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Necessary and sufficient conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized boson equations

Hichem Hajaiej,[†] **Luc Molinet,**[§] **Tohru Ozawa,**[♯] **Baoxiang Wang**[‡]

[†]*Dept of Mathematics, King Saud University, P.O. Box 2455, 11451 Riyadh, Saudi Arabia*

[§]*L.M.P.T., Université François Rabelais Tours, Parc Grandmont, 37200 Tours, France*

[♯]*Department of Applied Physics, Waseda University, Tokyo 169-8555, Japan*

[‡]*LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China*

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Abstract

Necessary and sufficient conditions for the generalized Gagliardo-Nirenberg inequalities are obtained. For $0 < q < \infty$, $0 < p, p_0, p_1 \leqslant \infty$, $s, s_0, s_1 \in \mathbb{R}$, $\theta \in (0,1)$,

$$\|u\|_{\dot B^{s}_{p,q}} \lesssim \|u\|^{1-\theta}_{\dot B^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{\dot B^{s_1}_{p_1,\infty}} \tag{0.1}$$

holds if and only if $n/p - s = (1-\theta)(n/p_0 - s_0) + \theta(n/p_1 - s_1)$, $s_0 - n/p_0 \neq s_1 - n/p_1$, $s \leqslant (1-\theta)s_0 + \theta s_1$, and $p_0 = p_1$ if $s = (1-\theta)s_0 + \theta s_1$. Applying this inequality, we show that the solution of the Navier-Stokes equation at finite blowup time T_m has a concentration phenomena in the critical space $L^3(\mathbb{R}^3)$. Moreover, we consider the minimization problem for the variational problem

$$M_c = \inf\left\{E(u): \ \|u_i\|_2^2 = c_i > 0, \ i = 1, ..., L\right\},$$

where

$$E(u) = \frac{1}{2}\|u\|^2_{\dot H^s} - \int_{\mathbb{R}^{2n}} G(u(x))V(x-y)G(u(y))dxdy$$

for $u = (u_1, ..., u_L) \in (H^s)^L$ and show that M_c admits a radial and radially decreasing minimizer under suitable assumptions on s, G and V.

Keywords. Fractional Gagaliardo-Nirenberg inequality, Besov spaces, Triebel-Lizorkin spaces, boson equation, minimizer.

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1 Introduction

The Gagliardo-Nirenberg (GN) inequality is a fundamental tool in the study of nonlinear partial differential equations, which was discovered by Gagliardo [28], Nirenberg [53] (see also [37]) in some special cases. Throughout this paper, we denote by $L^p := L^p(\mathbb{R}^n)$ the Lebesgue space, $\|\cdot\|_p := \|\cdot\|_{L^p}$. $C > 1$ will denote positive universal constants, which can be different at different places. $a \lesssim b$ stands for $a \leqslant Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We write $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. The classical integer version of the GN inequality can be stated as follows (see [26] for instance):

Theorem 1.1 *Let $1 \leqslant p, p_0, p_1 \leqslant \infty$, $\ell, m \in \mathbb{N} \cup \{0\}$, $\ell < m$, $\ell/m \leqslant \theta \leqslant 1$, and*

$$\frac{n}{p} - \ell = (1-\theta)\frac{n}{p_0} + \theta\left(\frac{n}{p_1} - m\right). \tag{1.1}$$

Then we have for all $u \in C_0^\infty(\mathbb{R}^n)$,

$$\sum_{|\alpha|=\ell} \|\partial^\alpha u\|_p \lesssim \|u\|_{p_0}^{1-\theta} \sum_{|\alpha|=m} \|\partial^\alpha u\|_{p_1}^{\theta}, \tag{1.2}$$

where we further assume $\ell/m \leqslant \theta < 1$ if $m - \ell - n/p_1$ is an integer.

The classical proof of the GN inequality is based on the global derivative analysis in L^p spaces, whose proof is rather complicated, cf. [26, 30]. On the basis of the harmonic analysis techniques, there are some recent works devoted to generalizations of the GN inequality, cf. [5, 9, 10, 16, 17, 18, 23, 26, 30, 31, 40, 44, 52, 54, 57].

In the first part of this paper, we consider the GN inequality with fractional order derivatives. First, we introduce some function spaces which will be frequently used, cf. [59]. We denote by $\dot H^s_p := (-\triangle)^{s/2} L^p$ the Riesz potential space, $\dot H^s = \dot H^s_2$, $H^s = L^2 \cap \dot H^s$ for any $s \geqslant 0$. Let ψ be a smooth cut-off function supported in the ball $\{\xi : |\xi| \leqslant 2\}$, $\varphi = \psi(\cdot) - \psi(2\,\cdot)$. We write $\varphi_k(\xi) = \varphi(2^{-k}\xi)$, $k \in \mathbb{Z}$. We see that

$$\sum_{k\in\mathbb{Z}} \varphi_k(\xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.3}$$

We introduce the homogeneous dyadic decomposition operators $\triangle_k = \mathscr{F}^{-1}\varphi_k\mathscr{F}$, $k \in \mathbb{Z}$. Let $-\infty < s < \infty$, $1 \leqslant p, q \leqslant \infty$. The space $\dot B^s_{p,q}$ equipped with norm

$$\|f\|_{\dot B^s_{p,q}} := \left(\sum_{k=-\infty}^{\infty} 2^{ksq} \|\triangle_k f\|_p^q\right)^{1/q} \tag{1.4}$$

is said to be a homogeneous Besov space (a tempered distribution $f \in \dot{B}^s_{p,q}$ modulo polynomials). Let

$$-\infty < s < \infty, \quad 1 \leqslant p < \infty, \quad 1 \leqslant q \leqslant \infty. \tag{1.5}$$

The space $\dot{F}^s_{p,q}$ equipped with norm

$$\|f\|_{\dot{F}^s_{p,q}} := \left\| \left(\sum_{k=-\infty}^{\infty} 2^{ksq} |\triangle_k f|^q \right)^{1/q} \right\|_p \tag{1.6}$$

is said to be a homogeneous Triebel-Lizorkin space (a tempered distribution $f \in \dot{F}^s_{p,q}$ modulo polynomials).

In this paper we will obtain necessary and sufficient conditions for the GN inequality in homogeneous Besov spaces $\dot{B}^s_{p,q}$ and Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$. As a corollary, we obtain that the GN inequality also holds in fractional Sobolev spaces $\dot{H}^s_p$. The fractional GN inequalities in Theorems 1.2, 1.3 and 1.4 below cover all of the available GN inequalities in [5, 9, 10, 16, 17, 18, 23, 26, 30, 31, 40, 44, 52, 54, 57] for both integer and fractional versions. Moreover, our results below clarify how the third indices q in $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ contribute the validity of the GN inequalities. We have

Theorem 1.2 *Let $0 < p, p_0, p_1, q, q_0, q_1 \leqslant \infty$, $s, s_0, s_1 \in \mathbb{R}$, $0 \leqslant \theta \leqslant 1$. Then the fractional GN inequality of the following type*

$$\|u\|_{\dot{B}^s_{p,q}} \lesssim \|u\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,q_0}} \|u\|^{\theta}_{\dot{B}^{s_1}_{p_1,q_1}} \tag{1.7}$$

holds for all $u \in \dot{B}^{s_0}_{p_0,q_0} \cap \dot{B}^{s_1}_{p_1,q_1}$ if and only if

$$\frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0} - s_0\right) + \theta\left(\frac{n}{p_1} - s_1\right), \tag{1.8}$$

$$s \leqslant (1-\theta)s_0 + \theta s_1, \tag{1.9}$$

$$\frac{1}{q} \leqslant \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \text{if } \ p_0 \neq p_1 \text{ and } s = (1-\theta)s_0 + \theta s_1, \tag{1.10}$$

$$s_0 \neq s_1 \quad \text{or} \quad \frac{1}{q} \leqslant \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \text{if } \ p_0 = p_1 \text{ and } s = (1-\theta)s_0 + \theta s_1, \tag{1.11}$$

$$s_0 - \frac{n}{p_0} \neq s - \frac{n}{p} \quad \text{or} \quad \frac{1}{q} \leqslant \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \text{if } \ s < (1-\theta)s_0 + \theta s_1. \tag{1.12}$$

Theorem 1.3 *Let $0 < q < \infty$, $0 < p, \ p_0, \ p_1 \leqslant \infty$, $0 < \theta < 1$, $s, s_0, s_1 \in \mathbb{R}$. Then the fractional GN inequality of the following type*

$$\|u\|_{\dot{B}^s_{p,q}} \lesssim \|u\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{\dot{B}^{s_1}_{p_1,\infty}} \tag{1.13}$$

holds if and only if

$$\frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0} - s_0\right) + \theta\left(\frac{n}{p_1} - s_1\right), \tag{1.14}$$

$$s_0 - \frac{n}{p_0} \neq s_1 - \frac{n}{p_1}, \tag{1.15}$$

$$s \leqslant (1-\theta)s_0 + \theta s_1, \tag{1.16}$$

$$p_0 = p_1 \quad \textit{if} \quad s = (1-\theta)s_0 + \theta s_1. \tag{1.17}$$

In homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$, we have the following

Theorem 1.4 *Let $0 < p, p_i, q < \infty$, $s, s_0, s_1 \in \mathbb{R}$, $0 < \theta < 1$. Then the fractional GN inequality of the following type*

$$\|u\|_{\dot{F}^s_{p,q}} \lesssim \|u\|^{1-\theta}_{\dot{F}^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{\dot{F}^{s_1}_{p_1,\infty}} \tag{1.18}$$

holds if and only if

$$\frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0} - s_0\right) + \theta\left(\frac{n}{p_1} - s_1\right), \tag{1.19}$$

$$s \leqslant (1-\theta)s_0 + \theta s_1, \tag{1.20}$$

$$s_0 \neq s_1 \quad \textit{if} \quad s = (1-\theta)s_0 + \theta s_1. \tag{1.21}$$

The following is the GN inequality with fractional derivatives.

Corollary 1.5 *Let $1 < p, p_0, p_1 < \infty$, s, $s_1 \in \mathbb{R}$, $0 \leqslant \theta \leqslant 1$. Then the fractional GN inequality of the following type*

$$\|u\|_{\dot{H}^s_p} \lesssim \|u\|^{1-\theta}_{L^{p_0}} \|u\|^{\theta}_{\dot{H}^{s_1}_{p_1}} \tag{1.22}$$

holds if and only if

$$\frac{n}{p} - s = (1-\theta)\frac{n}{p_0} + \theta\left(\frac{n}{p_1} - s_1\right), \quad s \leqslant \theta s_1. \tag{1.23}$$

We will prove Theorems 1.2–1.4 in Section 2. Relations with available GN inequalities are discussed in Section 3. We remark that analogous results to Theorems 1.2–1.4 and Corollary 1.5 also hold if one replaces all of the homogeneous spaces $\dot{B}^s_{p,q}$, $\dot{F}^s_{p,q}$, $\dot{H}^s_p$ by corresponding non-homogeneous spaces $B^s_{p,q}$, $F^s_{p,q}$, H^s_p, respectively. We will list those results in Section 4.

In the second part of this paper we consider some applications of the fractional GN inequality. First, We study the Cauchy problem for the Navier-Stokes (NS) equation

$$u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad u(0,x) = u_0(x), \tag{1.24}$$

where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$, $\nabla = (\partial_{x_1}, ..., \partial_{x_n})$, $\operatorname{div} u = \partial_{x_1} u_1 + ... + \partial_{x_n} u_n$, $u = (u_1, ..., u_n)$ and p are real-valued unknown functions of $(t,x) \in [0, T_m) \times \mathbb{R}^n$ for some $T_m > 0$, $u_0 = (u_0^1, ..., u_0^n)$ denotes the initial value of u at $t = 0$. It is known that NS equation is local well posed in L^n, namely, for initial data $u_0 \in L^n(\mathbb{R}^n)$, there exists a unique local solution $u \in C([0,T_m); L^n) \cap L^{2+n}_{\rm loc}(0, T_m; L^{2+n})$ (cf. [34, 35]). Whether the local solution can be extended to a global one is still open. Recently, Escauriaza, Seregin and Šverák [21] showed that any "Leray-Hopf" weak solution in 3D which remains bounded in $L^3(\mathbb{R}^3)$ cannot develop a singularity in finite time. Kenig and Koch [35] gave an alternative approach to this problem by substituting L^3 with $\dot{H}^{1/2}$. Dong and Du [20] generalized their results in higher spatial dimensions $n \geqslant 3$. Noticing that $L^3 \subset B^{-1}_{\infty,\infty}$ in 3D is a sharp embedding, for any solution u of the NS equation in $C([0,T^*); L^3)$, we see that $u \in C([0,T^*); B^{-1}_{\infty,\infty})$. May [51] (see also [39]) prove that if $T^* < \infty$, then there exists a constant $c > 0$ independent of the solution of NS equation such that $\limsup_{t \to T^*} \|u(t) - \omega\|_{B^{-1}_{\infty,\infty}} \geqslant c$ for all $\omega \in \mathscr{S}$. In this paper we will use the fractional GN inequality to study the finite time blowup solution and we have the following concentration result:

Theorem 1.6 *Let $n = 3$ and $u \in C([0,T_m); L^n \cap L^2) \cap L^{2+n}_{\rm loc}(0,T_m; L^{2+n})$ be the solution of NS equation with maximal existing time $T_m < \infty$. Then there exist $c_0 > 0$ and $\delta > 0$ such that*

$$\overline{\lim_{t \nearrow T_m}} \sup_{x_0 \in \mathbb{R}^n} \int_{|x - x_0| \le (T_m - t)^\delta} |u(t, x - x_0)|^n dx \geqslant c_0, \tag{1.25}$$

where the constant $c_0 > 0$ only depends on $\|u_0\|_n$, δ can be chosen as any positive constant less than $2/n^2$.

