ON THE GLOBAL CAUCHY PROBLEM FOR NON-LINEAR SCHRÖDINGER EQUATION WITH MAGNETIC POTENTIAL

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Abstract. We study the global Cauchy problem for the non-linear Schrödinger equation with time dependent magnetic field:

$$2i\partial_t u + \Delta_A u \pm |u|^{p-1}u = 0$$

in $\mathbb{R}^n$.

For power-like non linearities and possibly unbounded magnetic potential we prove global existence and uniqueness of solutions in the energy space. We also provide examples of solutions that blow-up in finite time.

1. Introduction

1.1. **Notation.** Below n is a non-zero integer[1].

For p in $[1,\infty]$ and a measurable function $u \in L^p(\mathbb{R}^n)$, that is such that $|u|^p$ is integrable, $\|u\|_p$ will denote its usual norm in $L^p(\mathbb{R}^n)$. Moreover $\|u\|$ will replace $\|u\|_2$.

The notation $C_0^k(\mathbb{R}^n)$ stands for the set of functions k-times continuously differentiable with compact support and their intersection is $C_0^\infty(\mathbb{R}^n)$.

The notation $H_c^1(\mathbb{R}^n)$ stands for the subset of $H^1(\mathbb{R}^n)$ made of compactly supported functions. Notice that for any bounded open subset Ω of $\mathbb{R}^n$, any extension by 0 of a distribution in $H_0^1(\Omega)$ gives a distribution in $H_c^1(\mathbb{R}^n)$.

The notation $B(x,R)$ stands for the open ball of $\mathbb{R}^n$ centered at x with radius R.

The set of functions from a set E to a set F is denoted by $\mathcal{F}(E,F)$.

1.2. **Settings.** We consider the non-linear Schrödinger equation with magnetic field in $\mathbb{R}^n$

$$i\partial_t u = \Delta_{A(t)}u - \mu^\gamma f(x,u) \tag{mNLS}$$

with initial condition

$$u_{|t=t_0} = \varphi. \tag{1.1}$$

Here the unknown function $u : (-T_*, T^*) \times \mathbb{R}^n \to \mathbb{C}$, the parameters $\mu \in \mathbb{R}$, $\gamma > 0$ and $f : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C}$ is some non-linear measurable function. The

[1] In dimension $n = 1$, the gauge invariance shows that the magnetic laplacian is unitarily equivalent to the free laplacian. The unitary equivalence is given by the conjugation with $e^{i\int^x A}$.

non-autonomous magnetic Laplacian $\Delta_{A(t)}$ is defined by

$$\begin{aligned}\Delta_{A(t)} &= \sum_{j=1}^{n}(i\partial_{x_j} - A_j(t,x))^2,\ t \in \mathbb{R},\ x \in \mathbb{R}^n \\ &= \Delta + 2iA\cdot\nabla + i\nabla\cdot A - |A|^2,\end{aligned}$$

where the real vector magnetic potential $A(t,x)$ is given by

$$A(t,x) = (A_1(t,x),\dots,A_n(t,x)).$$

We sometimes omit the space dependence and write $A(t)$ instead of $A(t,x)$. The magnetic Laplacian appears in many relevant physical phenomena like superconductivity and Coulomb gases. In particular, it is present in the famous Fröhlich polarons subject to the external fields in the limit of large electron phonon coupling [19]. Proving the presence of these polarons is equivalent to the establishment of minimizer of the following constrained variational problem:

$$\inf\left\{\int_{\mathbb{R}^3}|\nabla_A\varphi(x)|^2 + V(x)\varphi^2(x)dx - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\varphi^2(x)\varphi^2(y)}{|x-y|}dxdy:\ \|\varphi\|_{L^2} = 1\right\}$$

where V is a real-valued function and A is a vector potential

$$\nabla_A = -i\nabla + A.$$

Recently, M. Griesmer and D. Wellig [19] have been able to solve the above variational problem, taking advantage of many important previous results obtained by several authors, see [1,2,5,6,7,8] and [19].
Ground state solutions play a crucial role in the study of stable solutions of the associated nonlinear Schrödinger equation (1.2). One of the aims of this paper is to give a complete study of the minimization problem when the nonlocal nonlinearity is replaced by a local one and which is associated to the the magnetic nonlinear Schrödinger equation (mNLS). This is a very rich mathematical problem with still some challenging and interesting open questions.

The study of the Cauchy problem (mNLS), (1.1),(1.2) has a long and rich history especially in the flat case i.e. $A \equiv 0$. However, for arbitrary magnetic potentials, only special nonlinearities f have been treated so far. For example in [26], L. Michel has proved the existence and uniqueness of solutions of the Cauchy problem (mNLS) in the L^2-subcritical case. Moreover, we have also found some flaws in the blow-up result stated in the super-critical L^2 case in [16], while the literature is completely silent about the L^2-critical case. To our knowledge, the existence of ground state solution has been established via the Concentration-Compactness principle and only for particular nonlinearities and constant diamagnetic fields [12].
The goal of this article is to give the whole picture to all the situations described previously. This paper is organized as follows. First, we review some important properties of the diamagnetic potential. In Section 3, we establish the existence and uniqueness of the Cauchy problem. Here the choice of the appropriate function space is crucial as noticed in [26]. In this paper, we handle both the L^2-subcritical case (Section 4.2) and the L^2-critical case which is much more tricky and a much more careful analysis

is required there (Section 4.3). Our main result states that for an initial datum small enough, we have existence and uniqueness of (mNLS). Another stable and novel study is discussed in Section 5, in which we prove global well-posedness and blow-up in the L^2-supercritical case. In the appendix, we provide a concrete example of blow-up in dimension 2. Similar result has been stated in [15] but as we will show that the proof of their main result is erroneous.

2. The linear theory

In this section, we collect a few useful results for the linear flow.

2.1. The diamagnetic inequality.

For a fixed time t, let $A = A(t)$. Assume A is measurable then

$$-\Delta_A := \nabla_{\mathrm{A}}{}^2 \quad \text{where} \quad \nabla_{\mathrm{A}} := (i\nabla - A)$$

and define for $A \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ the subset of distributions :

$$H^1_A(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n), \nabla_{\mathrm{A}}\, u \in L^2(\mathbb{R}^n)\}.$$

The associated norm defines a closed quadratic form and the associated sefl-adjoint operator defines an extension (with form domain $H^1_A(\mathbb{R}^n)$) of $-\Delta_A$, this is the maximal form extension. Lemma 2.1) provides the density of $C_0^\infty(\mathbb{R}^n)$ in $H^1_A(\mathbb{R}^n)$ and thus the former is a form core.

Moreover if $A \in L^4_{loc}(\mathbb{R}^n)$ and $\nabla \cdot A \in L^4_{loc}(\mathbb{R}^n)$ then it is essentially self-adjoint in $C_0^\infty(\mathbb{R}^n)$, see [23].

Below we identify A with the associated 1-form $\sum A^i dx_i$ and we will denote by B the 2-form

$$B = dA = \sum_{j<k} (\partial_j A_k - \partial_k A_j) dx_j \wedge dx_k$$

which we will often identify with the skew-symmetric matrix

$$B = (B_{jk}) \quad B_{jk} = \partial_j A_k - \partial_k A_j.$$

In dimension 2 and 3, we can avoid the form language and define $B = \nabla \wedge A$ so that with respect to the previous notation, and with an abuse of notation,

$$BX = B \wedge X.$$

We have the following

Lemma 2.1 (Diamagnetic inequality). *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. For any $v \in H^1_A(\mathbb{R}^n)$,*

$$\| |v| \|_{\dot{H}^1} \leq \|(i\nabla - A)v\|.$$

Moreover the equality holds if and only if $B = 0$ in some open set Ω, v has support on Ω and for each $x \in \Omega$ there exists $V \subset \Omega$ a neighbourhood of x and $\phi \in \dot{H}^1(V)$ such that on V

$$A = \nabla\phi \quad v = |v|e^{i\phi}.$$

Proof. The following proof is from [3, proof of Lemma 1], see also [24, Theorem 7.21] and [15, Theorem 2.1.1].

