega}(x)}{|x-y_t|^N} \le G_\alpha(x, y_t) \le c_{59}\frac{\rho^\alpha_{\partial\Omega}(y_t)\rho^\alpha_{\partial\Omega}(x)}{|x-y_t|^N}.$$

From

$$\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}](x) = \lim_{t\to 0^+} G_\alpha(x, y_t), \qquad \forall x \in \Omega, \tag{5.9}$$

we deduce that

$$\frac{1}{c_{60}}\rho^\alpha(x)|x|^{-N} \le \mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}] \le c_{60}\rho^\alpha(x)|x|^{-N}, \qquad \forall x \in \Omega.$$

Lemma 5.2 *Assume that g is a continuous nondecreasing function satisfying $g(0) \ge 0$, (1.13) and (1.14). Then*

$$\lim_{\rho(x)\to 0^+} \frac{\mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}])](x)}{\rho(x)^{\alpha-1} + \rho^\alpha(x)|x|^{-N}} = 0. \tag{5.10}$$

Proof. From Lemma 3.1 and (1.13),

$$\lim_{\rho(x)\to 0^+} \mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}])](x)\rho^{1-\alpha}(x) = 0. \tag{5.11}$$

From [6, Lemma 4.1], we have that

$$\lim_{s\to 0^+} \mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}])](s\vec{n}_0)s^{N-\alpha} = 0. \tag{5.12}$$

By (1.14),

$$\mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\frac{\partial^\alpha \nu}{\partial \vec{n}^\alpha}])](x) \le \lambda\left[\mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}])](x) + \mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha}])](x)\right],$$

together with (5.11) and (5.12), we implies (5.10). □

Proof of Theorem 1.3. The existence of weak solution just follows the procedure of the proof of Theorem 1.1 by using Proposition 5.1 and Proposition 5.2. It is the same to prove the uniqueness and regularity of weak solution. Finally, plugging (2.3), (5.8) and (5.10) into (5.7) replaced g_n by g, we obtain (1.15). □

Acknowledge: H. Chen is supported by NSFC, No: 11401270.

References

[1] C. Bandle, M. Marcus, Asymptotic behaviour of solutions and derivatives for semilinear elliptic problems with blow-up on the boundary, *Ann. Inst. H. Poincar Anal. Non Linaire, 12,* 155C171 (1995).

[2] Ph. Bénilan, H. Brezis and M. Crandall, A semilinear elliptic equation in $L^1(\mathbb{R}^N)$, *Ann. Sc. Norm. Sup. Pisa Cl. Sci. 2*, 523-555 (1975).

[3] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equaitons, *Comm. Pure Appl. Math.*, 62(5), 597-638, 2009.

[4] Z. Chen and R. Song, Estimates on Green functions and poisson kernels for symmetric stable process, *Math. Ann. 312*, 465-501 (1998).

[5] H. Chen, P. Felmer and A. Quaas, Large solution to elliptic equations involving fractional Laplacian, accepted by *Ann. Inst. H. Poincaré, Analyse Non Linéaire*, DOI: 10.1016/j.anihpc.2014.08.001.

[6] H. Chen and H. Hajaiej, Existence, Non-existence, Uniqueness of solutions for semilinear elliptic equations involving measures concentrated on boundary, *arXiv:1410.2672* (2014).

[7] R. Cignoli and M. Cottlar, An Introduction to Functional Analysis, *North-Holland, Amsterdam*, 1974.

[8] H. Chen and L. Véron, Semilinear fractional elliptic equations involving measures, *J. Differential equations 257(5)*, 1457-1486 (2014).

[9] Y. Du, Z. Guo, Uniqueness and layer analysis for boundary blow-up solutions, *J. Math. Pures Appl., 83,* 739-763 (2004).

[10] M. del Pino and R. Letelier, The influence of domain geometry in boundary blow-up elliptic problems, *Nonlinear Analysis: Theory, Methods & Applications, 48(6),* 897-904, (2002).

[11] Y. Du, Z. Guo, F. Zhou, Boundary blow-up solutions with interior layers and spikes in a bistable problem, *Discrete Contin. Dyn. Syst. 19* 271C298 (2007).

[12] P. Felmer and A. Quaas, Fundamental solutions and Liouville type theorems for nonlinear integral operators, *Advances in Mathematics*, 226, 2712-2738, 2011.

[13] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.*, 10, 503-510, 1957.

[14] J. Garcia-Melin, R. Letelier, J. de Lis, Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up, *Proc. Amer. Math. Soc., 129*, 3593-3602 (2001)

[15] Z. Guo, F. Zhou, Exact multiplicity for boundary blow-up solutions, *J. Differential Equations 228,* 486-506 (2006).

[16] R. Osserman, On the inequality $\Delta u = f(u)$, *Pac. J. Math.* 7, 1641-1647, 1957.

[17] T. Ouyang, J. Shi, Exact multiplicity of positive solutions for a class of semilinear problems, *J. Differential Equations, 146,* 121-156 (1998)

[18] M. Pertti. Geometry of sets and measures in Euclidean spaces: fractals and rectifiability, *Cambridge University Press,* (1999).

[19] M. Marcus, and L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, *Ann. Inst. H. Poincaré, Analyse Non Linéaire, 14,* 237-274 (1997).

[20] M. Marcus and L. Véron, Existence and uniqueness results for large solutions of general nonlinear elliptic equation, *J. Evol. Equ.* 3, 637-652 (2003).

Huyuan Chen

Department of Mathematics, Jiangxi Normal University,
Nanchang, Jiangxi 330022, PR China
and
Institute of Mathematical Sciences, New York University Shanghai,
Shanghai 200120, PR China

Ying Wang

Department of Mathematics, Jiangxi Normal University,
Nanchang, Jiangxi 330022, PR China
and
Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático,
Universidad de Chile, Chile

Hichem Hajaiej

Institute of Mathematical Sciences, New York University Shanghai,
Shanghai 200120, PR China