As the second application of fractional GN inequalities, we consider the existence of the radial and radially decreasing non-negative solutions for the following system:

$$(m^2 - \triangle)^s u_i - [G(u) * V] \, \partial_i G(u) + r_i u_i = 0, \quad i = 1, ..., L, \tag{1.26}$$

where $m^2 \geqslant 0$, $u = (u_1, ..., u_L)$, $u_i \geqslant 0$ and $u \neq 0$, $G : \mathbb{R}^L_+ \to \mathbb{R}_+ = [0, \infty)$ is a differentiable function, $\partial_i G(v_1, ..., v_L) := \partial G(v_1, ..., v_L)/\partial v_i$. $V(x) = |x|^{-(n-\beta)}$, $*$

denotes the convolution in $\mathbb{R}^n$, $r_i > 0$. In order to work out a desired solution of (1.26), it suffices to consider the existence of the radial and radially decreasing non-negative and non-zero minimizers of the following variational problem. We write for $c_1, ..., c_L > 0$,

$$S_c = \left\{u = (u_1, ..., u_L) \in (H^s)^L : \ \|u_i\|_2^2 = c_i, \ i = 1, ..., L\right\}. \tag{1.27}$$

We will consider the variation problem

$$M_c = \inf\{E(u) : \ u \in S_c, \ c_1, ..., c_L > 0\}, \tag{1.28}$$

where

$$E(u) = \frac{1}{2}\sum_{i=1}^{L} \|(m^2 + |\xi|^2)^{s/2}\widehat{u}_i\|_2^2 - \int\int G(u(x))V(|x-y|)G(u(y))dxdy. \tag{1.29}$$

Fractional calculus has gained tremendous popularity during the last two decades thanks to its applications in widespread domains of sciences, economics and engineering, see [1, 6, 36, 38]. Fractional powers of the Laplacian arise in many areas. Some of the fields of applications of fractional Laplacian models include medicine where the equation of motion of semilunar heart value vibrations and stimuli of neural systems are modeled by a Capulo fractional Laplacian; cf. [22, 43]. It also appears in modeling populations [55], flood flow, material viscoelastic theory, biology dynamics, earthquakes, chemical physics, electromagnetic theory, optic, signal processing, astrophysics, water wave, bio-sciences dynamical process and turbulence; cf. [1, 2, 6, 7, 13, 14, 19, 25, 24, 36, 38, 41, 45, 46, 58].

In [41], Lieb and Yau studied the existence and symmetry of ground state solutions for the boson equation in three dimensions:

$$(m^2 - \triangle)^{1/2}u - \left(|x|^{-1} * u^2\right)u + ru = 0, \tag{1.30}$$

Taking $G(u) = u^2$ and $V(x) = |x|^{-1}$ in three dimensions, (1.26) is reduced to (1.30). The variational problem associated with (1.30) is

$$M_c^{(3)} = \inf\left\{\frac{1}{2}\|(m^2 + |\xi|^2)^{1/4}\widehat{u}\|_2^2 - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy : \ u \in H^{1/2}, \ \|u\|_2^2 = c\right\}. \tag{1.31}$$

As indicated in [41], (1.30) and (1.31) play a fundamental role in the mathematical theory of gravitational collapse of boson stars. Indeed, Lieb and Yau essentially showed that for $s = 1/2$, there exists $c_* > 0$, such that (1.30) has a non-negative radial solution if and only if $c = c_*$. It was proven in [41] that boson stars with total mass strictly less than c^* are gravitationally stable, whereas boson stars whose total mass exceed c^* may undergo a "gravitational collapse" based on variational arguments and many-body quantum theory. The main tools used by Lieb and Yau are the Hardy-Littlewood-Sobolev inequality together with some rearrangement inequalities. Inspired and motivated by Lieb and Yau's work, Frank and Lenzemann [27] recently showed the uniqueness of ground states to (1.30) in 1D.

Taking $G(u) = u^2$ and $V(x) = |x|^{-(n-2)}$ in n-dimensions with $n \geqslant 3$, (1.26) is reduced to the general Choquard-Peckard equation

$$(m^2 - \triangle)^s u - \left(|x|^{-(n-2)} * u^2\right) u + ru = 0. \tag{1.32}$$

The variational problem associated with (1.32) is

$$M_c^{(n)} = \inf \left\{ \frac{1}{2} \|(m^2 + |\xi|^2)^{s/2} \widehat{u}\|_2^2 - \Upsilon_2(u) : \, u \in H^s, \ \|u\|_2^2 = c \right\}, \tag{1.33}$$

where

$$\Upsilon_\beta(u) = \int \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{n-\beta}} dxdy. \tag{1.34}$$

Taking $G(u) = u_1^2 + u_2^2$ and $V(x) = |x|^{-1}$ in 3-dimensions, (1.26) is reduced to the following system

$$(m^2 - \triangle)^s u_i - \left(|x|^{-1} * (u_1^2 + u_2^2)\right) u_i + r_i u_i = 0, \ i = 1, 2, \tag{1.35}$$

which was studied in [4] and [27] in the cases $s = 1$ and $s = 1/2$, respectively. If we treat $u = (u_1, u_2)$ and $\|u\|_X^2 = \|u_1\|_X^2 + \|u_2\|_X^2$, we see that the variational problem associated with (1.35) is the same as in (1.33) if one constraint $\|u_1\|_2^2 + \|u_2\|_2^2 = c$ is considered.

Now we state our main result on the existence of the minimizer of (1.28). There are two kinds of basic nonlinearities, one is $G(u) = u_1^{\mu_1} ... u_L^{\mu_L}$ and another is $G(u) = u_1^\mu + ... + u_L^\mu$. For the former case, we need to use m-constraints $\|u_i\|_2^2 = c_i > 0$ to prevent the situation that the second term of $E(u)$ in (1.29) vanishes. For the later case, one can use one constraint $\|u_1\|_2^2 + ... + \|u_L\|_2^2 = c$. Let $s \geqslant (n - \beta)/2$. We first consider the former case and our main assumptions on G are the following:

(G1) $G : \mathbb{R}_+^L \ni (v_1, ..., v_L) \to G(v_1, ..., v_L) \in \mathbb{R}_+$ is a continuous function and there exists $\mu \in [2, 1 + (2s + \beta)/n)$ such that

$$G(v) \leqslant C(|v|^2 + |v|^\mu), \ v = (v_1, ..., v_L). \tag{1.36}$$

Moreover, there exist $\alpha_i > 0$ such that for all $0 < v_1, ..., v_L \ll 1$,

$$G(v) \geqslant c v_1^{\alpha_1} v_2^{\alpha_2} ... v_L^{\alpha_L}. \tag{1.37}$$

where $0 < n + \beta - n(\alpha_1 + ... + \alpha_L) + 2s$.

(G2) If v has a zero component, then $G(v) = 0$. The function $G \otimes G : \mathbb{R}_+^L \times \mathbb{R}_+^L \ni (u, v) \to G(u)G(v) \in \mathbb{R}_+$ is a super-modular[1].

(G3) $G(t_1 v_1, ..., t_L v_L) \geqslant t_{\max} G(v_1, ..., v_L)$ for any $t_i \geqslant 1$, where $t_{\max} = \max(t_1, ..., t_L)$.

Noticing that $v_1^{\alpha_1} v_2^{\alpha_2} ... v_L^{\alpha_L} \leq |v|^{\alpha_1 + ... + \alpha_L}$, we see that condition (1.36) covers the nonlinearity $G(v) = v_1^{\alpha_1} v_2^{\alpha_2} ... v_L^{\alpha_L}$ if $\alpha_1 + ... + \alpha_L \in [2, \mu]$. Our main result on the existence of the minimizer of (1.28) is the following:

Theorem 1.7 *Let $m^2 \geqslant 0$, $0 < \beta < n$, $s > (n-\beta)/2$. Assume that conditions* (G1)–(G3) *are satisfied. Then* (1.28) *admits a radial and radially decreasing minimizer in $(H^s)^L$.*

We point out that both conditions $s \geqslant (n-\beta)/2$ and $0 < n+\beta-n(\alpha_1+...+\alpha_L)+2s$ are necessary for Theorem 1.7. Indeed, we can give a counterexample to show that $M_c = -\infty$ if $s < (n-\beta)/2$ or $0 > n + \beta - n(\alpha_1 + ... + \alpha_L) + 2s$ for a class of nonlinearities $G(u)$.

The endpoint case $s = (n - \beta)/2$ can not be handled in Theorem 1.7. Note that for $s = (n - \beta)/2$, we have $\mu = 2$ in (1.36), a basic example is $G(u) = u_1^2 + ... + u_L^2$. Now we consider the variational problem

$$M_{c,\beta}^{(n)} = \inf \left\{ \frac{1}{2} \|(m^2 + |\xi|^2)^{s/2} \widehat{u}\|_2^2 - \Upsilon_\beta(u) : \; u \in (H^s)^L, \; \|u\|_2^2 = c > 0 \right\}. \tag{1.38}$$

where $u = (u_1, ..., u_L)$, $|u|^2 = u_1^2 + ... + u_L^2$ and $\|u\|_X^2 = \|u_1\|_X^2 + ... + \|u_L\|_X^2$. Using the definition of the Riesz potential, the Plancherel identity, the Hardy-Littlewood-Sobolev, and fractional GN inequalities, we have

$$\Upsilon_\beta(u) = C(n, \beta) \int |u(x)|^2 [(-\Delta)^{-\beta/2} |u|^2](x) dx = \|(-\Delta)^{-\beta/4} |u|^2\|_2^2$$

[1] F is said to be a supermodular if ([42])

$$F(y + he_i + ke_j) + F(y) \geqslant F(y + he_i) + F(y + ke_j) \; (i \neq j, \; h, k > 0),$$

where $y = (y_1, ..., y_L)$, and e_i denotes the i-th standard basis vector in $\mathbb{R}^L$. It is known that a smooth function is a supermodular if all its mixed second partial derivatives are nonnegative.

$$
\begin{aligned}
&\leqslant C\left(\|u_1\|^2_{4n/(n+\beta)} + \dots + \|u_L\|^2_{4n/(n+\beta)}\right)^2 \\
&\leqslant C\left(\|u_1\|_2\|u_1\|_{\dot H^{(n-\beta)/2}} + \dots + \|u_L\|_2\|u_L\|_{\dot H^{(n-\beta)/2}}\right)^2 \\
&\leqslant C\|u\|_2^2\|u\|^2_{\dot H^{(n-\beta)/2}}.
\end{aligned} \tag{1.39}
$$

Define

$$
C^* = \sup_{u\in H^{(n-\beta)/2}\setminus\{0\}} \frac{\Upsilon_\beta(u)}{\|u\|_2^2\|u\|^2_{\dot H^{(n-\beta)/2}}}. \tag{1.40}
$$

Theorem 1.8 *Let $m^2 = 0$, $0 < \beta < n$, $s = (n-\beta)/2$, $G(u) = u_1^2 + \dots + u_L^2$. Then* (1.38) *admits a radial and radially decreasing minimizer in $(H^s)^L$ if and only if $c = 1/2C^*$.*

As a straightforward consequence of Theorem 1.7, we see that (1.33) admits a radial and radially decreasing minimizer in $H^{(n-2)/2}$ if and only if $c = 1/2C^*$, where $\beta = 2$ in the definition of C^*.

In the case $m^2 > 0$ we have the following

Theorem 1.9 *Let $m^2 > 0$, $0 < \beta < n$, $s = (n-\beta)/2$, $c > 0$. Then we have*

(1) *If $n > 2+\beta$, then* (1.38) *has no minimizer in in $(H^s)^L$.*

(2) *If $n < 2+\beta$, then* (1.38) *admits a radial and radially decreasing minimizer in $(H^s)^L$ if and only if $0 < c < 1/2C^*$.*

(3) *If $n = 2+\beta$, then* (1.38) *admits a radial and radially decreasing minimizer in $(H^s)^L$ if and only if $c = 1/2C^*$.*

2 Proofs of the GN inequalities

The following is an interpolation inequality in Besov spaces, which is very useful in nonlinear estimates, see [29, 31].

Proposition 2.1 (Convexity Hölder's inequality) *Let $0 < p_i, q_i \leqslant \infty$, $0 \leqslant \theta_i \leqslant 1$, $\sigma_i, \sigma \in \mathbb{R}$ $(i = 1, \dots, N)$, $\sum_{i=1}^N \theta_i = 1$, $\sigma = \sum_{i=1}^N \theta_i\sigma_i$, $1/p = \sum_{i=1}^N \theta_i/p_i$, $1/q = \sum_{i=1}^N \theta_i/q_i$. Then $\cap_{i=1}^N \dot B^{\sigma_i}_{p_i,q_i} \subset \dot B^\sigma_{p,q}$ and for any $v \in \cap_{i=1}^N \dot B^{\sigma_i}_{p_i,q_i}$,*

$$
\|v\|_{\dot B^\sigma_{p,q}} \leqslant \prod_{i=1}^N \|v\|^{\theta_i}_{\dot B^{\sigma_i}_{p_i,q_i}}.
$$

This estimate also holds if one substitutes $\dot B^\sigma_{p,q}$ by $\dot F^\sigma_{p,q}$ $(p, p_i \neq \infty)$.

In the convexity Hölder inequality, condition $1/q = \sum_{i=1}^N \theta_i/q_i$ can be replaced by $1/q \leqslant \sum_{i=1}^N \theta_i/q_i$. Indeed, noticing that $\ell^q \subset \ell^p$ for all $q \leqslant p$, we see that Proposition 2.1 still holds if $1/q < \sum_{i=1}^N \theta_i/q_i$. In [29, 31], Proposition 2.1 was stated as the case $1 \le p_i, q_i \le \infty$, however, the proof in [31] is also adapted to the case $0 < p_i, q_i \le \infty$.

Proof of Theorem 1.2 (Sufficiency) First, we consider the case $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$. By (1.9), we have

$$\frac{1}{p} - \frac{1-\theta}{p_0} - \frac{\theta}{p_1} = \frac{s}{n} - (1-\theta)\frac{s_0}{n} - \theta\frac{s_1}{n} := -\eta \leqslant 0. \tag{2.1}$$

Take p^* and s^* satisfying

$$\frac{1}{p^*} = \frac{1}{p} + \eta, \quad s^* = s + n\eta.$$

Applying the convexity Hölder inequality, we have

$$\|f\|_{\dot{B}^{s^*}_{p^*,q}} \leqslant \|f\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,q_0}} \|f\|^{\theta}_{\dot{B}^{s_1}_{p_1,q_1}}. \tag{2.2}$$

Using the inclusion $\dot{B}^{s^*}_{p^*,q} \subset \dot{B}^{s}_{p,q}$, we get the conclusion.