Let $v \in H^1_A(\mathbb{R}^n)$ below we will identify it with a representative such that the interior of $\{v = 0\}$ is exactly the complementary set to the support of the associated distribution, this is thus the closure of the biggest open set where the distribution vanishes. To do so it is enough to change the values of v on a negligible set. Thus as $\{v = 0\} = \{|v| = 0\}$, over the interior, in the distributional sense $\nabla|v| = \nabla v = 0$ and $Av = 0$ so that it remains to prove the inequality in the set $\{v \neq 0\}$. This is the interior of the complementary set of $\{v = 0\}$ actually in the interior as in the boundary due to traces of Sobolev functions, this is still 0

1. For any $v \in C_0^\infty(\mathbb{R}^n)$
$$2|v|\partial_{x_j}|v| = \partial_{x_j}|v|^2 = 2\,\mathrm{Im}\,\bar{v} \cdot i\partial_{x_j} v = 2\,\mathrm{Im}\,\bar{v}(i\partial_{x_j} - A_j)v$$
and so multiply by a function $w : \mathbb{R}^n \to \mathbb{R}^n$
$$2|v|w \cdot \nabla|v| = 2\,\mathrm{Im}\,\bar{v}w \cdot (i\nabla - A)v.$$
Thus as $|Imz| \leq |z|$ this leads on $\{v \neq 0\}$ to
$$|w \cdot \nabla|v|| \leq |w \cdot (i\nabla - A)v|\,.$$
2. Let $w = \nabla|v|$ then
$$|\nabla|v||^2 \leq |\nabla|v||\,|(i\nabla - A)v|\,.$$
and thus using the Cauchy-Schwarz inequality
$$|\nabla|v|| \leq |(i\nabla - A)v|\,. \tag{2.1}$$
3. The equality holds for $v \in H^1_A(\mathbb{R}^n)$ if at step of the previous analysis the equality holds that is
$$\mathrm{Re}\,\bar{v}w \cdot (i\nabla - A)v = 0,\ w = \nabla|v|,\ \exists\mu \in \mathcal{F}(\mathbb{R}^n, \mathbb{C}),\ \nabla|v| = \mu(i\nabla - A)v.$$
Notice that from the equality case in (2.1), μ has modulus 1.

We can write
$$\mathrm{Re}\,\bar{\mu}\bar{v}|(i\nabla - A)v|^2 = 0$$
so that almost everywhere on $\{v \neq 0\}$
$$\mu = \pm i\frac{\bar{v}}{|v|}$$
and almost everywhere on $\{v \neq 0\}$
$$\nabla|v| = \mp\frac{\bar{v}}{|v|}(\nabla + iA)v.$$
and thus almost everywhere on $\{v \neq 0\}$ by equating real and imaginary parts
$$\nabla|v| = \mp\,\mathrm{Re}\,\frac{\bar{v}}{|v|}\nabla v$$
$$0 = \mp\,\mathrm{Im}\,\frac{\bar{v}}{|v|}\nabla v \mp A|v|.$$

and thus up to changing v to $e^{i\pi/2}v$ in the $-$ case almost everywhere on $\{v \neq 0\}$

$$\nabla_{\mathrm{A}} |v| = \frac{\bar{v}}{|v|}\nabla v$$

giving almost everywhere on $\{v \neq 0\}$

$$A = i\frac{v}{|v|}(\nabla \frac{\bar{v}}{|v|}).$$

As

$$\nabla \frac{\bar{v}}{|v|} = -i\frac{\bar{v}}{|v|}A$$

we obtain $\frac{v}{|v|} \in L^\infty(\{v \neq 0\}) \cap \dot{H}^1_{loc}(\{v \neq 0\})$.

Then as $\frac{\bar{v}}{|v|} = \left(\frac{v}{|v|}\right)^{-1}$, we have in the distributional sense

$$dA = 0$$

in the distributional sense. Notice that locally A will be a gradient, the gradient of the phase of v. Indeed, for some x_0 in the support of v, we define

$$\beta(x) = \int_{\mathbb{R}^n}\int_0^1 A(a + t(x-a)) \cdot (x-a)\phi(a)\, dt\, da$$

where ϕ is some non negative smooth function with compact support in a ball around x_0 and integral mean equal to 1. The function β is continuous and thus locally square integrable. We have

$$\beta(x) = \int_0^1 \int_{\mathbb{R}^n} A(b) \cdot (x-b)\phi(x + \frac{1}{1-t}(b-x))\,(1-t)^{-n-1} db\, dt$$

with the change of variable $b = a + t(x-a)$. The integrand

$$u(t,x) := \int_{\mathbb{R}^n} A(b) \cdot (x-b)\phi(x + \frac{1}{1-t}(b-x))\,(1-t)^{-n-1} db$$

appears as a smooth function for $t \neq 1$ and

$$\begin{aligned}\partial_x^\alpha u(t,x) &= \langle t^{|\alpha|}\partial_x^\alpha A(\cdot + t(x-\cdot)) \cdot (x-\cdot), \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} \\ &\quad + \sum_{i=1,\alpha_i\neq 0}^{n} \langle t^{|\alpha|-1}\partial_x^{\alpha-1_i} A_i(\cdot + t(x-\cdot)), \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)}\end{aligned}$$

where 1_i stands for the multi-index which is 0 everywhere but at the index i which is 1.

In the other hand if

$$v_i(t,x) = \int_{\mathbb{R}^n} A_i(a + t(x-a))\phi(a)\, da$$

we have that v_i is continuous and apart from $t = 1$, v_i is smooth and

$$\partial_t v_i(t,x) = \sum_{k=1}^{n} \langle t\partial_k A_i(\cdot + t(x-\cdot))(x_k - \cdot), \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)}$$

Its the gradient is given by

$$\begin{aligned}\partial_i u(t,x) &= \sum_{k=1}^{n} \langle t\partial_i A_k(\cdot + t(x-\cdot))(x_k - \cdot), \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} \\ &+ \langle A_i(\cdot + t(x-\cdot)), \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} \\ &= \sum_{k=1}^{n} \langle t\partial_k A_i(\cdot + t(x-\cdot))(x_k - \cdot), \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} \\ &+ \langle A_i(\cdot + t(x-\cdot)), \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} \\ &= t\partial_t v_i(t,x) + \int_{\mathbb{R}^n} A_i(a + t(x-a))\phi(a)\, da\end{aligned}$$

Integrating over the interval $[0, s]$ and taking the limit, we obtain that the distributional derivative of β satisfies after an integration by parts

$$\partial_i \beta(x) = \lim_{s\to 1} s v_i(s,x) - \lim_{s\to 1} \int_0^s v_i(t,x)\, dt + \int_0^1 \int_{\mathbb{R}^n} A_i(a + t(x-a))\phi(a)\, da$$

that is

$$\partial_i \beta(x) = A_i(x).$$

4. The converse implication is a verification. □

If $A \in L^n_{loc}(\mathbb{R}^n)$ (or $A \in L^\alpha_{loc}(\mathbb{R}^2)$ with $\alpha > 2$), following [12] we have the density of $C_0^\infty(\mathbb{R}^n)$ in $H^1_A(\mathbb{R}^n)$ we thus have the following immediate corollary.