Next, we need to consider the following two cases: (i) $s = (1-\theta)s_0 + \theta s_1$, $p_0 = p_1$ and $s_0 \neq s_1$; (ii) $s < (1-\theta)s_0 + \theta s_1$ and $s - n/p \neq s_0 - n/p_0$. We can show that

$$\|f\|_{\dot{B}^{s}_{p,q}} \leqslant \|f\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,\infty}} \|f\|^{\theta}_{\dot{B}^{s_1}_{p_1,\infty}}, \tag{2.3}$$

see below, the proof of Theorem 1.3. (2.3) implies the result, as desired.

(Necessity) By scaling,

$$\|f(\lambda\cdot)\|_{\dot{B}^{s}_{p,q}} \sim \lambda^{s-n/p}\|f\|_{\dot{B}^{s}_{p,q}}, \quad \lambda \in 2^{\mathbb{Z}}.$$

Hence, if (1.7) holds, then

$$\lambda^{s-n/p-[(1-\theta)(s_0-n/p_0)+\theta(s_1-n/p_1)]} \leqslant C.$$

Letting $\lambda \to 0$ or $\lambda \to \infty$, we immediately obtain that $s - n/p - [(1-\theta)(s_0 - n/p_0) + \theta(s_1 - n/p_1) = 0$.

Next, we show that $s - s_0 \leqslant \theta(s_1 - s_0)$. Assume on the contrary that $s - s_0 > \theta(s_1 - s_0)$. Assume that $s_0 = 0$. Let φ satisfy supp $\varphi \subset \{\xi : 1/2 \leqslant |\xi| \leqslant 3/2\}$ and $\varphi(\xi) = 1$ for $3/4 \leqslant |\xi| \leqslant 1$. So, $\varphi(2^{-j}\xi) = 1$ if $3 \cdot 2^{j-2} \leqslant |\xi| \leqslant 2^j$. Denoting

$$\rho_j(\xi) = \varphi(2(\xi - \xi^{(j)})), \quad \xi^{(j)} = (7 \cdot 2^{j-3}, 0, ..., 0). \tag{2.4}$$

and for sufficiently small $\varepsilon > 0$, we write

$$\hat{f}(\xi) = \sum_{j=100}^{N} 2^{\varepsilon j} \rho_j(\xi). \tag{2.5}$$

This leads to

$$\|f\|^q_{\dot{B}^s_{p,q}} = \sum_{j=100}^{N} 2^{(s+\varepsilon)qj} \|\mathscr{F}^{-1}(\varphi_j \rho_j)\|^q_p.$$

Noticing that $\varphi_j(\xi) = 1$ for $\xi \in \operatorname{supp} \rho_j$, we have

$$\|\mathscr{F}^{-1}(\varphi_j \rho_j)\|_p = \|\mathscr{F}^{-1} \rho_j\|_p = \|\mathscr{F}^{-1} \rho_0\|_p.$$

Hence,

$$\|f\|_{\dot{B}^s_{p,q}} \sim 2^{(s+\varepsilon)N}.$$

Similarly,

$$\|f\|_{\dot{B}^0_{p_0,q_0}} \sim 2^{\varepsilon N}, \quad \|f\|_{\dot{B}^{s_1}_{p_1,q_1}} \sim 2^{(s_1+\varepsilon)N}.$$

By (1.7), we obtain that $2^{(s+\varepsilon)N} < 2^{\varepsilon N} 2^{s_1 \theta N}$. However, for sufficiently large N, it contradicts the fact $s > \theta s_1$. Substituting s by $s - s_0$, we get the proof in the case $s_0 \neq 0$.

Thirdly, we consider the case $p_0 \neq p_1$ and $s = (1-\theta)s_0 + \theta s_1$ and show that $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$. Put

$$\lambda = \frac{s_1 - s_0}{n(1/p_0 - 1/p_1)}. \tag{2.6}$$

We see that

$$s + n\lambda \left(\frac{1}{p} - 1 \right) = s_0 + n\lambda \left(\frac{1}{p_0} - 1 \right) = s_1 + n\lambda \left(\frac{1}{p_1} - 1 \right). \tag{2.7}$$

Case 1. We consider the case $\lambda \geqslant 0$. Let φ and $\xi^{(j)}$ be as in (2.4). Put

$$\varrho^\lambda_j := \varphi(2^{\lambda j}(\xi - \xi^{(j)}))$$

and

$$\widehat{F} = \sum_{j=100}^{J} 2^{-sj - n\lambda(1/p-1)j} \varrho^\lambda_j. \tag{2.8}$$

Since $\operatorname{supp} \widehat{F}$ overlaps only one $\operatorname{supp} \varphi_j$ for all $j \in \mathbb{Z}$ and for $j \geqslant 100$,

$$\|\triangle_j \mathscr{F}^{-1} \varrho^\lambda_j\|_p = \|\mathscr{F}^{-1} \varrho^\lambda_j\|_p \sim 2^{n\lambda j(1/p-1)},$$

we have

$$\begin{aligned}
\|F\|^q_{\dot B^s_{p,q}} &= \sum_{j\in\mathbb{Z}} (2^{sj}\|\triangle_j F\|_p)^q \\
&= \sum_{j=100}^{J} (2^{sj}\|\triangle_j F\|_p)^q \\
&= \sum_{j=100}^{J} (2^{-n\lambda(1/p-1)j}\|\triangle_j \mathscr{F}^{-1}\varrho^\lambda_j\|_p)^q \\
&\sim J, \qquad (2.9)
\end{aligned}$$

which means that $\|F\|_{\dot B^s_{p,q}} \sim J^{1/q}$. On the other hand, in view of (2.7) and (2.8), we see that

$$\widehat{F} = \sum_{j=100}^{J} 2^{-s_0 j - n\lambda(1/p_0-1)j}\varrho^\lambda_j = \sum_{j=100}^{J} 2^{-s_1 j - n\lambda(1/p_1-1)j}\varrho^\lambda_j. \qquad (2.10)$$

In an analogous way to (2.9), we find that

$$\|F\|_{\dot B^{s_0}_{p_0,q_0}} \sim J^{1/q_0}, \quad \|F\|_{\dot B^{s_1}_{p_1,q_1}} \sim J^{1/q_1}, \qquad (2.11)$$

By (1.7), we have $J^{1/q} \lesssim J^{(1-\theta)/q_0} J^{\theta/q_1}$ for any $J \gg 1$. It follows that $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$.

Case 2. We consider the case $\lambda < 0$. Denote

$$\varphi^{(N)} = \varphi(2^{-N}\,\cdot), \quad \varphi^{(N)}_j = \varphi(2^{-j-N}\,\cdot), \quad \triangle_{j,N} = \mathscr{F}^{-1}\varphi^{(N)}_j\mathscr{F}.$$

It is easy to see that

$$\|f\|^{(N)}_{\dot B^s_{p,q}} = \left(\sum_j (2^{sj}\|\triangle_{j,N}\|_p)^q\right)^{1/q}$$

is an equivalent norm on $\dot B^s_{p,q}$ (see also [59]). Let

$$\widehat{F} = \sum_{j=100}^{J} 2^{-sj-n\lambda(1/p-1)j}\varphi(2^{\lambda j}\,\cdot). \qquad (2.12)$$

Assuming that $N \geqslant 100(|\lambda|+1)$, analogously to the above, we have from the definition of $\|\cdot\|^{(N)}_{\dot B^s_{p,q}}$ that

$$\|F\|^{(N)}_{\dot B^s_{p,q}} \sim J^{1/q}, \quad \|F\|^{(N)}_{\dot B^{s_0}_{p_0,q_0}} \sim J^{1/q_0}, \quad \|F\|^{(N)}_{\dot B^{s_1}_{p_1,q_1}} \sim J^{1/q_1}. \qquad (2.13)$$

By (1.7) we have $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$.

Fourthly, we show the necessity of (1.11). If not, then we have $p_0 = p_1 = p$, $s_0 = s_1 = s$ and $1/q > (1-\theta)/q_0 + \theta/q_1$. Let

$$\widehat{F} = \sum_{j=100}^{J} 2^{-sj+n(1/p-1)j} \varphi(2^{-j}\,\cdot). \tag{2.14}$$

We easily see that for $N \gg 1$,

$$\|F\|^{(N)}_{\dot{B}^s_{p,q}} \sim J^{1/q}, \quad \|F\|^{(N)}_{\dot{B}^s_{p,q_0}} \sim J^{1/q_0}, \quad \|F\|^{(N)}_{\dot{B}^s_{p,q_1}} \sim J^{1/q_1}. \tag{2.15}$$

We have $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$, which is a contradiction.

Finally, we show the necessity of (1.12). Assume for a contrary that $s - n/p = s_0 - n/p_0$ and $1/q > (1-\theta)/q_0 + \theta/q_1$. Using the same way as in (2.14) and (2.15), we have a contraction. $\square$

Proof of Theorem 1.3. (Sufficiency) We can assume that $s_0 = 0$ and the case $s_0 \neq 0$ can be shown by a similar way.

Step 1. We consider the case $p \geqslant p_0 \vee p_1$. By definition,

$$\|u\|_{\dot{B}^s_{p,q}} = \left(\sum_{N \text{ dyadic}} N^{sq} \|\triangle_N u\|_p^q \right)^{1/q}. \tag{2.16}$$

From (4.8), it follows that

$$\theta \left(\frac{n}{p} - \frac{n}{p_1} + s_1 - s \right) = (1-\theta) \left(s + \frac{n}{p_0} - \frac{n}{p} \right). \tag{2.17}$$

Since $0 < \theta < 1$, (4.7) implies that $\left(\frac{n}{p} - \frac{n}{p_1} + s_1 - s\right)\left(s + \frac{n}{p_0} - \frac{n}{p}\right) > 0$.

Case 1. We consider the case

$$s_1 - s + \frac{n}{p} - \frac{n}{p_1} > 0, \quad s + \frac{n}{p_0} - \frac{n}{p} > 0. \tag{2.18}$$

Using the inclusion $\dot{B}^s_{p,r_1} \subset \dot{B}^s_{p,r_2}$ for any $r_1 \leq r_2$, it suffices to consider the case $q < 1/2$, $q^{-1} \in \mathbb{N}$. For brevity, we write $K := q^{-1}$.

$$\begin{aligned} \|u\|_{\dot{B}^s_{p,q}} \leqslant & \sum_{N_1 \geqslant \ldots \geqslant N_K} (N_1^s \ldots N_K^s \|\triangle_{N_1} u\|_p \ldots \|\triangle_{N_K} u\|_p)^{q^2} \\ & \times (N_1^s \ldots N_K^s \|\triangle_{N_1} u\|_p \ldots \|\triangle_{N_K} u\|_p)^{q(1-q)}. \end{aligned} \tag{2.19}$$

In view of Bernstein's inequality,

$$\|\triangle_N u\|_p \leqslant N^{\frac{n}{p_0}-\frac{n}{p}}\|\triangle_N u\|_{p_0}, \quad \|\triangle_N u\|_p \leqslant N^{\frac{n}{p_1}-\frac{n}{p}}\|\triangle_N u\|_{p_1}. \tag{2.20}$$

We can choose $a \in (0,1], k \geqslant 1$ satisfying $\theta K = k-1+a$. Hence,

$$\begin{aligned}
&\|\triangle_{N_1} u\|_p ... \|\triangle_{N_K} u\|_p \\
&\quad =(\|\triangle_{N_1} u\|_p ... \|\triangle_{N_{k-1}} u\|_p \|\triangle_{N_k} u\|_p^a)(\|\triangle_{N_k} u\|_p^{1-a}\|\triangle_{N_{k+1}} u\|_p \dots \|\triangle_{N_K} u\|_p) \\
&\quad \lesssim N_k^{(1-a)(\frac{n}{p_0}-\frac{n}{p})} N_{k+1}^{\frac{n}{p_0}-\frac{n}{p}} \dots N_K^{\frac{n}{p_0}-\frac{n}{p}} \|\triangle_{N_k} u\|_{p_0}^{1-a}\|\triangle_{N_{k+1}} u\|_{p_0} \dots \|\triangle_{N_K} u\|_{p_0} \\
&\qquad \times N_1^{\frac{n}{p_1}-\frac{n}{p}} \dots N_{k-1}^{\frac{n}{p_1}-\frac{n}{p}} N_k^{a(\frac{n}{p_1}-\frac{n}{p})} \|\triangle_{N_1} u\|_{p_1} \dots \|\triangle_{N_{k-1}} u\|_{p_1}\|\triangle_{N_k} u\|_{p_1}^a.
\end{aligned} \tag{2.21}$$

Inserting (2.21) into (2.19), we have

$$\begin{aligned}
\|u\|_{\dot B^s_{p,q}} \lesssim & \sum_{N_1 \geqslant \dots \geqslant N_K} (N_1^s \dots N_K^s \|\triangle_{N_1} u\|_p \dots \|\triangle_{N_K} u\|_p)^{q^2} \\
& \times \Lambda(N_1, ..., N_K)\|u\|_{\dot B^{s_1}_{p_1,\infty}}^{q(1-q)\theta K}\|u\|_{\dot B^{0}_{p_0,\infty}}^{(1-\theta)Kq(1-q)},
\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}
\Lambda(N_1, \dots N_K) = \Big(& N_1^{-\frac{n}{p}+\frac{n}{p_1}-s_1+s} \dots N_{k-1}^{-\frac{n}{p}+\frac{n}{p_1}-s_1+s} N_k^{a(-\frac{n}{p}+\frac{n}{p_1}-s_1+s)} \\
& \times N_k^{(1-a)(-\frac{n}{p}+\frac{n}{p_0}+s)} N_{k+1}^{-\frac{n}{p}+\frac{n}{p_0}+s} \dots N_K^{-\frac{n}{p}+\frac{n}{p_0}+s}\Big)^{q(1-q)}.
\end{aligned} \tag{2.23}$$

By (2.22), we have

$$\|u\|_{\dot B^s_{p,q}} \lesssim \sum_{N_1 \geqslant \dots \geqslant N_K} \Lambda(N_1, \dots N_K) \sum_{i=1}^K (N_i^s \|\Delta_i u\|_p)^q \tag{2.24}$$

$$\times \|u\|_{\dot B^{s_1}_{p_1,\infty}}^{(1-q)\theta}\|u\|_{\dot B^{0}_{p_0,\infty}}^{(1-\theta)(1-q)}. \tag{2.25}$$

So, it suffices to prove

$$\sum_{N_1 \geqslant \dots \geqslant N_K} \Lambda(N_1, \dots N_K) \sum_{i=1}^K (N_i^s \|\Delta_i u\|_p)^q \lesssim \|u\|^q_{\dot B^s_{p,q}}. \tag{2.26}$$

In fact, (2.23)–(2.26) imply the result. Finally, we prove (2.26). Applying the condition (2.18), we have

$$\begin{aligned}
&\sum_{N_1 \geqslant \dots \geqslant N_K} \Lambda(N_1, \dots N_K)(N_k^s \|\Delta_k u\|_p)^q \\
&\lesssim \sum_{N_{k-1} \geqslant N_k} \Big(N_{k-1}^{(k-1)(s-s_1+\frac{n}{p_1}-\frac{n}{p})} N_k^{(K-k+1-a)(s+\frac{n}{p_0}-\frac{n}{p})+a(s-s_1+\frac{n}{p_1}-\frac{n}{p})}\Big)^{q(1-q)} N_k^{sq}\|\Delta_k u\|_p^q
\end{aligned}$$

$$\lesssim \sum_{N_{k-1}\geqslant N_k} \left(\frac{N_{k-1}}{N_k}\right)^{(k-1)(s-s_1+\frac{n}{p_1}-\frac{n}{p})q(1-q)} N_k^{sq}\|\Delta_k u\|_p^q$$

$$\lesssim \|u\|^q_{\dot B^s_{p,q}}. \tag{2.27}$$

Case 2. We consider the case

$$s_1 - s + \frac{n}{p} - \frac{n}{p_1} < 0, \quad s + \frac{n}{p_0} - \frac{n}{p} < 0. \tag{2.28}$$

Substituting the summation $\sum_{N_1\geqslant\ldots\geqslant N_K}$ by $\sum_{N_1\leqslant\ldots\leqslant N_K}$ in (2.19) and repeating the procedure as in Case 1, we can get the result, as desired.