Lemma 2.2. *Assume that $A \in L^n_{loc}(\mathbb{R}^n)$ (or $A \in L^\alpha_{loc}(\mathbb{R}^2)$ with $\alpha > 2$) then $H^1_A(\mathbb{R}^n)$ is embedded in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Moreover for any bounded open subset Ω of $\mathbb{R}^n$ and any $u \in H^1_c(\Omega)$

$$\|\nabla u\| \le (1 + \|A\|_{L^n(\Omega)})\|\nabla_A u\|.$$

Proof. First notice that from the diamagnetic inequality and Sobolev embedding one deduces

$$u \in H^1_A(\mathbb{R}^n) \Rightarrow |u| \in L^{2^*}(\mathbb{R}^n) \text{ and } \|u\|_{2^*} \le C\|\nabla_A u\|$$

where C is the critical Sobolev constant.

Then if $u \in H^1_A(\mathbb{R}^n)$ and θ is a smooth compactly supported function

$$\|\nabla u\| \le \|\theta\, \nabla_{\mathrm{A}}\, u\| + \|(\nabla\theta)u\| + \|(\theta A)u\|$$

and the lemma follows from Hölder inequality making a crucial use of the diamagnetic inequality. □

Remark 2.3. *This lemma is valid on any set on which $|A|^n$ is integrable.*

Notice that the Lemma fails for $A = \nabla\phi$ where ϕ is some real function with distributional derivative not in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$. Indeed for some $u \in H^1(\mathbb{R}^n)$; $e^{i\phi}u$ is not in $H^1(\mathbb{R}^n)$ while it is in $H^1_A(\mathbb{R}^n)$.

2.2. **Gauge.** To allow the existence of stationary solutions below we assume that f enjoys the following gauge invariance :

$$\forall x \in \mathbb{R}^n,\ \forall z \in \mathbb{C}, f(x,z) = (z/|z|)f(x,|z|).$$

Let ϕ be a square integrable solution of (mNLS) and $g : \mathbb{R}^4 \mapsto \mathbb{R}$ a smooth function then $\psi = e^{-ig}\phi$ is a solution of

$$2i(\partial_t + i\partial_t g(t,x))\psi + \Delta_{A+\nabla g}\psi - \mu^\gamma f(x,\psi) = 0.$$

As a simple consequence the phase invariance is clear : if ϕ is a square integrable solution of (mNLS) then for any real η : $e^{-i\eta}\psi$ is also a solution.

This kind of remarks allows the inclusion of an electric field in the following way: If ϕ is a solution of

$$2i\partial_t\phi + \Delta_{A(t)}\phi + V(t)\phi - \mu^\gamma f(x,\psi) = 0.$$

with V smooth then for $g(t,x) = \frac{1}{2}\int_0^t V(s,x)\,ds$, the function $\psi = e^{-ig}\phi$ is a solution of

$$2i\partial_t\psi + \Delta_{A+\nabla g}\psi - \mu^\gamma f(x,\psi) = 0.$$

The magnetic field $B = dA = d(A + \nabla g)$ is gauge invariant. On the other hand we can impose a condition which determines A. Instead of the Coulomb Gauge, $\nabla\cdot A = 0$, we follow [25] and consider the transversal gauge (also known as the line gauge, point gauge, multipolar gauge or Poincaré gauge) :

$$A(t,x)\cdot x = 0, \quad \forall x \in \mathbb{R}^n$$

so that if $B(t,x) = o(|x|^{2-n})$ as $x \to 0$ for $t \in \mathbb{R}$

$$A(t,x) = \frac{1}{n-2}\int_0^1 s^{n-2}B(t,sx)x\,ds.$$

Before going any further, some comments are in place. In the dispersive literature, the quantity

$$B_\tau := B \wedge \frac{x}{|x|}$$

was first identified by Fanelli and Vega [14, 13] to be the main obstruction to dispersion is known after [11] as the trapping component. This is sometimes assumed to be small or fast decaying [4, 5, 16, 10].

Notice that in any case, the gauge does not determine uniquely the magnetic field. For instance in the transversal gauge, the addition of the gradient of a radially constant function does not change the gauge or the potential (one has to allow a singularity, though). In the Coulomb gauge the addition of the gradient of a harmonic function does not change the gauge or the magnetic potential either.

2.3. **The time dependent case.** We suppose that the magnetic potential is a smooth function $\partial_x^\alpha A \in C^1(\mathbb{R}_t \times \mathbb{R}_x^n, \mathbb{R}^n)$ for any $\alpha \in \mathbb{N}^n$ and that it satisfies the following assumption.

Assumption 1. *There exists a map $\alpha \in \mathbb{N}^n \mapsto C_\alpha \in [0,+\infty)$ such that*

(1) $\forall|\alpha| \geq 1, \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} |\partial_x^\alpha A| + \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} |\partial_x^\alpha \partial_t A| \leq C_\alpha.$

(2) $\exists\epsilon > 0, \forall|\alpha| \geq 1, \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} |\partial_x^\alpha B| \leq C_\alpha\langle x\rangle^{-1-\epsilon}.$

Recall that $B(t,x)$ is the matrix defined by $B_{jk} = \partial_{x_j} A_k - \partial_{x_k} A_j$.

Remark 2.4. *These assumptions are introduced in order to apply* [32, Theorem 1] *which provides local in time Strichartz estimates.*

Any smooth compactly supported perturbation of linear (with respect to x) magnetic potentials satisfies the above hypothesis.

Notice that as long as one is interested in local in time results the above assumptions can be restricted to finite time interval. The results below will hold locally in time just by considering a smooth time compactly supported extension of B and A.

Lemma 2.5. *Let $\beta \in \mathbb{N}$. Under Assumption 1, the domain $D(\Delta^{\beta}_{A(t)}) = \{u \in L^2(\mathbb{R}^n_x), (-\Delta_{A(t)})^{\beta} u \in L^2(\mathbb{R}^n_x)\}$ does not depend on t.*

We define the magnetic Sobolev norm by

$$\|u\|_{H^{\beta}_{mg}} = \|u\|_{H^{\beta}_{A(t_0)}}.$$

Proof. For $t, t' \in \mathbb{R}$ one has

$$\Delta_{A(t')} = \Delta_{A(t)} + W(t,t')(i\nabla_x - A(t)) + (i\nabla_x - A(t))W(t,t') + W(t,t')^2 \tag{2.2}$$

with $x \mapsto W(t,t',x) = \int_t^{t'} \partial_s A(s,x) ds$ bounded as well as its x-derivatives uniformly with respect to t, t' in any compact set. In fact, the above identity shows that the space

$$H^{\beta}_{mg}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n), (1 + \Delta_{A(t)})^{\beta/2} u \in L^2(\mathbb{R}^n)\}$$

does not depend on $t \in \mathbb{R}$. As $D(\Delta_{A(t)}) = H^2_{mg}(\mathbb{R}^n)$, the above statement is straightforward. Moreover, the natural norms on this space are equivalent and this equivalence is controlled by the difference $|t - t'|$. More precisely, it is proven in [26] that, denoting $m_A = \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^n} |\partial_t A(t,x)|$, for all $u \in H^{\beta}_{mg}$ we have

$$\|(\Delta_{A(t')} + 1)^{\beta} u\|_{L^2} \leq (1 + 2m_A|t - t'| + m_A^2 |t - t'|^2)^{\beta} \|(\Delta_{A(t)} + 1)^{\beta} u\|_{L^2}.$$

For $\beta \in \mathbb{N}$ we set

$$\|u\|_{H^{\beta}_{A(t)}} = \|(i\nabla_x - bA(t))^{\beta} u\|_{L^2} + \|u\|_{L^2}.$$

This norm is clearly equivalent (uniformly with respect to b) to $\|(1 + \Delta^{\beta/2}_{A(t)} u\|_{L^2}$. □

Under Assumption 1 it is well-known (see [28, Theorem 4.6, p143], [32] or [33]) that

Proposition 2.6. *For $\varphi \in H^1_{mg}$, the linear Schrödinger equation*

$$i\partial_t u = \Delta_{A(t)} u, \quad u_{|t=s} = \varphi \tag{2.3}$$

has a solution $U_0(t,s)\varphi$. The operator $U_0(t,s)$ maps H^1_{mg} into itself, is continuous from L^2 into L^2 and from H^1_{mg} into H^1_{mg}. Moreover, $U_0(t,s)\varphi$ is the unique H^1_{mg} valued solution of (2.3) *and $U_0(t,s)$ is unitary.*

Remark 2.7. *In the above cited works the time dependence is at least absolute continuous. One can also refer to* [21] *for bounded variation time dependence.*

Actually Strichartz estimates are proved in [32] for the linear problem.