Up to now, we have shown the results for the following two cases: (i) $s = (1-\theta)s_0 + \theta s_1$ and $p_0 = p_1$; (ii) $s < (1-\theta)s_0 + \theta s_1$ and $p \geqslant p_0 \vee p_1$.

Step 2. We consider the case $p < p_0 \vee p_1$ and $s < (1-\theta)s_0 + \theta s_1$. Due to $\theta \in (0,1)$ and $1/p \leqslant (1-\theta)/p_0 + \theta/p_1$, we see that $p_0 \neq p_1$ and $p_0 \wedge p_1 < p < p_0 \vee p_1$. Let $0 < \varepsilon \ll 1$. In view of the result as in Step 1, we see that

$$\|f\|_{\dot B^s_{p,q}} \lesssim \|f\|^{1/2}_{\dot B^{s-\varepsilon}_{p,\infty}} \|f\|^{1/2}_{\dot B^{s+\varepsilon}_{p,\infty}}. \tag{2.29}$$

Since $s_0 - n/p_0 \neq s_1 - n/p_1$, we can assume that $s_0 - n/p_0 < s_1 - n/p_1$. It follows that $1/p - s/n \in (1/p_0 - s_0/n,\ 1/p_1 - s_1/n)$. Hence, for sufficiently small $\varepsilon > 0$,

$$\frac{1}{p} - \frac{s\pm\varepsilon}{n} \in \left(\frac{1}{p_0} - \frac{s_0}{n},\ \frac{1}{p_1} - \frac{s_1}{n}\right).$$

It follows that there exist $\theta_\pm \in (0,1)$ satisfying

$$\frac{1}{p} - \frac{s\pm\varepsilon}{n} = (1-\theta_\pm)\left(\frac{1}{p_0} - \frac{s_0}{n}\right) + \theta_\pm\left(\frac{1}{p_1} - \frac{s_1}{n}\right).$$

Due to $\lim_{\varepsilon\to 0}\theta_\pm = \theta$, we see that for sufficiently small $\varepsilon > 0$,

$$s \pm \varepsilon \leqslant (1-\theta_\pm)s_0 + \theta_\pm s_1.$$

Therefore, by Theorem 1.2, we have

$$\|f\|_{\dot B^{s-\varepsilon}_{p,\infty}} \lesssim \|f\|^{1-\theta_-}_{\dot B^{s_0}_{p_0,\infty}} \|f\|^{\theta_-}_{\dot B^{s_1}_{p_1,\infty}}, \tag{2.30}$$

$$\|f\|_{\dot B^{s+\varepsilon}_{p,\infty}} \lesssim \|f\|^{1-\theta_+}_{\dot B^{s_0}_{p_0,\infty}} \|f\|^{\theta_+}_{\dot B^{s_1}_{p_1,\infty}}. \tag{2.31}$$

We easily see that $\theta = (\theta_+ + \theta_-)/2$. Inserting (2.30) and (2.31) into (2.29), we have the result, as desired.

(Necessity) First, we show the necessity for $s - n/p \neq s_0 - n/p_0$. If not, then $s - n/p = s_0 - n/p_0 = s_1 - n/p_1$. Let

$$\hat{f}(\xi) = \sum_{j=100}^{N} 2^{(n/p-s)j} \varphi_j(\xi). \tag{2.32}$$

We see that $\|f\|_{B^s_{p,q}} \sim N^{1/q}$, $\|f\|_{B^s_{p,\infty}} \sim 1$, which contradicts (4.5).

Next, we show the necessity of $p_0 = p_1$ when $s = (1-\theta)s_0 + \theta s_1$. Assume for a contrary that $p_0 \neq p_1$. By Theorem 1.2, we have $1/q \leqslant (1-\theta)/\infty + \theta/\infty = 0$. This contradicts the condition $q < \infty$. □

Proof of Theorem 1.4 (Sufficiency) First, we consider the case $s < (1-\theta)s_0 + \theta s_1$. We can take sufficiently small $\varepsilon > 0$ satisfying

$$s \leqslant (1-\theta)s_0^* + \theta s_1^*, \quad s_0^* := s_0 - \varepsilon, \; s_1^* := s_1 - \varepsilon.$$

Since $\varepsilon \ll 1$, we can assume that

$$\frac{1}{p_0^*} := \frac{1}{p_0} - \frac{\varepsilon}{n} > 0, \quad \frac{1}{p_1^*} := \frac{1}{p_1} - \frac{\varepsilon}{n} > 0.$$

Hence,

$$\frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0^*} - s_0^*\right) + \theta\left(\frac{n}{p_1^*} - s_1^*\right), \tag{2.33}$$

which implies that

$$\frac{1}{p} - \frac{1-\theta}{p_0^*} - \frac{\theta}{p_1^*} = \frac{s}{n} - (1-\theta)\frac{s_0^*}{n} - \theta\frac{s_1^*}{n} := -\eta \leqslant 0. \tag{2.34}$$

Putting

$$\frac{1}{p^*} = \frac{1}{p} + \eta, \quad s^* = s + n\eta, \tag{2.35}$$

we see that

$$\frac{1}{p^*} = \frac{1-\theta}{p_0^*} + \frac{\theta}{p_1^*}, \quad s^* = (1-\theta)s_0^* + \theta s_1^*. \tag{2.36}$$

Using Hölder's inequality, in an analogous way as in Besov spaces, we have

$$\|f\|_{\dot{F}^{s^*}_{p^*,q}} \lesssim \|f\|^{1-\theta}_{\dot{F}^{s_0^*}_{p_0^*,q}} \|f\|^{\theta}_{\dot{F}^{s_1^*}_{p_1^*,q}}.$$

Recalling the inclusions (see Triebel [59])

$$\dot F^{s_0}_{p_0,\infty} \subset F^{s_0^*}_{p_0^*,q}, \quad \dot F^{s_1}_{p_1,\infty} \subset F^{s_1^*}_{p_1^*,q}$$

we immediately get the conclusion.

Next, we consider the case $s = (1-\theta)s_0 + \theta s_1$ and $s_0 \neq s_1$. In this case we easily see that $1/p = (1-\theta)/p_0 + \theta/p_1$. The result has been shown in [54] and [11] and we omit the details of the proof.

(Necessity) It suffices to consider the necessity in the case $s = (1-\theta)s_0 + \theta s_1$. If not, then $s_0 = s_1 = s$. Let ρ_j be as in (2.4) and

$$\hat f(\xi) = \sum_{j=100}^{N} 2^{-sj}\rho_j(\xi). \tag{2.37}$$

We easily see that

$$\|f\|_{\dot F^s_{p,\infty}} = \|\mathscr{F}^{-1}(\rho_0)\|_p \sim 1.$$

But

$$\|f\|_{\dot F^s_{p,q}} \sim N^{1/q},$$

which contradicts the GN inequality. □

3 Corollaries of the GN inequalities

In this section we give some corollaries of our main results. Noticing that $BMO = \dot F^0_{\infty,2} \subset \dot B^0_{\infty,\infty}$ and $\|\nabla^s u\|_{\dot B^0_{p,\infty}} \lesssim \|\nabla^s u\|_p$, we can deduce the following useful interpolation inequalities:

$$\|u\|_{L^{10}(\mathbb{R}^3)} \leqslant C\|u\|^{2/3}_{\dot B^{-1/2}_{\infty,\infty}(\mathbb{R}^3)}\|u\|^{1/3}_{\dot B^1_{10/3,10/3}(\mathbb{R}^3)}, \tag{3.1}$$

$$\|u\|_{L^4} \lesssim \|\nabla u\|^{1/2}_{L^2}\|u\|^{1/2}_{\dot B^{-1}_{\infty,\infty}}, \tag{3.2}$$

$$\|\nabla u\|_{L^4} \lesssim \|\nabla^2 u\|^{1/2}_{L^2}\|u\|^{1/2}_{BMO}, \tag{3.3}$$

$$\|u\|_{L^q} \lesssim \|\nabla u\|^{\theta}_{L^p}\|u\|^{1-\theta}_{\dot B^{-\theta/(1-\theta)}_{\infty,\infty}}, \quad 1 \le p < q < \infty, \theta = p/q. \tag{3.4}$$

$$\|\nabla^m u\|_{L^q} \lesssim \|\nabla^k u\|^{\theta}_{L^p}\|u\|^{1-\theta}_{BMO}, \quad 1 \leqslant m < k,\ q = kp/m,\ \theta = m/k. \tag{3.5}$$

Following Bourgain [8], we can show (3.1), which is useful for the concentration phenomena for the solutions of nonlinear Schrödinger equations. Meyer and Rivière [52] studied the partial regularity of solutions for the stationary Yang-Mills fields by using (3.2) and (3.3). (3.4) and (3.5) are generalized versions of (3.2) and (3.3), respectively (see Ledoux [40], Strzelecki [57]). Machihara and Ozawa [44] showed that

Proposition 3.1 *Let* $1 \le p_0 \vee p_1 \leqslant p \leqslant \infty$, $0<\theta<1$, $s_0, s_1 \in \mathbb{R}$. *Assume that*

$$\frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0} - s_0\right) + \theta\left(\frac{n}{p_1} - s_1\right),$$
$$s_0 < \frac{n}{p_0} - \frac{n}{p}, \quad s_1 > \frac{n}{p_1} - \frac{n}{p}. \tag{3.6}$$

Then

$$\|u\|_{\dot B^0_{p,1}} \lesssim \|u\|^{1-\theta}_{\dot B^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{\dot B^{s_1}_{p_1,\infty}} \tag{3.7}$$

Oru [54] obtained that (see also [11])

Proposition 3.2 *Let* $0 < p_0, p_1, p < \infty$, $0 < r < \infty$, $-\infty < s_0, s_1, s < \infty$, $0 < \theta < 1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1, \quad s_0 \neq s_1. \tag{3.8}$$

Then

$$\|u\|_{\dot F^s_{p,r}(\mathbb{R}^n)} \leqslant C\|u\|^{1-\theta}_{\dot F^{s_0}_{p_0,\infty}(\mathbb{R}^n)} \|u\|^{\theta}_{\dot F^{s_1}_{p_1,\infty}(\mathbb{R}^n)}. \tag{3.9}$$

The following interpolation inequality was shown in [60].

Proposition 3.3 *Let* $0 < p_0 < p < \infty$, $0 < r \leqslant \infty$, $-\infty < s_1 < s < s_0 < \infty$, $0 < \theta < 1$ *and*

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{\infty}, \qquad s = \theta s_0 + (1-\theta)s_1. \tag{3.10}$$

Then

$$\|u\|_{\dot F^s_{p,r}(\mathbb{R}^n)} \leqslant C\|u\|^{1-\theta}_{\dot B^{s_1}_{\infty,\infty}(\mathbb{R}^n)} \|u\|^{\theta}_{\dot B^{s_0}_{p_0,p_0}(\mathbb{R}^n)}. \tag{3.11}$$

4 GN inequalities in nonhomogeneous spaces

We denote by $H^s_p := (I-\triangle)^{s/2}L^p$ the Bessel potential space, $H^s = H^s_2$. Let ψ be a smooth cut-off function supported in the ball $\{\xi : |\xi| \leqslant 2\}$, $\varphi = \psi(\cdot) - \psi(2\,\cdot)$. We write $\psi_0 := \psi$ and $\psi_k(\xi) = \varphi(2^{-k}\xi)$, $k \in \mathbb{N}$. We see that

$$\sum_{k=0}^{\infty} \psi_k(\xi) = 1, \quad \xi \in \mathbb{R}^n. \tag{4.1}$$

We introduce the dyadic decomposition operators $\triangle_k = \mathscr{F}^{-1}\varphi_k\mathscr{F}, \quad k \in \mathbb{Z}_+$. Let $-\infty < s < \infty,\ 1 \leqslant p, q \leqslant \infty$. The space $B^s_{p,q}$ equipped with norm

$$\|f\|_{B^s_{p,q}} := \left(\sum_{k=0}^{\infty} 2^{ksq}\|\triangle_k f\|_p^q\right)^{1/q} \tag{4.2}$$

is said to be a Besov space. Let

$$-\infty < s < \infty, \quad 1 \leqslant p < \infty, \quad 1 \leqslant q \leqslant \infty. \tag{4.3}$$

The space $F^s_{p,q}$ equipped with norm

$$\|f\|_{F^s_{p,q}} := \left\|\left(\sum_{k=0}^{\infty} 2^{ksq}|\triangle_k f|^q\right)^{1/q}\right\|_p \tag{4.4}$$

is said to be a homogeneous Triebel-Lizorkin space. For Besov spaces and Triebel spaces, we have similar results as in Theorems 1.2, 1.3 and 1.4. In this paper, we will use the following

Theorem 4.1 *Let $0 < q < \infty$, $0 < p,\ p_0,\ p_1 \leqslant \infty$, $0 < \theta < 1$, $-\infty < s, s_0, s_1 < \infty$. Then the GN inequality of the following type*

$$\|u\|_{B^s_{p,q}} \lesssim \|u\|^{1-\theta}_{B^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{B^{s_1}_{p_1,\infty}} \tag{4.5}$$

holds if

$$\frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0} - s_0\right) + \theta\left(\frac{n}{p_1} - s_1\right), \tag{4.6}$$

$$s_0 - \frac{n}{p_0} \neq s_1 - \frac{n}{p_1}, \tag{4.7}$$

$$s \leqslant (1-\theta)s_0 + \theta s_1, \tag{4.8}$$

$$p_0 = p_1 \quad \text{if} \quad s = (1-\theta)s_0 + \theta s_1. \tag{4.9}$$

Proposition 4.2 *Let $1 < p, p_0, p_1 < \infty$, $s,\ s_1 \in \mathbb{R}$, $0 \leqslant \theta \leqslant 1$. Then the GN inequality of the following type*

$$\|u\|_{H^s_p} \lesssim \|u\|^{1-\theta}_{L^{p_0}} \|u\|^{\theta}_{H^{s_1}_{p_1}} \tag{4.10}$$

holds if

$$\frac{n}{p} - s = (1-\theta)\frac{n}{p_0} + \theta\left(\frac{n}{p_1} - s_1\right), \quad s \leqslant \theta s_1. \tag{4.11}$$

The proofs of these results are the same as those in homogeneous spaces and the details of the proofs are omitted.

5 Concentration of solutions of NS equation

The local well posedness in L^n for the NS equation is well-known; cf. Kato [34]. Here we need the following result (see for instance [35] in 3D and [61] in higher spatial dimensions).

Theorem 5.1 *Let $u_0 \in L^n$ with $\mathrm{div} u_0 = 0$. Then there exists a $T_m > 0$ such that the NS equation (1.24) has a unique solution u satisfying*

$$u \in C([0, T_m);\ L^n)\ \cap L^{2+n}_{\mathrm{loc}}(0, T_m;\ L^{2+n}). \tag{5.1}$$

If $T_m < \infty$, then we have $\|u\|_{L^{2+n}(0,T_m;\ L^{2+n})} = \infty$. Moreover, if $u_0 \in L^2$, then

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds = \frac{1}{2}\|u_0\|_2^2, \quad 0 < t < T_m. \tag{5.2}$$

We will sketch the proof of Theorem 5.1 in the Appendix. In the sequel, we will write $\|u\|_2^2 := \sum_{i=1}^n \|u_i\|_2^2$, $\|\nabla u\|_2^2 := \sum_{i,j=1}^n \|\partial_{x_j} u_i\|_2^2$ for $u = (u_1, ..., u_n)$. We have the following

Proposition 5.2 *Let $\sigma \geqslant 1$ and u be the smooth solution of NS equation. Then we have*

$$\begin{aligned} \frac{1}{2+\sigma}\frac{d}{dt}\|u(t)\|_{2+\sigma}^{2+\sigma} &+ \frac{1}{2}\int_{\mathbb{R}^n} (\nabla |u|^\sigma \cdot \nabla |u|^2)(x)dx \\ &+ \int_{\mathbb{R}^n} |u|^\sigma |\nabla u|^2(x)dx - \int_{\mathbb{R}^n} (\nabla p \cdot |u|^\sigma u)(x)dx = 0. \end{aligned} \tag{5.3}$$

Proof. The first equation in (1.24) is multiplied by $|u|^\sigma u$, we have

$$|u|^\sigma u \cdot \left(\partial_t u - \Delta u + \sum_{j=1}^n u_j \partial_{x_j} u + \nabla p\right) = 0. \tag{5.4}$$

We have

$$\sum_{i=1}^n |u|^\sigma u_i \partial_t u_i = \frac{1}{2}|u|^\sigma \partial_t |u|^2 = \frac{1}{2+\sigma}\partial_t |u|^{\sigma+2}, \tag{5.5}$$

$$|u|^\sigma u_i \Delta u_i = \nabla(|u|^\sigma u_i \nabla u_i) - \frac{1}{2}(\nabla |u|^\sigma \cdot \nabla u_i^2) - |u|^\sigma |\nabla u_i|^2. \tag{5.6}$$

It follows that

$$\sum_{i=1}^n |u|^\sigma u_i \Delta u_i = \sum_{i=1}^n \nabla(|u|^\sigma u_i \nabla u_i) - \frac{1}{2}(\nabla |u|^\sigma \cdot \nabla |u|^2) - |u|^\sigma |\nabla u|^2. \tag{5.7}$$

Noticing that $\mathrm{div} u = 0$, we have

$$\sum_{i=1}^{n} |u|^{\sigma} u_i \sum_{j=1}^{n} u_j \partial_j u_i = \frac{1}{2} \sum_{i,j=1}^{n} |u|^{\sigma} u_j \partial_j u_i^2 = \frac{1}{2} \sum_{j=1}^{n} |u|^{\sigma} u_j \partial_j |u|^2$$
$$= \frac{1}{2+\sigma} \sum_{j=1}^{n} \partial_j (|u|^{\sigma+2} u_j). \tag{5.8}$$

We obtain that

$$\frac{1}{2+\sigma} \partial_t |u|^{\sigma+2} - \frac{1}{2} \nabla (|u|^{\sigma} \nabla |u|^2) + \frac{1}{2} (\nabla |u|^{\sigma} \cdot \nabla |u|^2)$$
$$+ |u|^{\sigma} |\nabla u|^2 + |u|^{\sigma} u \nabla p + \frac{1}{2+\sigma} \nabla (|u|^{\sigma+2} u) = 0. \tag{5.9}$$

Integrating (5.9) over $\mathbb{R}^n$, we immediately obtain the result, as desired. $\square$

Recall that by (1.24),

$$-\Delta p = \sum_{i,j=1}^{n} \partial_{x_i x_j} (u_i u_j). \tag{5.10}$$

Let us denote

$$E(u,v) = \sum_{i,j=1}^{n} \mathscr{F}^{-1} |\xi|^{-2} \xi_i \xi_j \mathscr{F}(u_i v_j). \tag{5.11}$$

From the Hörmander-Mikhlin multiplier theorem, we obtain that for any $p \in (1, \infty)$,

$$\|E(u,v)\|_p \lesssim \sum_{i,j=1}^{n} \|u_i v_j\|_p. \tag{5.12}$$

Putting $\sigma = n-2$ and integrating (5.3) over $[t_1, t_2]$, we have

$$\|u(t_2)\|_n^n + 2(n-2) \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left| \nabla |u|^{n/2} \right|^2 dx dt$$
$$+ n \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u|^{n-2} |\nabla u|^2 dx dt \leq \|u(t_1)\|_n^n + n \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |p \nabla (|u|^{n-2} u)| dx dt. \tag{5.13}$$

Applying (5.2) and (A.7), we obtain that

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left| p \nabla (|u|^{n-2} u) \right| dx dt$$
$$\lesssim \int_{t_1}^{t_2} \| |u|^{(n-2)/2} |\nabla u| \|_2 \| |u|^{(n-2)/2} E(u,u) \|_2 dt$$

$$\lesssim \int_{t_1}^{t_2} \||u|^{(n-2)/2}|\nabla u|\|_2 \|u\|_{2+n}^{(n-2)/2} \|E(u,u)\|_{(n+2)/2} dt$$

$$\lesssim \frac{1}{100}\int_{t_1}^{t_2} \||u|^{(n-2)/2}|\nabla u|\|_2^2 dt + C_n \int_{t_1}^{t_2} \|u\|_{2+n}^{2+n} dt. \tag{5.14}$$

Inserting the estimate as in (5.14) into (5.15), we have

Lemma 5.3 *Let u be the solution of the NS equation* (1.24) *in* $[0, T_m)$ *and* $t_1, t_2 \in [0, T_m)$*. We have*

$$\|u(t_2)\|_n^n + 2(n-2)\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left|\nabla |u|^{n/2}\right|^2 dxdt$$
$$+ \frac{99n}{100}\int_{t_1}^{t_2}\int_{\mathbb{R}^n} |u|^{n-2}|\nabla u|^2 dxdt \le \|u(t_1)\|_n^n + C_n \int_{t_1}^{t_2} \|u\|_{2+n}^{2+n} dt. \tag{5.15}$$

Proof of Theorem 1.6. By the local well posedness result and Lemma 5.3, we see that if $T_m < \infty$, then we have

$$\|u\|_{L^{2+n}_{x,t\in[0,T_m)}} = \infty. \tag{5.16}$$

In the following we give the details of the analysis to $\|u\|_{L^{2+n}_{x,t\in[0,T]}}$. We have

$$\int_S^T \|u(t)\|_{2+n}^{2+n} dt = \int_S^T \left\||u(t)|^{n/2}\right\|_{2(2+n)/n}^{2(2+n)/n} dt$$
$$\le \int_S^T \left\|P_{\le N}|u(t)|^{n/2}\right\|_{2(2+n)/n}^{2(2+n)/n} dt + \int_S^T \left\|P_{\geqslant N}|u(t)|^{n/2}\right\|_{2(2+n)/n}^{2(2+n)/n} dt. \tag{5.17}$$

For convenience, we write

$$P_{\le N} f := \mathscr{F}^{-1}\psi(2^{-N}\xi)\mathscr{F},$$

where ψ is the smooth cut-off function supported in $\{\xi : |\xi| \le 2\}$ as before. Using Bernstein's estimates and the L^2 bound of solutions, we see that

$$\int_S^T \left\|P_{\le N}|u(t)|^{n/2}\right\|_{2(2+n)/n}^{2(2+n)/n} dt$$
$$\lesssim (T-S)2^{Nn^2/2} \max_{t\in[S,T]} \left\|P_{\le N}|u(t)|^{n/2}\right\|_{4/n}^{2(n+2)/n}$$
$$\lesssim (T-S)2^{Nn^2/2} \max_{t\in[S,T]} \|u(t)\|_2^{(n+2)/2} \lesssim (T-S)2^{Nn^2/2}. \tag{5.18}$$

Let $T_k \nearrow T_m$, we see that $\|u\|_{L^{2+n}_{x,t\in[0,T_k]}} \nearrow \infty$. We can assume, by passing to a subsequence of $\{T_k\}$ that

$$\|u\|_{L^{2+n}_{x,t\in[T_{k-1},T_k]}} \geqslant \|u\|_{L^{2+n}_{x,t\in[0,T_{k-1}]}}. \tag{5.19}$$

Let $N_k \nearrow \infty$ satisfy

$$c\|u\|^{2+n}_{L^{2+n}_{x,t\in[T_{k-1},T_k]}} \leq C(T_k - T_{k-1})2^{N_k n^2/2} \leq \frac{1}{2}\|u\|^{2+n}_{L^{2+n}_{x,t\in[T_{k-1},T_k]}}. \tag{5.20}$$

We have

$$\int_{T_{k-1}}^{T_k} \left\|P_{\leq N_k}|u(t)|^{n/2}\right\|^{2(2+n)/n}_{2(2+n)/n} dt \leq \frac{1}{2}\|u\|^{2+n}_{L^{2+n}_{x,t\in[T_{k-1},T_k]}}. \tag{5.21}$$

It follows from (5.17), (5.19) and (5.21) that

$$\frac{1}{4}\|u\|^{2+n}_{L^{2+n}_{x,t\in[0,T_k]}} \leq \int_{T_{k-1}}^{T_k} \left\|P_{\geqslant N_k}|u(t)|^{n/2}\right\|^{2(2+n)/n}_{2(2+n)/n} dt. \tag{5.22}$$

In view of the fractional GN inequality, we have

$$\|v\|_{L^{2(2+n)/n}_{x,t\in[T_{k-1},T_k]}} \lesssim \|v\|^{2/(n+2)}_{L^\infty(T_{k-1},T_k;\dot B^{-n/2}_{\infty,\infty})}\|\nabla v\|^{n/(n+2)}_{L^2_{x,t\in[T_{k-1},T_k]}}. \tag{5.23}$$

Taking $v = P_{\geqslant N_k}|u|^{n/2}$, by (5.22) and (5.23) we have

$$\frac{1}{4}\|u\|^{2+n}_{L^{2+n}_{x,t\in[0,T_k]}} \lesssim \|P_{\geqslant N_k}|u|^{n/2}\|^{4/n}_{L^\infty(T_{k-1},T_k;\dot B^{-n/2}_{\infty,\infty})}\|\nabla P_{\geqslant N_k}|u|^{n/2}\|^2_{L^2_{x,t\in[T_{k-1},T_k]}}. \tag{5.24}$$

By Lemma 5.3, we see that

$$\|\nabla P_{\geqslant N_k}|u|^{n/2}\|^2_{L^2_{x,t\in[0,T_k]}} \lesssim \|u_0\|^n_n + \|u\|^{2+n}_{L^{2+n}_{x,t\in[0,T_k]}}. \tag{5.25}$$

Hence, it follows from (5.24) and (5.25) that

$$\|P_{\geqslant N_k}|u|^{n/2}\|_{L^\infty(T_{k-1},T_k;\dot B^{-n/2}_{\infty,\infty})} \gtrsim 1. \tag{5.26}$$