Theorem 1. [32, Theorem 1] *Let I be a finite real interval, (q,r) and $(\gamma_j,\rho_j), j=1,2$ be such that $r,\rho_j \in [2,\frac{2n}{n-2}[$, $\frac{2}{q}=n(\frac{1}{2}-\frac{1}{r})$ and $\frac{2}{\gamma_j}=n(\frac{1}{2}-\frac{1}{\rho_j})$. Let $g_j \in L^{\gamma_j'}(I,L^{\rho_j'}(\mathbb{R}^n_x)), j=1,2$, where γ_j',ρ_j' are the conjugate exponentsF of γ_j,ρ_j. Then the solution u to*

$$i\partial_t u = \Delta_{A(t)}u + g_1(t) + g_2(t)$$

with initial condition

$$u_{|t=t_0} = \varphi.$$

satisfies

$$\|u\|_{L^q(I,L^r(\mathbb{R}^n_x))} \le C(\|g_1\|_{L^{\gamma_1'}(I,L^{\rho_1'}(\mathbb{R}^n_x))} + \|g_2\|_{L^{\gamma_2'}(I,L^{\rho_2'}(\mathbb{R}^n_x))} + \|\varphi\|_{L^2(\mathbb{R}^n)})$$

where the constant C depends only on the length of I and the constant C_α of Assumption 1.

Remark 2.8. *Notice that the endpoint $(1,\frac{2n}{n-2})$ are not included here.*

2.4. **Energy.** Let us introduce the energy functional associated to these non-linearities. We define $F(x,\cdot)$ as the antiderivative (in u) of f that vanishes at 0

$$F(x,z) = \int_0^{|z|} f(x,s)ds, \; G(u) = \int_{\mathbb{R}^n} F(x,u(x))dx$$

and for $t \in \mathbb{R}$ and $u \in H^1_{mg}$ we define the energy

$$E_A(t,u) = \int_{\mathbb{R}^n} \frac{1}{2}|(i\nabla_x - A(t,x))u(x)|^2 dx - \mu^\gamma G(u).$$

If A is constant in time, this is a conserved quantity.

2.5. **Momentum conservation law.** Beside the charge (or mass) and in the time-independent case the energy we have another set of conserved quantities in the time-independent case again. Following [10], we consider the momentum

$$I_j(\psi) := \operatorname{Im} \int \overline{u}(\partial_j - iA_j)u$$

2.6. **The spectrum.** In the time independent linear case the spectrum may depend dramatically on the dimension, let us cite a result of [1] for the constant magnetic field case.

Theorem 2 ([1, Proposition 2.5])**.** *Assume B is constant*

$N=3n$ *For n three-dimensional particles, $n\in\mathbb{N}$, the spectrum of $-\Delta_A$ is absolutely continuous and given by $[\frac{N}{3}|B|,+\infty]$.*

$N=2n$ *For n two-dimensional particles, $n\in\mathbb{N}$, the spectrum of $-\Delta_A$ purely point-wise and accumulates as infinity, each eigenvalue is infinite dimensional, the smallest being $\frac{N}{2}|B|$.*

We rephrased the assertion in the case of identical particles. A rescaling allows to restore the original statement of [2]. In any dimension other than $3n$ or $2n$ the spectrum can be obtained by means of tensor sums and products. For instance from [31, Theorem VII.33] we can deduce that the spectrum is an infinite positive interval as long as N is odd.

We emphasize that in even dimension the previous result will later imply the existence of small solitary waves so that classical scattering results cannot hold in the homogeneous field case.

Note that the degeneracy can be obtained from the invariance with respect to $e^{iA(x_0)\cdot x}u(x+x_0)$.

In dimension 3, the operator is the tensor sum of the two-dimensional case with a one-dimensional free Shrödinger operator. A partial Fourier transform with respect to this variable provides the spectrum.

3. On the minimizers of a diamagnetic Gagliardo-Nirenberg inequality

Our aim in this section is to generalize to magnetic fields the classical Gagliardo-Nirenberg inequality

Recall that for any $u \in H^1(\mathbb{R}^n)$, we have

$$\|u\|_p^p \le C_{p,n}^p \|\nabla u\|_2^{\frac{(p-2)}{2}n} \|u\|_2^{2+\frac{(p-2)}{2}(2-n)} \tag{GN}$$

for $2 < p < \frac{2n}{n-2}$ with equality at the minimal L^2-norm solutions of

$$\frac{(p-2)n}{4}\Delta Q - \left(1 + \frac{(p-2)}{4}(2-n)\right) Q + Q^{p-1} = 0$$

and $C_{p,n} = \left(\frac{p+1}{2\|Q\|_2^{p-1}}\right)^{\frac{1}{p}}$.

Inequality (GN) is obtained by interpolation between the critical Sobolev inequality and the L^2-norm.

Proposition 3.1. *Let $A \in L^n_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. Then for any $u \in \mathcal{D}(\Delta_A)$, the following estimate holds*

$$\|u\|_p^p \le C_{p,n}^p \| \nabla_{\mathrm{A}}\, u\|_2^{\frac{(p-2)}{2}n} \|u\|_2^{2+\frac{(p-2)}{2}(2-n)} \tag{3.1}$$

with a best constant $C_{p,n}$ defined in (GN)*. Moreover the equality is achieved if and only if* $\operatorname{curl} A = 0$.

Proof. First, it is clear from Lemma 2.1 that $C_{p,n} \ge C_A$. the inequality follows from (GN). Then, as noticed in [12, Theorem 3.7] by rescaling

$$\|u\|_p^p \le C_{p,n}^p \|\nabla_{A^\sigma} u\|_2^{\frac{(p-2)}{2}n} \|u\|_2^{2+\frac{(p-2)}{2}(2-n)}$$

where $A^\sigma(x) = \sigma A(\sigma x)$. As for any u smooth with compact support we have

$$\|\nabla_{A^\sigma} u\|_2 \le \|\nabla u\|_2 + \|A^\sigma\|_n \|\nabla u\|_2,$$

which after taking limit yields

$$\limsup_{\sigma\to 0} \|\nabla_{A^\sigma} u\|_2 \le \|\nabla u\|_2,$$

and therefore $C_{p,n}^p$ is the best constant. Finally, clearly the inequality cannot be achieved unless $\operatorname{curl} A = 0$ is a gradient as if is there is some w with $\|w\|_p = 1$ such that

$$\begin{aligned} 1 &= C_{p,n}^p \| \nabla_{\mathrm{A}} w\|_2^{\frac{(p-2)}{2}n} \|w\|_2^{2+\frac{(p-2)}{2}(2-n)} \\ &\geq C_{p,n}^p \|\nabla |w|\|_2^{\frac{(p-2)}{2}n} \|w\|_2^{2+\frac{(p-2)}{2}(2-n)} \geq \|w\|_p = 1 \end{aligned}$$

so that

$$\| \nabla_{\mathrm{A}} w\|_2 = \|\nabla |w|\|_2$$

and thus from the equality case in Lemma 2.1, we deduce that $\operatorname{curl} A = 0$. □

4. Wellposedness in the L^2-critical/subcritical case

4.1. The local wellposedness in $H^1_{mg}(\mathbb{R}^n)$. Let us set our assumption on f.

Assumption 2. *(1) $f(x,0) = 0$ almost every where.*

(2) $\exists M \geq 0, \alpha \in [0, \frac{4}{n-2}[$ ($\alpha \in [0,\infty[$ if $n = 1,2$) such that

$$|f(x,z_1) - f(x,z_2)[\leq M(1 + |z_1|^\alpha + |z_2|^\alpha)|z_1 - z_2|$$

for almost all $x \in \mathbb{R}^n$ and all $z_1, z_2 \in \mathbb{C}$.

(3) $\forall z \in \mathbb{C}, f(x,z) = (z/|z|)f(x,|z|)$.

As mention in the introduction our analysis is an addendum to [26] where the author studied the local Cauchy problem. In [26] the following theorem is proved.

Theorem 3 ([26, Theorem 1])**.** *Suppose that Assumptions 1 and 2 are satisfied and let $\varphi \in H^1_{mg}$. Then, there exists $T_b, T^b > 0$ and a unique $u \in C(] - T_b, T^b[, H^1_{mg}) \cap C^1(] - T_b, T^b[, H^{-1}_{mg})$ solution of* (mNLS)*. Moreover, either $T_b = \infty$ (resp. $T^b = \infty$), or $\lim_{t\to -T_b} \|u(t)\|_{H^1_{mg}} = \infty$ (resp. $\lim_{t\to T^b} \|u(t)\|_{H^1_{mg}} = \infty$) and*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2},$$

$$E_A(b,t,u) = E_A(b,0,\varphi) - \mathrm{Re} \int_0^t \langle \partial_s A(s)u(x), (i\nabla - A(s))u(s)\rangle_{L^2} ds,$$

for all $t \in] - T_b, T^b[$. Additionally, there exists $\epsilon > 0$ such that, for all $b > 0$ and $\varphi \in H^1_{mg}$ such that $\|\varphi\|_{H^1_{mg}} \leq Cb$, we have $T_b, T^b \geq \epsilon b^{-\delta}$ with $\delta = \max(1, 2\gamma, \frac{2\gamma}{\alpha})$.

Unlike [26], the proof is based on the Strichartz estimates for magnetic Schrödinger operator proved in [32], following the strategy of [7] or [8].

4.2. **The global wellposedness in the L^2-subcritical case.** Here we analyze the global well-posedness of (mNLS). Notice that from the assumption we have

$$|f(x,z)[\leq M_f(1+|z|^\alpha)|z|$$

where we introduced

$$M_f = \sup_{r>0} \frac{|f(x,r)|}{(1+r^\alpha)r}, \quad \text{and,} \quad g(t) = \sup_{x\in\mathbb{R}^n} |\partial_t A(t,x)|.$$

Concerning the L^2-subcritical case we first introduce the function

$$g(t) = \sup_{x\in\mathbb{R}^d} |\partial_t A(t,x)|.$$

Then we have have the following proposition.

Proposition 4.1. *Suppose that $0<\alpha<\frac{4}{n}$ and g is in belongs to $L^\infty(\mathbb{R}_t)$. Then, for any initial condition $u_0 \in H^1_{mg}$, equation* (mNLS) *has a unique solution u in $C^0(\mathbb{R}, H^1_{mg}) \cap C^1(\mathbb{R}, L^2)$.*

Proof. The proof is based on (3.1).We assume by contradiction that T^* is finite. Then, the solution u enjoys $\lim_{t\to T^*} \|u\|_{H^1_{mg}} = +\infty$. Moreover, its L^2 norm is constant

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2} := d_0$$

and u enjoys the energy evolution law:

$$|E_A(t,u)| \leq E_0 + \int_0^t |\partial_s A| \|u\|_{L^2} \|\nabla_{A(s)} u\|_{L^2} ds,$$

where we set $E_0 := E_A(t=0)$. Since $E_A(t,u(t)) = \frac{1}{2}\|\nabla_{A(t)} u\|^2_{L^2} - G(u)$ it follows that

$$\frac{1}{2}\|\nabla_{A(t)} u\|^2_{L^2} \leq E_0 + G(u) + \int_0^t |\partial_s A| \|u\|_{L^2} \|\nabla_{A(s)} u\|_{L^2} ds \tag{4.1}$$

On the other hand, it follows from Hölder inequality and the above lemma that for $p=\alpha+2<\frac{2n}{n-2}$,

$$\begin{aligned} G(u) &\leq |b|^\gamma M_f \left(\frac{1}{2}\|u\|_2^2 + \frac{1}{p} C^p_{p,n} \|u\|_{H^1_{mg}}^{n\frac{(p-2)}{2}} \|u\|_2^{2+(2-n)\frac{(p-2)}{2}} \right) \\ &\leq |b|^\gamma M_f \left(\frac{1}{2}d_0^2 + \frac{1}{p} C^p_{p,n} \|u\|_{H^1_{mg}}^{n\frac{(p-2)}{2}} d_0^{2+(2-n)\frac{(p-2)}{2}} \right) \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we get for $g(t) = \sup_{x\in\mathbb{R}^d} |\partial_t A(t,x)|$

$$\begin{aligned} \frac{1}{2}\|\nabla_{A(t)} u(t)\|^2_{L^2} &\leq E_0 \\ &+ |b|^\gamma M_f \left(\frac{1}{2}d_0^2 + \frac{1}{p} C^p_{p,n} \|\nabla_{A(t)} u(t)\|_2^{n\frac{(p-2)}{2}} d_0^{2+(2-n)\frac{(p-2)}{2}} \right) \\ &+ 2d_0 \int_0^t \frac{1}{2}\|\nabla_{A(s)} u(t)\|_{L^2} g(s) ds \end{aligned} \tag{4.3}$$

or for $m(T) = \sup_{t\in[0,T]} \frac{1}{2}\|\nabla_{A(t)}u(t)\|_{L^2}^2$ this leads if $T < T^*$ to

$$\begin{aligned} m(t) &\le E_0 \\ &+ |b|^\gamma M_f \left(\frac{1}{2}d_0^2 + \frac{1}{p}C_{p,n}^p 2^{n\frac{(p-2)}{2}} m(t)^{n\frac{(p-2)}{2}} d_0^{2+(2-n)\frac{(p-2)}{2}}\right) \\ &+ 2d_0 C_\alpha T^* m(t). \end{aligned}$$

If $\frac{(p-2)}{2} < \frac{1}{n}$, this implies that $\|\nabla_{A(t)}u\|_{L^2}$ remains bounded when t goes to T^*. Consequently, $T^* = +\infty$. $\square$

Remark 4.2. *If $f(x,r) = |r|^\alpha r$ then $M_f = 1$ and if $b = 1$ then we find that the condition is $\|\phi\|_{L^2} < \|Q\|_{L^2}$ exactly as in the case $A = 0$. This fact was already noticed in* [6, Corollary 4.1] *for quadratic potential perturbation of the L^2-critical NLS. In the latter analysis this follows from by change of variables that relates the two mentioned equations. In the present case the same considerations will hold for constant magnetic fields.*

4.3. **The global wellposedness in the L^2-critical case below the Gagliardo-Nirenberg thresholds.** In the L^2 critical case we need a more careful analysis.