We remark that the constant in the right hand side of (5.26) only depends on n and $\|u_0\|_n$. So, there exist $x_k \in \mathbb{R}^n$, $t_k \in [T_{k-1}, T_k]$ and $j_k \geqslant N_k$ such that

$$2^{-nj_k/2}|(\Delta_{j_k}|u|^{n/2})(x_k,t_k)| \gtrsim 1. \tag{5.27}$$

Let ψ be as in (4.1), $0 < \varepsilon \ll 1$. It follows that

$$1 \lesssim 2^{nj_k/2}\left|\int (\mathscr{F}^{-1}\psi)(2^{j_k}(x_k - y))|u(t_k,y)|^{n/2}dy\right|$$

$$\lesssim 2^{nj_k/2}\left|\left(\int_{|y-x_k|\le 2^{(\varepsilon-1)j_k}}+\int_{|y-x_k|>2^{(\varepsilon-1)j_k}}\right)(\mathscr{F}^{-1}\psi)(2^{j_k}(x_k-y))|u(t_k,y)|^{n/2}dy\right|$$
$$:= I + II. \tag{5.28}$$

By Hölder's inequality, we have

$$\begin{aligned} II &\lesssim 2^{nj_k/2}\|(\mathscr{F}^{-1}\psi)(2^{j_k}\cdot)\|_{L^{4/(4-n)}(|\cdot|>2^{(\varepsilon-1)j_k})}\|u(t_k,\cdot)\|_2^{n/2} \\ &\lesssim 2^{nj_k/2}\|(\mathscr{F}^{-1}\psi)(2^{j_k}\cdot)\|_{L^{4/(4-n)}(|\cdot|>2^{(\varepsilon-1)j_k})} \\ &\lesssim 2^{nj_k(n/4-1/2)}\|(\mathscr{F}^{-1}\psi)\|_{L^{4/(4-n)}(|\cdot|>2^{\varepsilon j_k})}. \end{aligned} \tag{5.29}$$

Since ψ is a Schwartz function, for fixed $\varepsilon>0$, we have

$$II \ll 1/2, \quad if \quad k \gg 1.$$

Hence,

$$1/2 \lesssim I \lesssim \|u\|^{n/2}_{L^n(|\cdot-x_k|\le 2^{(\varepsilon-1)j_k})}\|\mathscr{F}^{-1}\psi\|_2 \lesssim \|u\|^{n/2}_{L^n(|\cdot-x_k|\le 2^{(\varepsilon-1)j_k})}.$$

By (5.20), we see that $2^{(\varepsilon-1)j_k}\lesssim (T_m-T_{k-1})^{\delta}$ for any $\delta<2/n^2$. □

6 Proof of Theorem 1.7

Let $q=n/(n-\beta)$. First, we consider the case $m^2=0$. We divide the proof into the following five steps.

Step 1. We show that $M_c>-\infty$. Applying Hardy-Littlewood-Sobolev's inequality, we have

$$\begin{aligned} &\int\int G(u_1(x),...,u_L(x))V(|x-y|)G(u_1(y),...,u_L(y))dxdy \\ &\lesssim (\|u\|^2_{2(2q)'}+\|u\|^{\mu}_{(2q)'\mu})^2, \end{aligned} \tag{6.1}$$

where $(2q)'$ is the dual exponent to $2q$. In view of the fractional Gagliardo-Nirenberg inequality, we have

$$\|u\|_{2(2q)'} \lesssim \|u\|_2^{1-\theta_2}\|u\|^{\theta_2}_{\dot H^s}, \tag{6.2}$$
$$\|u\|_{\mu(2q)'} \lesssim \|u\|_2^{1-\theta_\mu}\|u\|^{\theta_\mu}_{\dot H^s}, \tag{6.3}$$

where

$$\frac{s\theta_\lambda}{n}=\frac{1}{2}-\frac{1}{\lambda(2q)'}.$$

We consider the following two cases. First, if $\mu < 2+2s/n-1/q$, we easily see that $2\theta_2, \mu\theta_\mu < 1$. It follows from $u \in S_c$ that

$$\begin{aligned}&\int\int G(u_1(x),...,u_L(x))V(|x-y|)G(u_1(y),...,u_L(y))dxdy\\&\lesssim \|u\|_{\dot{H}^s}^{2\theta_\mu} + \|u\|_{\dot{H}^s}^{2\theta_2} \lesssim 1+\varepsilon\|u\|_{\dot{H}^s}^2\end{aligned} \tag{6.4}$$

for some sufficiently small $\varepsilon > 0$. Next, if $\mu = 2+2s/n-1/q$, applying the condition $u \in S_c$ and $c_1,...,c_L$ are sufficiently small, we see that (6.4) also holds. So, we have shown that

$$E(u) \geqslant \left(\frac{1}{2} - C\varepsilon\right)\|u\|_{\dot{H}^s}^2 - C. \tag{6.5}$$

Therefore, we have $M_c > -\infty$ and all of the minimizing sequence of (1.28) are bounded in $(H^s)^L$.

Step 2. We show the existence of the Schwarz symmetric (=radial and radially decreasing) sequence. Let u^* be the monotone rearrangement of u. By the supermodularity of G (see Proposition 3.13 of [32]) and Theorem 1.2 in [12],

$$\begin{aligned}&\int\int G(u_1(x),...,u_L(x))V(|x-y|)G(u_1(y),...,u_L(y))dxdy\\&\leqslant \int\int G(u_1^*(x),...,u_L^*(x))V(|x-y|)G(u_1^*(y),...,u_L^*(y))dxdy.\end{aligned} \tag{6.6}$$

On the other hand, we know thanks to (cf. Appendix of [3])

$$\|u^*\|_{\dot{H}^s} \leqslant \|u\|_{\dot{H}^s}. \tag{6.7}$$

It follows that $E(u^*) \leqslant E(u)$. Hence, we obtain the existence of the Schwarz minimizing sequence. So, it suffices to consider the Schwarz minimizing sequence below.

Step 3. We show the lower semi-continuity of $E(\cdot)$ under the Schwarz minimizing sequence. Let $u_k = (u_{k,1},...,u_{k,L})$ be a Schwarz symmetric minimizing sequence. We show that if u_k weakly converges to u in $(H^s)^L$, then

$$E(u) \leqslant \liminf E(u_k). \tag{6.8}$$

Since the minimizing sequence in $(H^s)^L$ is bounded, we see that there exists a subsequence, which is still written by u_k such that u_k weakly converges to $u = (u_1,...,u_L)$ in $(\dot{H}^s)^L$. It follows that

$$\|u\|_{\dot{H}^s}^2 \leqslant \lim\inf_{k\to\infty} \|u_k\|_{\dot{H}^s}^2. \tag{6.9}$$

In the following we show that

$$\lim_{k\to\infty}\int\int G(u_k(x))V(|x-y|)G(u_k(y))dxdy=\int\int G(u(x))V(|x-y|)G(u(y))dxdy. \tag{6.10}$$

The sequence u_k is bounded in $(H^s)^L$, so is in $L^{(2q)'\mu}\cap L^{2(2q)'}$. Since u_k is a symmetric sequence, we can certainly find a subsequence of u_k still written by u_k such that $u_k\to u$ and $|u_{k,j}|\leqslant a_j$ for some $a_j\in L^{(2q)'\mu}\cap L^{2(2q)'}$. By the continuity of G we have

$$G(u_k(x))V(|x-y|)G(u_k(y))\to G(u(x))V(|x-y|)G(u(y)),\quad k\to\infty$$

for all $x,y\in\mathbb{R}^n$. On the other hand, since G is non-decreasing with respect to all variables, we have from condition (G1) that

$$\begin{aligned}G(u_k(x))V(|x-y|)G(u_k(y))&\leqslant G(a(x))V(|x-y|)G(a(y))\\&\lesssim(|a(x)|^2+|a(x)|^\mu)V(|x-y|)(|a(y)|^2+|a(y)|^\mu).\end{aligned} \tag{6.11}$$

It follows that

$$\begin{aligned}&\int\int G(u_k(x))V(|x-y|)G(u_k(y))dxdy\\&\lesssim\int\int(|a(x)|^2+|a(x)|^\mu)V(|x-y|)(|a(y)|^2+|a(y)|^\mu)\\&\lesssim(\|a\|^2_{2(2q)'}+\|a\|^\mu_{\mu(2q)'})<\infty.\end{aligned} \tag{6.12}$$

In view of the dominated convergence theorem, we immediately have (6.10).

Step 4. We show the strict negativity of M_c. Let $\varphi:\mathbb{R}^n\to(0,1)$ be a Schwarz radial function satisfying $\|\varphi\|_2=1$. Taking $\varphi_i=c_i\varphi$, $i=1,...,L$ and $\Phi_\lambda=\lambda^{n/2}\Phi(\lambda\cdot)=\lambda^{n/2}(\varphi_1(\lambda\cdot),...,\varphi_L(\lambda\cdot))$. Clearly, we have $\Phi_\lambda\in S_c$. For convenience, we write $\alpha=\alpha_1+...+\alpha_L$. We have from the second growth condition in (G1) that for $0<\lambda\ll 1$,

$$\begin{aligned}E(\Phi_\lambda)&=\frac{1}{2}\|\Phi_\lambda\|^2_{\dot{H}^s}-\int\int G(\Phi_\lambda(x))V(|x-y|)G(\Phi_\lambda(y))dxdy\\&=\frac{1}{2}\lambda^{2s}\|\Phi_1\|^2_{\dot{H}^s}-\lambda^{-2n}\int\int G(\lambda^{n/2}\Phi(x))V(|x-y|/\lambda)G(\lambda^{n/2}\Phi(y))dxdy\\&\leqslant\frac{1}{2}\lambda^{2s}\|\Phi_1\|^2_{\dot{H}^s}-C\lambda^{-n+\alpha n-\beta}\int\int\varphi(x)^\alpha V(|x-y|)\varphi(y)^\alpha dxdy\\&\leqslant\lambda^{2s}(C_1-C_2\lambda^{-n+\alpha n-\beta-2s})\end{aligned} \tag{6.13}$$

for some $C_1, C_2 > 0$. Noticing that $n + \beta - n\alpha + 2s > 0$ and taking $0 < \lambda \ll 1$, we immediately have $E(\Phi_\lambda) < 0$. It follows that $M_c < 0$.

Step 5. We show that M_c is achieved. Notice that $M_c = E(u)$. It suffices to show that $\|u_i\|_2^2 = c_i$. The strict negativity of M_c and condition (G2) imply that $u_i \neq 0$ for all $i = 1, .., L$. Let $t_i = c_i/\|u_i\|_2^2$, $i = 1, ..., L$. We have $t_i \geqslant 1$ and $(t_1 u_1, ..., t_L u_L) \in S_c$. Therefore,

$$M_c \leqslant E(t_1 u_1, ..., t_L u_L) \leqslant t_{\max}^2 E(u) = t_{\max}^2 M_c.$$

Since $M_c < 0$, we immediately have $t_{\max} = 1$ and so, $t_1 = ... = t_L = 1$. It follows that u is a minimizer.

Next, we consider the case $m^2 > 0$. Since $\|u\|_{\dot{H}^s} \leq \|u\|_{H^s}$, $s > 0$, we see that the proof in Steps 1–3 and 5 holds if we substitute $\dot{H}^s$ by H^s. Moreover, noticing that $\|u_\lambda\|_{H^s}^2 \lesssim \|u\|_2^2 + \lambda^{2s}\|u\|_{\dot{H}^s}^2$, we see that the result in Step 4 is also true. □

In the proof of Step 4, we easily see that for the single power case $G(|u|^2) = u^\alpha$ with $2s + n + \beta - \alpha < 0$, $E(\Phi_\lambda) \to -\infty$ as $\lambda \to \infty$. Moreover, taking $\alpha = 2$, we see that the condition $s \geqslant (n - \beta)/2$ is also necessary.

7 Proof of Theorem 1.8

(Necessity) Put $u_\lambda = \lambda^{n/2} u(\lambda\,\cdot)$, $s = (n - \beta)/2$. For any $\phi \in (H^s)^L$, we write

$$I_{c,\beta}^{(n)}(\phi) = \frac{1}{2}\|\phi\|_{\dot{H}^s}^2 - \Upsilon_\beta(\phi). \tag{7.1}$$

we have

$$I_{c,\beta}^{(n)}(\phi_\lambda) = \lambda^{n-\beta}\left(\frac{1}{2}\|\phi\|_{\dot{H}^s}^2 - \Upsilon_\beta(\phi)\right). \tag{7.2}$$

By (1.40)

$$\Upsilon_\beta(u) = \int_{\mathbb{R}^{2n}} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{n-\beta}} dxdy \leq C^* c \|u\|_{\dot{H}^s}^2.$$

If $C^* c < 1/2$, then

$$\left(\frac{1}{2} - C^* c\right)\|u\|_{\dot{H}^s}^2 \leq I_{c,\beta}^{(n)}(u) \leqslant \frac{1}{2}\|u\|_{\dot{H}^s}^2. \tag{7.3}$$

It follows that $M_{c,\beta}^{(n)} \geqslant 0$. On the other hand, noticing that $\|\phi_\lambda\|_2 = \|\phi\|_2$, we see that

$$M_{c,\beta}^{(n)} \leq \inf\{I_{c,\beta}^{(n)}(\phi_\lambda) : \|\phi\|_2^2 = c\} \leqslant \frac{\lambda^{n-\beta}}{2}\|\phi\|_{\dot{H}^s}^2$$

holds for all $\phi \in H^s$ with $\|\phi\|_2^2 = c$. Hence, $M_{c,\beta}^{(n)} = 0$. For any minimizing sequence u_k, we have $I_{c,\beta}^{(n)}(u_k) \sim \|u_k\|_{\dot{H}^s}^2 \to 0$. It follows that $u_k \to 0$ in $(\dot{H}^s)^L$. But this contradicts the fact $\|u_k\|_2^2 = c$.

If $C^* c > 1/2$, we have $(C^* - \varepsilon)c > 1/2$ for sufficiently small $\varepsilon > 0$. By the definition of C^* we can choose some $\phi \in (H^s)^L$ such that

$$\Upsilon_\beta(\phi) \geqslant (C^* - \varepsilon)\|\phi\|_2^2 \|\phi\|_{\dot{H}^s}^2.$$

However,

$$I_{c,\beta}^{(n)}(\phi_\lambda) \leq \lambda^{n-\beta}\left(\frac{1}{2} - (C^* - \varepsilon)c\right)\|\phi\|_{\dot{H}^s}^2. \tag{7.4}$$

Taking $\lambda \to \infty$, we immediately have $M_{c,\beta}^{(n)} = -\infty$.

(Sufficiency) First, we show that $M_{c,\beta}^{(n)} = 0$. Since $C^* c = 1/2$, we have

$$\Upsilon_\beta(u) \leqslant \frac{1}{2}\|u\|_{\dot{H}^s}^2.$$

It follows that $M_{c,\beta}^{(n)} \geqslant 0$. On the other hand, for any $\varepsilon > 0$, we find some $\phi \in (\dot{H}^s)^L$ satisfying

$$\Upsilon_\beta(\phi) \geqslant \frac{1-\varepsilon}{2}\|\phi\|_{\dot{H}^s}^2.$$

For $s = (n-2)/2$, the above inequality is invariant under the scaling $\phi \mapsto \lambda^{n/2}\phi(\lambda\,\cdot)$, which implies that we can assume that $\|\phi\|_{\dot{H}^s} = 1$. It follows that $I_{c,\beta}^{(n)}(\phi) \leqslant \varepsilon$. Hence $M_{c,\beta}^{(n)} = 0$.

Now, let u_k be a sequence verifying

$$\frac{\Upsilon_\beta(u_k)}{\|u_k\|_2^2 \|u_k\|_{\dot{H}^s}^2} \geqslant C^*\left(1 - \frac{1}{k}\right). \tag{7.5}$$

Let u_k^* be the rearrangement of u_k. Using the fact that

$$\Upsilon_\beta(u_k) \leq \Upsilon_\beta(u_k^*), \ \|u^*\|_{\dot{H}^s} \leqslant \|u\|_{\dot{H}^s}, \ \|u^*\|_2 = \|u\|_2,$$

we see that (7.5) also holds if u_k is replaced by u_k^*, i.e.,

$$\frac{\Upsilon_\beta(u_k^*)}{\|u_k^*\|_2^2 \|u_k^*\|_{\dot{H}^s}^2} \geqslant C^*\left(1 - \frac{1}{k}\right). \tag{7.6}$$

One can find $\lambda_k > 0$ such that $\|\lambda_k^{n/2} u_k^*(\lambda_k\,\cdot)\|_{\dot{H}^s} = 1$. Since (7.6) is invariant under the scaling $u_k^* \mapsto \lambda^{n/2} u_k^*(\lambda\,\cdot)$, we see that for $v_k = \lambda_k^{n/2} u_k^*(\lambda_k\,\cdot)$,

$$\frac{\Upsilon_\beta(v_k)}{\|v_k\|_2^2 \|v_k\|_{\dot{H}^s}^2} \geqslant C^*\left(1 - \frac{1}{k}\right) \tag{7.7}$$

and $\|v_k\|_2^2 = c$, $\|v_k\|_{\dot H^s} = 1$. The inequality (7.7) also implies that $I^{(n)}_{c,\beta}(v_k) \leqslant 1/2k \to 0$. It follows that v_k is a radial and radially decreasing minimizing sequence. In view of $\|v_k\|^2_{H^s} \leqslant 1 + c$ we see that v_k has a subsequence which is still written by v_k such that v_k converges to v with respect to the weak topology in $(H^s)^L$. On the other hand, the embedding $H^s \subset L^q$ with $s = (n-\beta)/2$, $2 < q < 2n/\beta$ is compact for the class of radial functions, we see that v_k strongly converges to v (up to a subsequence) in $(L^q)^L$ for all $2 < q < 2n/\beta$. By (7.6) and Theorem 1.3, we have for $k \geqslant 2$,

$$1/4 \leqslant \Upsilon_\beta(v_k) \leqslant C\||v_k|^2\|^2_{2n/(n+\beta)} \leq C\|v_k\|^2_{B^s_{2,\infty}}\|v_k\|^2_{B^{-n/2}_{\infty,\infty}}. \tag{7.8}$$

It follows that $\|v_k\|_{B^{-n/2}_{\infty,\infty}} \geqslant c_0$, where $c_0 := 1/2\sqrt{C}$ is independent of k. Let $v_k = (v_k^1, ..., v_k^L)$. It is easy to see that there exist $i \in \{1, 2, ..., L\}$ and a subsequence of v_k^i which is still written by v_k^i verifying $\|v_k^i\|_{B^{-n/2}_{\infty,\infty}} \geqslant c_0/L$. From the definition of $B^a_{\infty,\infty}$ we can choose $j_k \in \mathbb{Z}_+$ and $x_k \in \mathbb{R}^n$,

$$c_0/2L \leqslant 2^{-nj_k/2}|(\triangle_{j_k} v_k^i)(x_k)|. \tag{7.9}$$

Denoting

$$\mathbb{A}(j_k) := \{x : \ |x_k - x| \leq A2^{-j_k}\},$$

$$\begin{aligned} c_0/2m &\leqslant 2^{-nj_k/2}|(\mathscr{F}^{-1}\varphi_{j_k}) * v_k^i(x_k)| \\ &= 2^{nj_k/2}\int_{\mathbb{R}^n}(\mathscr{F}^{-1}\varphi_{j_k})(2^{j_k}(x_k - z))v_k^i(z)dz \\ &= 2^{nj_k/2}\left(\int_{\mathbb{A}(j_k)} + \int_{\mathbb{R}^n\setminus\mathbb{A}(j_k)}\right)(\mathscr{F}^{-1}\varphi_{j_k})(2^{j_k}(x_k - z))v_k^i(z)dz \\ &:= I + II. \end{aligned} \tag{7.10}$$

Taking $A := A(\varphi, c) \gg 1$, we see that

$$II \leqslant \|v_k^i\|_2\|\mathscr{F}^{-1}\varphi\|_{L^2(|\cdot - x_k| \geqslant A)} \leq c_0/4.$$

By Hölder's inequality, we have

$$I \leqslant C\|v_k^i\|_{L^2(|\cdot - x_k| \leq A2^{-j_k})} \leqslant C\|v_k^i\|_{L^2(|\cdot - x_k| \leq A)}.$$

We have

$$\|v_k^i\|_{L^2(|\cdot - x_k| \leq A)} \geqslant c_0/4C.$$

Since v_k^i is radial, we have $|x_k| \leq X_0 := X_0(c_0, C, A)$. Indeed, in the opposite case we will have $\|v_k^i\|_2^2 > c$ if $|x_k| \gg 1$. So, we further have

$$\|v_k^i\|_{L^2(|\cdot| \leq X_0 + A)} \geqslant c_0/4C.$$

By Hölder's inequality,

$$\|v_k^i\|_{L^q(|\cdot|\leq X_0+A)} \geqslant \tilde{c}_0, \quad \tilde{c}_0 := \tilde{c}_0(A, X_0, c_0).$$

Since $v_k \to v$ in $(L^q)^L$, $2 < q < 2n/\beta$, we immediately have $v \neq 0$. Using the same way as in the proof of Theorem 1.7, we can get that

$$0 \leq I_{c,\beta}^{(n)}(v) \leq I_{c,\beta}^{(n)}(v_k) \to 0.$$

It follows that $I_{c,\beta}^{(n)}(v) = 0$. To finish the proof, it suffices to show that $\|v\|_2^2 = c$. If not, then we have $\|v\|_2^2 < c$. Putting $\tilde{v} = \sqrt{c}v/\|v\|_2$, we have

$$\begin{aligned} I_{c,\beta}^{(n)}(\tilde{v}) &= \frac{c}{\|v\|_2^2}\left(\frac{1}{2}\|v\|_{\dot{H}^s}^2 - \frac{c}{\|v\|_2^2}\Upsilon(v)\right) \\ &= \frac{c}{\|v\|_2^2} I_{c,\beta}^{(n)}(v) - \left(\frac{c}{\|v\|_2^2} - 1\right)\Upsilon_\beta(v) < 0, \end{aligned} \tag{7.11}$$

which contradicts the fact that $I_{c,\beta}^{(n)}(u) \geqslant 0$ for all $u \in (H^s)^L$.

8 Proof of Theorem 1.9

We consider the variational problem

$$M_{c,\beta,m}^{(n)} = \inf\{I_{c,\beta,m}^{(n)}(u) : u \in (H^s)^L, \ \|u\|_2^2 = c > 0\}, \tag{8.1}$$

$$I_{c,\beta,m}^{(n)}(u) = \frac{1}{2}\int (m^2 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi - \Upsilon_\beta(u). \tag{8.2}$$

Lemma 8.1 *Let $s = (n-\beta)/2$. If $C^*c > 1/2$, then $M_{c,\beta,m}^{(n)} = -\infty$.*

Proof. By Theorem 1.8, there exists $\phi \in (H^s)^L$ with $\|\phi\|_2^2 = c$ satisfying

$$\Upsilon_\beta(\phi) = C^* c \|\phi\|_{\dot{H}^s}^2.$$

It follows that

$$\begin{aligned} I_{c,\beta,m}^{(n)}(\phi_\lambda) &= \frac{1}{2}\int (m^2 + |\lambda\xi|^2)^s |\widehat{\phi}(\xi)|^2 d\xi - \lambda^{n-\beta}\Upsilon_\beta(\phi) \\ &= \frac{1}{2}\int (m^2 + |\lambda\xi|^2)^s |\widehat{\phi}(\xi)|^2 d\xi - \lambda^{n-\beta} C^* c \|\phi\|_{\dot{H}^s}^2. \end{aligned} \tag{8.3}$$

If $s \leqslant 1$, then

$$I_{c,\beta,m}^{(n)}(\phi_\lambda) \leq \frac{1}{2} m^{2s} + \lambda^{n-\beta}\left(\frac{1}{2} - C^* c\right)\|\phi\|_{\dot{H}^s}^2. \tag{8.4}$$

Taking $\lambda \to \infty$, we immediately have $M^{(n)}_{c,\beta,m} = -\infty$.

Next, we consider the case $s > 1$. Denote

$$\mathbb{A} = \{\xi : \ \lambda|\xi| > m/\varepsilon\}.$$

We have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{A}} (m^2 + |\lambda\xi|^2)^s |\widehat{\phi}(\xi)|^2 d\xi &\leqslant \frac{1}{2}\lambda^{2s}(1+\varepsilon^2)^s \int_{\mathbb{A}} |\xi|^{2s}|\widehat{\phi}(\xi)|^2 d\xi \\ &\leqslant \frac{1}{2}\lambda^{2s}(1+\varepsilon^2)^s \|\phi\|^2_{\dot{H}^s}. \end{aligned} \tag{8.5}$$

On the other hand,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n\setminus\mathbb{A}} (m^2 + |\lambda\xi|^2)^s |\widehat{\phi}(\xi)|^2 d\xi &\leqslant \frac{1}{2}m^{2s}(1+1/\varepsilon^2)^s \int_{\mathbb{R}^n\setminus\mathbb{A}} |\widehat{\phi}(\xi)|^2 d\xi \\ &\leqslant \frac{1}{2}m^{2s}(1+1/\varepsilon^2)^s \|\phi\|^2_2. \end{aligned} \tag{8.6}$$

Collecting the estimates as in (8.3), (8.5) and (8.6), we have

$$I^{(n)}_{c,\beta,m}(\phi_\lambda) \le C_\varepsilon - \lambda^{n-\beta}\left(C^*c - \frac{1}{2}(1+\varepsilon^2)^s\right)\|\phi\|^2_{\dot{H}^s}. \tag{8.7}$$

By taking $\varepsilon > 0$ small enough and $\lambda \to \infty$, we immediately have $M^{(n)}_{c,\beta,m} = -\infty$. $\square$

Lemma 8.2 *Let $s = (n-\beta)/2 \geqslant 1$. If $C^*c \le 1/2$, then $M^{(n)}_{c,\beta,m} = cm^{2s}/2$.*

Proof. If $s \geqslant 1$, then we have

$$(m^2 + |\xi|^2)^s \geqslant m^{2s} + |\xi|^{2s}.$$

It follows that

$$I^{(n)}_{c,\beta,m}(\phi) \geqslant \frac{1}{2}m^{2s}\|\phi\|^2_2 + \frac{1}{2}\|\phi\|^2_{\dot{H}^s} - \Upsilon_\beta(\phi).$$

If $C^*c \le 1/2$ and $\|\phi\|^2_2 = c$, then we have

$$\Upsilon_\beta(\phi) \le C^*\|\phi\|^2_2\|\phi\|^2_{\dot{H}^s} \le \frac{1}{2}\|\phi\|^2_{\dot{H}^s}.$$

Hence, we have $M^{(n)}_{c,\beta,m} \geqslant cm^{2s}/2$.