Theorem 4. *Suppose that $\alpha = \frac{4}{n}$ and that the function $g(t) = \sup_{x\in\mathbb{R}^n} |\partial_t A(t,x)|$ belongs to $L^1(\mathbb{R}_t)$. Then, there exists $r_0 > 0$ such that for any $\varphi \in H^1_{mg}$ such that*

$$\|\varphi\|_{L^2} < \frac{1}{(|b|^\gamma M_f)^{\frac{n}{4}}} \left(\frac{2n+4}{2nC_{2+\frac{4}{n},n}^{2+\frac{4}{n}}}\right)^{\frac{n}{4}}$$

the equation (mNLS) *has a unique solution u in $C^0(\mathbb{R}, H^1_{mg}) \cap C^1(\mathbb{R}, L^2)$.*

Proof. As the function $g(t) = \sup_{x\in\mathbb{R}^d} |\partial_t A(t,x)|$ belongs to $L^1(\mathbb{R}_t)$. Then (4.3)

$$\begin{aligned} \frac{1}{2}\|\nabla_{A(t)}u(t)\|_{L^2}^2 &\le E_0 \\ &+ |b|^\gamma M_f \left(\frac{1}{2}d_0^2 + \frac{n}{2n+4}C_{2+\frac{4}{n},n}^{2+\frac{4}{n}} \|\nabla_{A(t)}u(t)\|_2^2 d_0^{\frac{4}{n}}\right) \\ &+ 2d_0 \int_0^t \frac{1}{2}\|\nabla_{A(s)}u(s)\|_{L^2} g(s)ds \end{aligned}$$

or by Gronwall's lemma

$$\frac{1}{2}\|\nabla_{A(t)}u(t)\|_{L^2}^2 \le E_0 + |b|^\gamma M_f \left(\frac{1}{2}d_0^2 + \frac{n}{2n+4}C_{2+\frac{4}{n},n}^{2+\frac{4}{n}} \|\nabla_{A(t)}u(t)\|_2^2 d_0^{\frac{4}{n}}\right)$$
$$+ \int_0^t 2d_0 g(s) \left(E_0 + |b|^\gamma M_f \left(\frac{1}{2}d_0^2 + \frac{n}{2n+4}C_{2+\frac{4}{n},n}^{2+\frac{4}{n}} \|\nabla_{A(s)}u(s)\|_2^2 d_0^{\frac{4}{n}}\right)\right) e^{2d_0\int_s^t g(u)du} ds.$$

Hence as

$$\sup_{t\in\mathbb{R}} |b|^\gamma M_f \frac{n}{2n+4} C_{2+\frac{4}{n},n}^{2+\frac{4}{n}} d_0^{\frac{4}{n}} < \frac{1}{2}$$

this gives

$$\|\nabla_{A(t)}u(t)\|_{L^2}^2 \leq \frac{1}{\frac{1}{2}-\sup_{t\in\mathbb{R}}|b|^\gamma M_f \frac{n}{2n+4}C_{2+\frac{4}{n},n}^{2+\frac{4}{n}} d_0^{\frac{4}{n}}}\Bigg[E_0+|b|^\gamma M_f\frac{1}{2}d_0^2+$$

$$+\int_0^t 2d_0 g(s)\left(E_0+|b|^\gamma M_f\left(\frac{1}{2}d_0^2+\frac{n}{2n+4}C_{2+\frac{4}{n},n}^{2+\frac{4}{n}}\|\nabla_{A(t)}u(t)\|_2^2 d_0^{\frac{4}{n}}\right)\right)e^{2d_0\int_s^t g(u)du}ds\Bigg].$$

Then the Gronwall's lemma provides

$$\|\nabla_{A(t)}u(t)\|_{L^2}^2 \leq \frac{1}{\frac{1}{2}-\sup_{t\in\mathbb{R}}|b|^\gamma M_f \frac{n}{2n+4}C_{2+\frac{4}{n},n}^{2+\frac{4}{n}} d_0^{\frac{4}{n}}}\times$$

$$\times\left[E_0+|b|^\gamma M_f\frac{1}{2}d_0^2++\int_0^t 2d_0 g(s)\left(E_0+|b|^\gamma M_f\frac{1}{2}d_0^2\right)e^{2d_0\int_s^t g(u)du}ds\right]$$

$$\exp\left\{\frac{1}{\frac{1}{2}-\sup_{t\in\mathbb{R}}|b|^\gamma M_f \frac{n}{2n+4}C_{2+\frac{4}{n},n}^{2+\frac{4}{n}} d_0^{\frac{4}{n}}}\times\right.$$

$$\left.\times\left[\int_0^t 2d_0 g(s)\left(|b|^\gamma M_f\frac{n}{2n+4}C_{2+\frac{4}{n},n}^{2+\frac{4}{n}} d_0^{\frac{4}{n}}\right)e^{2d_0\int_s^t g(u)du}ds\right]\right\}.$$

This shows that $\|\nabla_{A(t)}u\|_{L^2}$ remains bounded for any time. This complete the proof of the theorem. □

5. Constant magnetic field

In this section we will only be interested by B constant.

5.1. Blow-up and Global Wellposedness in the L^2-supercritical case. The idea is to prove blow-up by virial type arguments, this is an idea that already appeared in [17]. Unfortunately the inhomogeneity introduced by the magnetic field does not allow to provide bounds in term of the energy. It seems that it has been attempted by [16] but the bounds in terms of the energy for homogeneous potentials is only stated and not proved, at least we don't understand the argument. The inhomogeneity in the analysis of the terms in the virial identity appears the so-called transversal component of the magnetic field. This can be set to be 0 as in [16] but in our case (transversal gauge) it will impose to A to vanish when B is not too singular.

5.2. Conservation of the angular momentum. Below we will use another conserved quantity : The angular momentum. The angular momentum operator is defined as

$$L := X \wedge \nabla u$$

and has $\frac{n(n-1)}{2}$ components. It is the generator of spatial rotations. The angular momentum is defined as the bilinear quantity

$$\langle u, Lu\rangle$$

it is conserved by the linear flow of (2.3) and consequently by the flow of (mNLS) for the constant magnetic field when the transversal gauge is fixed i.e. $AX = 0$ for any $X \in \mathbb{R}^n$.

5.3. **The fundamental example : The L^2-critical 2-dimensional.** In dimension 2, the magnetic field is a scalar, constant in this section and the associated smooth magnetic field in the transversal gauge is up to a constant

$$A(x, y) = \frac{b}{2}(-y, x).$$

Let us consider $u_0 \in H^1(\mathbb{R}^2)$ such that $|x|u_0$ is square integrable. Then $u_0 \in \mathrm{H}^1_A$ and the corresponding solution u of (mNLS) is in H^1 and $|x|u$ is square integrable during its lifespan as

$$\|\nabla_A u\|^2 = \|\nabla u\|^2 + \frac{b^2}{4}\|Xu\|^2 + b\langle u, Lu\rangle,$$

where $< u, Lu >= i \int_{\mathbb{R}^n} \bar{u}(X^\perp \cdot \nabla)u \, dX$. Let

$$Q(t) = \|Xu(t)\|^2$$

then

$$\dot{Q}(t) = 4\,\mathrm{Im}\langle u(t), X \cdot \nabla u(t)\rangle$$

and

$$\ddot{Q}(t) = 8\|\nabla_A u(t)\|^2 - 8\frac{p-1}{p+1}\|u(t)\|_{p+1}^{p+1} - 4b^2\|Xu(t)\|^2 + 8b\langle u(t), Lu(t)\rangle.$$

So that for $p = 3$

$$\ddot{Q}(t) + 4b^2 Q(t) = 16E_A(u_0) + 8b\langle u_0, Lu_0\rangle.$$

the solution of which is given by

$$Q(t) = \frac{16E_A(u_0) + 8b\langle u_0, Lu_0\rangle}{4b^2} + \left(Q(0) - \frac{16E_A(u_0) + 8b\langle u_0, Lu_0\rangle}{4b^2}\right)\cos(2bt) + \frac{\dot{Q}(0)}{2b}\sin(2bt).$$

Hence Q takes non positive values if and only if

$$16E_A(u_0) + 8b\langle u_0, Lu_0\rangle \leq 0$$

or

$$\left(\frac{16E_A(u_0) + 8b\langle u_0, Lu_0\rangle}{4b^2}\right)^2 \leq \left(Q(0) - \frac{16E_A(u_0) + 8b\langle u_0, Lu_0\rangle}{4b^2}\right)^2 + \frac{\dot{Q}(0)^2}{4b^2}.$$

The latter can be rewritten

$$\Phi(u_0) := 4b^2 Q(0)^2 + 2Q(0)\left(16E_A(u_0) + 8b\langle u_0, Lu_0\rangle\right) + \dot{Q}(0)^2 \geq 0.$$

The blow up condition is thus linked to the value of the energy and the angular momentum together.

5.3.1. *The limit case $b = 0$.* In the limit case $b = 0$, this is the non-linear Schrödinger equation with no magnetic field, the solution is given by

$$Q(t) = 8E_A(u_0)t^2 + \dot{Q}(0)t + Q(0).$$

Our first blow-up condition will correspond to the usual Glassey condition namely $E_A(u_0) \leq 0$. While the other will be

$$\dot{Q}(0)^2 - 32E_A(u_0)Q(0) \geq 0$$

and gives a blow up time in one of the interval

$$\begin{cases} [\frac{-\dot{Q}(0)+\sqrt{\dot{Q}(0)^2-32E_A(u_0)Q(0)}}{16E_A(u_0)}, 0) & \text{if } \dot{Q}(0) \geq 0 \\ (0, \frac{-\dot{Q}(0)-\sqrt{\dot{Q}(0)^2-32E_A(u_0)Q(0)}}{16E_A(u_0)}] & \text{if } \dot{Q}(0) \leq 0 \end{cases}.$$

5.4. **The L^2-supercritical 2-dimensional.** In the case $p > 3$, we have

$$\ddot{Q}(t) = 16E_A(u_0) - 8\frac{p-3}{p+1}\|u(t)\|_{p+1}^{p+1} - 4b^2\|Xu(t)\|^2 + 8b\langle u(t), Lu(t)\rangle.$$

Let

$$Q_1(t) = \frac{16E_A(u_0) + 8b\langle u_0, Lu_0\rangle}{4b^2} + \left(Q(0) - \frac{16E_A(u_0) + 8b\langle u_0, Lu_0\rangle}{4b^2}\right)\cos(2bt) + \frac{\dot{Q}(0)}{2b}\sin(2bt).$$

then by comparison

$$Q \leq Q_1$$

so that the same blow up criterion holds.

5.5. **The L^2-critical/supercritical 3-dimensional.** In dimension 3, the magnetic field is a scalar, constant in this section and the associated smooth magnetic field in the transversal gauge is up to a constant

$$A(x, y) = \frac{b}{2}(-y, x, 0).$$

Let us consider $u_0 \in H^1(\mathbb{R}^2)$ such that $|x|u_0$ is square integrable. Then $u_0 \in \mathrm{H}^1_A$ the corresponding solution u of (mNLS) is in H^1 and $|x|u$ is square integrable during its lifespan as

$$\|\nabla_A u\|^2 = \|\nabla u\|^2 + \frac{b^2}{4}(\|xu\|^2 + \|yu\|^2) + b\langle u, L_z u\rangle.$$

Let

$$Q(t) = \|Xu(t)\|^2$$

then

$$\dot{Q}(t) = 4\,\mathrm{Im}\langle u(t), X \cdot \nabla u(t)\rangle$$

or

$$\ddot{Q}(t) = 8\|\nabla_A u(t)\|^2 - 12\frac{p-1}{p+1}\|u(t)\|_{p+1}^{p+1} - 4b^2(\|xu\|^2 + \|yu\|^2) + 8b\langle u(t), L_z u(t)\rangle.$$

that is

$$\ddot{Q}(t) + 4b^2Q(t) = 8\|\nabla_A u(t)\|^2 - 2\frac{p-1}{p+1}\|u(t)\|_{p+1}^{p+1} + 4b^2\|zu\|^2 + 8b\langle u(t), L_z u(t)\rangle.$$

We thus have

$$\ddot{Q}(t) \leq 8\|\nabla_A u(t)\|^2 - 12\frac{p-1}{p+1}\|u(t)\|_{p+1}^{p+1} + 8b\langle u(t), L_z u(t)\rangle$$
$$\leq 16E_A(u_0) - \frac{16-12(p-1)}{p+1}\|u(t)\|_{p+1}^{p+1} + 8b\langle u(t), L_z u(t)\rangle.$$

So that for $p \leq 1 + 4/3$ as $16 - 12(p-1) \geq 0$

$$16E_A(u_0) + 8b\langle u_0, L_z u_0\rangle \leq 0$$

is a sufficient condition for blow up.

5.6. **The $\mathcal{K}_A$-functional.** Let us define the following (time dependent) quantities. The action

$$\mathcal{S}_{A,\omega}(u) := \omega \|u\|_{L^2}^2 + 2E_A(u) + b\langle u, Lu\rangle.$$

and

$$m_{A,\omega} := \inf\left\{\mathcal{S}_{A,\omega}(u) \;\middle|\; u \in H^1(\mathbb{R}^n)\setminus\{0\},\ \mathcal{K}_A(u) = 0\right\},$$

where

$$\mathcal{K}_A(u) = 2\int_{\mathbb{R}^n} |\nabla u|^2\,dx + b^2\int_{\mathbb{R}^n}|xu|^2\,dx - \frac{n(p-1)}{(p+1)}\int_{\mathbb{R}^n}|u|^{p+1}\,dx.$$

$$\widetilde{m}_{A,\omega} = \inf\left\{\mathcal{I}_{A,\omega}(u) \;\middle|\; u \in H^1(\mathbb{R}^d)\setminus\{0\},\ \mathcal{K}_A(u) \leq 0\right\},$$

where

$$\mathcal{I}_{A,\omega}(u) := \mathcal{S}_{A,\omega}(u) - \frac{2}{n(p-1)}\mathcal{K}_A(u)$$
$$= \omega\|u\|_{L^2}^2 + \frac{n(p-1)-4}{2n(p-1)}\|\nabla u\|_{L^2}^2 + \frac{n(p-1)-4}{2n(p-1)}\frac{b^2}{2}\|xu\|_{L^2}^2.$$

Note that the nonlinear term disappears in the functional $\mathcal{I}_{A,\omega}$. We have the following immediate result.

Lemma 5.1. *Let $\epsilon > 0$ then there exists $\delta > 0$ such that*

$$\mathcal{I}_{A,\omega}(u) < \delta \Rightarrow \|u\|_{H^1_{mg}} < \epsilon$$

Using Sobolev embedding it implies the following

Lemma 5.2. *Let $\epsilon > 0$ then there exist $\delta > 0$ and $c > 0$ such that*

$$\mathcal{I}_{A,\omega}(u) < \delta \Rightarrow \mathcal{K}_A(u) > c\int_{\mathbb{R}^n}|\nabla u|^2\,dx + c\int_{\mathbb{R}^n}|xu|^2\,dx + c\int_{\mathbb{R}^n}|u|^2\,dx.$$