Now let $\phi \in (H^s)^L$ with $\|\phi\|^2_2 = c$. We have

$$I^{(n)}_{c,\beta,m}(\phi_\lambda) = \frac{1}{2}\int (m^2 + |\lambda\xi|^2)^s |\widehat{\phi}(\xi)|^2 d\xi - \lambda^{n-\beta}\Upsilon_\beta(\phi). \tag{8.8}$$

We denote by [s] the largest integer which is less than or equals to s, $\{s\} = s - [s]$. It suffices to consider the case that s is not an integer. Since

$$(a+b)^s = (a+b)^{[s]}(a+b)^{\{s\}} \leqslant \sum_{j=0}^{[s]} \begin{pmatrix} j \\ [s] \end{pmatrix} a^j b^{(s-j)} + \sum_{j=0}^{[s]} \begin{pmatrix} j \\ [s] \end{pmatrix} a^{j+\{s\}} b^{([s]-j)},$$

we have

$$\begin{aligned}(m^2 + |\lambda\xi|^2)^s &\leqslant m^2 + \sum_{j=0}^{[s]} \begin{pmatrix} j \\ [s] \end{pmatrix} m^{2j} (|\lambda\xi|^2)^{(s-j)} + \sum_{j=0}^{[s]-1} \begin{pmatrix} j \\ [s] \end{pmatrix} m^{2(j+\{s\})} (|\lambda\xi|^2)^{([s]-j)} \\ &:= m^2 + \lambda^{2\{s\}} P(\lambda, m, |\xi|). \end{aligned} \tag{8.9}$$

Noticing that for $\lambda \le 1$, we have $P(\lambda, m, |\xi|) \lesssim 1 + |\xi|^2 s$, which implied that

$$\int \lambda^{2\{s\}} P(\lambda, m, |\xi|) |\widehat{\phi}(\xi)|^2 d\xi \to 0, \quad \lambda \to 0.$$

Hence, we have

$$cm^{2s}/2 \leqslant I^{(n)}_{c,\beta,m}(\phi_\lambda) \leqslant cm^{2s}/2 + O(\lambda^{2\{s\}}), \tag{8.10}$$

which yields $M^{(n)}_{c,\beta,m} = cm^{2s}/2$. □

Lemma 8.3 *Let $s = (n-\beta)/2 > 1$. If $C^* c \le 1/2$, then $M^{(n)}_{c,\beta,m}$ is not achieved.*

Proof. Suppose on the contrary that there exists $u > 0$ satisfying

$$\frac{1}{2} cm^{2s} = I^{(n)}_{c,\beta,m}(u) \geqslant \frac{1}{2} \int (m^2 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi - \frac{1}{2} \|\phi\|^2_{\dot{H}^s}.$$

By the mean value theorem, there exits $\theta(t) \in (0, t)$ such that

$$f(t) := (m^2 + t)^s - t^s - m^{2s} = st \left((m^2 + \theta(t))^{s-1} - \theta(t)^{s-1} \right) > 0$$

for any $t > 0$. It follows that

$$\frac{1}{2} cm^{2s} \geqslant \frac{1}{2} \int (m^2 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi - \frac{1}{2} \|\phi\|^2_{\dot{H}^s} = \frac{1}{2} cm^{2s} + \int f(|\xi|^2) |\widehat{u}(\xi)|^2 d\xi. \tag{8.11}$$

Noticing that $f(|\xi|^2)$ is a continuous functions of $\xi \in \mathbb{R}^n$ and $f(|\xi|^2) > 0$ if $\xi \neq 0$, we immediately have $\int f(|\xi|^2) |\widehat{u}(\xi)|^2 d\xi > 0$. A contraction. □

Up to now, we have shown that for any $s > 1$, $I^{(n)}_{c,\beta,m}(\cdot)$ has no minimizer. In the following we consider the case $0 < s < 1$.

Lemma 8.4 *Let $s = (n-\beta)/2 < 1$. If $C^* c < 1/2$, then $M^{(n)}_{c,\beta,m} \in (0,\ cm^{2s}/2)$.*

Proof. Let us denote $u_R := \mathscr{F}^{-1}\chi_{|\xi|\le R}\mathscr{F}u$. Let $\phi \in (H^s)^L$ with $\|\phi\|_2^2 = c$ satisfy

$$\Upsilon_\beta(\phi) = C^*\|\phi\|_2^2\|\phi\|^2_{\dot H^s}.$$

Then we have some $R > 0$ satisfying

$$\Upsilon_\beta(\phi_R) \geqslant \frac{1}{2}C^*\|\phi\|_2^2\|\phi\|^2_{\dot H^s} \geqslant \frac{1}{2}C^*\|\phi_R\|_2^2\|\phi_R\|^2_{\dot H^s}.$$

Taking $v = \sqrt{c}\phi_R/\|\phi_R\|_2$, we see that $\|v\|_2^2 = c$ and

$$\Upsilon_\beta(v) \geqslant \frac{1}{2}C^*\|v\|_2^2\|v\|^2_{\dot H^s}.$$

Moreover, the above inequality is invariant under the scaling $v \mapsto v_\lambda$, i.e.,

$$\Upsilon_\beta(v_\lambda) \geqslant \frac{1}{2}C^*\|v_\lambda\|_2^2\|v_\lambda\|^2_{\dot H^s} = \frac{a}{2}\|v_\lambda\|^2_{\dot H^s}, \quad a = C^*c.$$

Moreover, we have

$$I^{(n)}_{c,\beta,m}(v_\lambda) \leqslant \frac{1}{2}cm^{2s} + \frac{1}{2}\int_{|\xi|\le R}\left((m^2+|\lambda\xi|^2)^s - a|\lambda\xi|^{2s} - m^{2s}\right)|\widehat{v}(\xi)|^2 d\xi. \qquad (8.12)$$

Using the mean value theorem, for any $t > 0$, we have some $\theta(t) \in (0,t)$ verifying

$$f(t) := (m^2+t)^s - at^s - m^{2s} = st\left((m^2+\theta(t))^{s-1} - a\theta(t)^{s-1}\right).$$

Noticing that $s < 1$, it follows that for $0 < t \ll 1$, one has that

$$(m^2+\theta(t))^{s-1} - a\theta(t)^{s-1} < 0.$$

Hence, taking $\lambda > 0$ such that $\lambda R \ll 1$, we obtain that

$$(m^2+|\lambda\xi|^2)^s - a|\lambda\xi|^{2s} - m^{2s} < 0, \quad \forall\ 0 < |\xi| \le R.$$

Since $\xi \mapsto f(|\xi|^2)$ is continuous and $v \neq 0$, we immediately have $I^{(n)}_{c,\beta,m}(v_\lambda) < cm^{2s}/2$. Due to $C^*c < 1/2$, we easily see that

$$I^{(n)}_{c,\beta,m}(\phi) > (1/2 - C^*c)\|\phi\|^2_{\dot H^s} > 0.$$

The result follows. □

Lemma 8.5 *Let $s = (n-\beta)/2 < 1$. If $C^*c = 1/2$, then $M^{(n)}_{c,\beta,m} = 0$.*

Proof. Clearly, we have $M^{(n)}_{c,\beta,m} \geqslant 0$. Let us recall that for any minimizer ϕ of the functional $I^{(n)}_{c,\beta}(\cdot)$, we have for any $\varepsilon > 0$,

$$\begin{aligned} I^{(n)}_{c,\beta,m}(\phi_\lambda) &= \frac{1}{2}\int ((m^2+|\lambda\xi|^2)^s - |\lambda\xi|^{2s})|\widehat{\phi}(\xi)|^2 d\xi \\ &= \frac{1}{2}\left(\int_{|\lambda\xi|>m/\varepsilon} + \int_{|\lambda\xi|\le m/\varepsilon}\right)((m^2+|\lambda\xi|^2)^s - |\lambda\xi|^{2s})|\widehat{\phi}(\xi)|^2 d\xi \\ &:= I + II. \end{aligned} \tag{8.13}$$

We estimate I. We may assume that $m/\varepsilon \gg 1$. Recall that

$$(m^2+|\lambda\xi|^2)^s - |\lambda\xi|^{2s} = |\lambda\xi|^{2s}\left(\left(1+\frac{m^2}{|\lambda\xi|^2}\right)^s - 1\right) < sm^2\frac{|\lambda\xi|^{2s}}{|\lambda\xi|^2} \le s\, m^{2s}\varepsilon^{2(1-s)}.$$

It follows that

$$I \le cs\, m^{2s}\varepsilon^{2(1-s)}.$$

On the other hand, due to $(a+b)^s \le a^s + b^s$ and $\phi \in L^2$,

$$II \le \frac{1}{2}m^{2s}\int_{|\xi|\le m/\lambda\varepsilon} |\widehat{\phi}(\xi)|^2 d\xi \to 0, \quad \lambda \to \infty.$$

Hence, $I^{(n)}_{c,\beta,m}(\phi_\lambda) \to 0$ as $\lambda \to \infty$. □

Lemma 8.6 *Let $s = (n-\beta)/2 = 1$. Then $M^{(n)}_{c,\beta,m}$ is achieved if and only if $C^*c = 1/2$.*

Proof. Noticing that for $s = 1$

$$I^{(n)}_{c,\beta,m}(\phi) = \frac{1}{2}m^2c + \frac{1}{2}\int |\xi|^2|\widehat{\phi}(\xi)|^2 d\xi - \Upsilon_\beta(\phi) = \frac{1}{2}m^2c + I^{(n)}_{c,\beta}(\phi), \tag{8.14}$$

we can obtain the result, as desired. □

By Lemma 8.6 and Theorem 1.8, we can prove Theorem 1.9 in the case $s = 1$.

Proof of Theorem 1.9. In view of the discussions above, it suffices to consider the case $0 < s < 1$. Now let u_k be a minimizing sequence. By Lemma 8.4, we see that u_k is bounded in $(H^s)^L$. Following the proof as in Theorem 1.8, we can assume that u_k is radial and radially decreasing. We have

$$M^{(n)}_{c,\beta,m} \le I^{(n)}_{c,\beta,m}(u_k) \to M^{(n)}_{c,\beta,m}.$$

Now we claim that $\inf\{\Upsilon_\beta(u_k):\ k \geqslant 0\} \geqslant c_0$ for some $c_0 > 0$. If not, then we have $\Upsilon_\beta(u_k) \to 0$ up to a subsequence. By Lemma 8.4,

$$\frac{1}{2}m^{2s}c \le \lim_{k\to\infty}\frac{1}{2}\|(m^2+|\xi|^2)^{s/2}\widehat{u}_k\|_2^2 = \lim_{k\to\infty} I^{(n)}_{c,\beta,m}(u_k) = M^{(n)}_{c,\beta,m} < \frac{1}{2}m^{2s}c.$$

This is a contradiction.

Now we can repeat the same procedure as in the proof of Theorem 1.8 to show that $u_k \to u \geqslant 0$ and $u \neq 0$, with a minimizer u, as desired.

Finally, we show the necessity of $C^*c < 1/2$. If not, then $C^*c \geqslant 1/2$. If $C^*c > 1/2$, be Lemma 8.1 we have $M^{(n)}_{c,\beta,m} = -\infty$. If $C^*c = 1/2$, in view of Lemma 8.5 we have $M^{(n)}_{c,\beta,m} = 0$. If $u \neq 0$ is a minimizer, then $I^{(n)}_{c,\beta,m}(u) = 0$. On the other hand, from the definition of $I^{(n)}_{c,\beta,m}(\cdot)$ we have $I^{(n)}_{c,\beta,m}(u) > 0$. A contradiction. $\square$

A Proof of Theorem 5.1

The proof of Theorem 5.1 is essentially known and we now sketch its proof by following [61], Section 2.4 (see also [35] in 3D).

Proposition A.1 *Let $H(t) = e^{t\Delta}$, $\mathscr{A} f = \int_0^t H(t-s)f(s)ds$. We have*

$$\|H(t)u_0\|_{L^{n+2}_{x,t\in[0,T]}} \lesssim \|u_0\|_n, \tag{A.1}$$

$$\|H(t)u_0\|_{L^\infty(0,T;\ L^n)} \lesssim \|u_0\|_n, \tag{A.2}$$

$$\|\nabla \mathscr{A} f\|_{L^{n+2}_{x,t\in[0,T]}} \lesssim \|f\|_{L^{(n+2)/2}_{x,t\in[0,T]}}, \tag{A.3}$$

$$\|\nabla \mathscr{A} f\|_{L^\infty(0,T;\ L^n)} \lesssim \|f\|_{L^{(2+n)/2}_{x,t\in[0,T]}}. \tag{A.4}$$

Put

$$\mathfrak{D} = \left\{ u : \|u\|_{L^{2+n}_{x,t\in[0,T]}} \leq \delta,\ \|u\|_{L^\infty([0,T];L^n)} \leq 2C\|u_0\|_n \right\}, \tag{A.5}$$

$$d(u,v) = \|u-v\|_{L^{2+n}_{x,t\in[0,T]}}. \tag{A.6}$$

We consider the mapping:

$$\mathfrak{M} : u(t) \to H(t)u_0 + \mathscr{A}\mathbb{P}\,\mathrm{div}\,(u \otimes u), \tag{A.7}$$

where

$$\mathbb{P} = I + (-\Delta)^{-1}\nabla \mathrm{div}. \tag{A.8}$$

By Proposition A.1, we have

$$\begin{aligned} \|\mathfrak{M}u\|_{L^{2+n}_{x,t\in[0,T]}} &\lesssim \|H(t)u_0\|_{L^{2+n}_{x,t\in[0,T]}} + \|u\otimes u\|_{L^{(2+n)/2}_{x,t\in[0,T]}} \\ &\lesssim \|H(t)u_0\|_{L^{2+n}_{x,t\in[0,T]}} + \delta^2, \end{aligned} \tag{A.9}$$

$$\begin{aligned}\|\mathfrak{M}u\|_{L^\infty(0,T;\ L^n)} &\lesssim \|u_0\|_n + \|u\otimes u\|_{L^{(2+n)/2}_{x,t\in[0,T]}} \\ &\lesssim \|u_0\|_n + \delta^2. \end{aligned} \tag{A.10}$$

If $C\delta \le 1/4$, we can show that $\mathfrak{M}$ is a contraction mapping from $\mathfrak{D}$ into itself. So, there exists a u satisfying

$$u(t) = H(t)u_0 + \mathscr{A}\mathbb{P}\nabla\cdot(u\otimes u). \tag{A.11}$$

By a standard argument, we see that u is unique in $L^{2+n}(0,T;\ L^{2+n})$. Moreover, one can extend the solution step by step and find a maximal T_m such that $u \in C([0,T_m);\ L^n)\cap L^{2+n}_{\rm loc}(0,T_m;\ L^{2+n})$. In the following we show that

$$\|u\|_{L^{2+n}(0,T_m;\ L^{2+n})} = \infty.$$

Assume for a contrary that $\|u\|_{L^{2+n}(0,T_m;\ L^{2+n})} < \infty$. In view of the first inequality in (A.10) we see that

$$\|u\|_{C([0,T_m);\ L^n)\ \cap\ L^{2+n}(0,T_m;\ L^{2+n})} < \infty.$$

Using the same idea as in [15] for the nonlinear Schrödinger equation, we now extend the solution beyond T_m. We have for $0 < T_m - T \ll 1$,

$$u(t) = H(t-T)u(T) + \int_T^t H(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)d\tau.$$

It follows that

$$\|H(t-T)u(T)\|_{L^{n+2}_{x,t\in(T,T_m)}} \le \|u\|_{L^{n+2}_{x,t\in(T,T_m)}} + \|u\|^2_{L^{n+2}_{x,t\in(T,T_m)}} \to 0, \quad T\to T_m. \tag{A.12}$$

Replacing $[0,T]$ by $[T,T_m]$ and $\|u_0\|_n$ by $\|u(T)\|_n$ in the definition of $(\mathfrak{D}, d)$, we can find that the solution can be extended to $C([T,T_m], L^n)$ if T is sufficiently close to T_m. It follows that the solution exists beyond T_m. A contradiction.

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