We deduce that

$$m_{A,\omega} \geq \widetilde{m}_{A,\omega} \geq \delta/2.$$

Now let u be such that

$$\mathcal{S}_{A,\omega}(u) < m_{A,\omega} \text{ and } \mathcal{K}_A(u) \geq 0$$

then $\mathcal{S}_{A,\omega}(u)$ is conserved and $\mathcal{K}_A(u(t)) \geq 0$ for all time in the lifespan of u. Hence

$$2\int_{\mathbb{R}^n} |\nabla u|^2\,dx + b^2\int_{\mathbb{R}^n} |xu|^2\,dx \geq \frac{n(p-1)}{(p+1)}\int_{\mathbb{R}^n} |u|^{p+1}\,dx$$

and thus

$$\begin{aligned}\mathcal{S}_{A,\omega}(u) \geq{}& \frac{\omega}{2}\int_{\mathbb{R}^n} |u|^2\,dx + \int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{b^2}{2}\int_{\mathbb{R}^n} |xu|^2\,dx \\ &- \frac{2}{n(p-1)}\left(2\int_{\mathbb{R}^n} |\nabla u|^2\,dx + b^2\int_{\mathbb{R}^n} |xu|^2\,dx\right)\end{aligned}$$

that is

$$\mathcal{S}_{A,\omega}(u) \geq \frac{\omega}{2}\int_{\mathbb{R}^n} |u|^2\,dx + (1-\frac{4}{n(p-1)})\int_{\mathbb{R}^n} |\nabla u|^2\,dx + (1-\frac{4}{n(p-1)})\frac{b^2}{2}\int_{\mathbb{R}^n} |xu|^2\,dx$$

as $p < 1 + \frac{4}{n}$

$$b < u, Lu > + \frac{m_{A,\omega}}{(\frac{1}{2} - \frac{2}{n(p-1)})} \geq \int_{\mathbb{R}^n} |\nabla_A u|^2\,dx$$

so that the solution is globally wellposed.

Remark 5.3. *One can notice that in the L^2-critical case $p = 1 + \frac{4}{n}$ with $A = 0$, $K = 4E$ but when $B \neq 0$ the relation between K and E is not clear.*

In Appendix A we show that it possible to have $E_A(u) < 0$ and $\mathcal{K}_A(u) > 0$.

5.7. **Existence of small stationary solutions.** In the even dimensional case, it is interesting to notice that Equation mNLS can be isometrically mapped to the corresponding nonlinear Shrödinger equation with a repulsive harmonic oscillator potential when f does not depend on x. For simplicity consider $n = 2$, then as

$$\Delta_A = \Delta + \frac{b^2}{4}x^2 + bL_z$$

if u is a solution of (mNLS), then $v = e^{itbL_z}u$, rotations of angle tb, is a solution of

$$i\partial_t v = -\Delta v + \frac{b^2}{4}x^2 v - \mu^\gamma f(v)$$

with initial condition

$$v_{|t=t_0} = \varphi.$$

Recall that in the L^2-critical case, following [6] this in turn can be mapped

$$u(t,x) = \frac{1}{(1+(bt)^2)^{n/4}} e^{i\frac{b^2 t}{(1+(bt)^2)^{n/4}}\frac{x^2}{2}} v(\frac{\arctan(bt)}{b}, \frac{x}{\sqrt{1+(bt)^2}})$$

to the usual NLS.

The linear operator $H = -\Delta + \frac{b^2}{4}x^2$ is the tensor sum of two one dimensional harmonic oscillators whose spectrum is thus

$$|b|(\mathbb{N}+1)$$

the smallest eigenvalue is simple let us denote by ϕ_0 an associated normalized eigenvector.

As an immediate consequence the associated nonlinear equation has small stationary solutions if Assumption 2 is satisfied and thus preventing scattering of small solutions for (mNLS). Let is state the following proposition.

Proposition 5.4. *There exist* Ω *a neighborhood of* $0 \in \mathbb{C}$*, a* $\mathcal{C}^\infty$ *map*

$$h : \Omega \mapsto \{\phi_0\}^\perp \cap H^1(\mathbb{R}^2)$$

and a $\mathcal{C}^\infty$ *map* $E : \Omega \mapsto \mathbb{R}$ *such that* $S(u) = u\phi_0 + h(u)$ *satisfy, for all* $u \in \Omega$*, the identity*

$$HS(u) + f(S(u)) = E(u)S(u),$$

with the following properties

$$\begin{cases} h(e^{i\theta}u) = e^{i\theta}h(u), \quad \forall \theta \in \mathbb{R}, \\ h(u) = O(|u|^2), \\ E(u) = E(|u|), \\ E(u) = \lambda_0 + O(|u|^2). \end{cases}$$

Proof. This kind of results is now classical and left to the reader. For more details, and is an obvious adaptation of the one of [30, Proposition 2.2], and we don't repeat it here. One can also obtain it by means of the Crandall-Rabinowitz theorem but it doesn't give immediately the decomposition associated to the spectrum of H. □

6. Asymptotically small magnetic potentials

In this section we will consider magnetic fields tending to 0 at infinity. For instance following [18], we define the norm

$$\|g\|_X := \left(\sum_{j=0}^{\infty} 2^{3j} \|g\|^2_{L^\infty(D_j)} \right)^{1/2}$$

where $D_j = \{x \in R^n : 2^{j-1} \leq |x| \leq 2^j\}$.and consider magnetic potentials as bounded functions satisfying

$$A \text{ is continuous, and } \|A\|_X < \infty.$$

Up to a rescalling we can assume that $M_f = \mu^\gamma = 1$ and the thresholds in Proposition 4 is thus

$$\|\phi\|_2 < \|Q\|_2.$$

As proved in the previous section, beyond this threshold there is blow-up solutions. A natural and classical question is what happens to solution below this thresholds, here as expected they scatters.

To this end we need global in times Strichartz estimates provided by [18]. Our first task is to exclude the occurence of localized solutions at the threshold of the spectrum which is $\mathbb{R}^+$.

6.1. Magnetic Strichartz type estimates.

6.1.1. *Absence of* 0*-eigenvalue and* 0*-resonances.* The definition of resonances introduced in [18] is

Definition 6.1 (Resonances)**.** *A λ-resonance is a nontrivial function $f \in L^2_{-1}(\mathbb{R}^n)$ such that*

$$\sup_{j\geq 0} 2^{-\frac{j}{2}} \| |\nabla |f\|_{L^2(D_j)} < \infty$$

which is a distributional solution of equation

$$\Delta_A f = \lambda f.$$

As mentionned in [18], the combined efforts of [22] and [20] rule out their existence for each $\lambda > 0$. Below we rule out the case $\lambda = 0$.

Lemma 6.2 (Absence of 0-eigenvalue)**.** *Let $A \in L^2_{loc}(\mathbb{R}^n)$. Let $u \in H^1_A(\mathbb{R}^n)$ such that $\Delta_A u = 0$ then $u = 0$ in $L^2(\mathbb{R}^n)$.*

Proof. Let $u \in H^1_A(\mathbb{R}^n)$ such that $\Delta_A u = 0$ then

$$\| \nabla_{\mathrm{A}} u\| = 0$$

and hence $\nabla_{\mathrm{A}} u = 0$ almost everywhere so that by the Lemma 2.1 $\nabla |u| = 0$ almost everywhere and thus $u = 0$ in $L^2(\mathbb{R}^n)$. □

To prove the absence of 0-resonances we need the boundedness of the Riesz transforms R_k, the Fourier multiplier of symbol

$$R_k(\xi) = i\frac{\xi_k}{|\xi|},$$

in some weighted $L^2_w(\mathbb{R}^n)$ space, the statements are from [29]. To state the corresponding result we first introduce a class of weights satisfying the Muckenhoupt condition [27].

Definition 6.3 (A_p weights)**.** *A positive function $w \in L^1_{loc}(\mathbb{R}^n)$ is a weight of class A_p if and only if*

$$C_p(w) := \sup_{C\in Cu(\mathbb{R}^n)} [w]_C [w^{-\frac{1}{p-1}}]_C^{p-1} < \infty$$

where $Cu(\mathbb{R}^n)$ is the set of cubes of $\mathbb{R}_n$ and

$$[w]_C := \frac{1}{|C|}\int_C w$$

is the average of w over C, $|C| := \int_C 1$.

Notice that below we only consider A_2 and that $w \in A_2$ implies $w^{-1} \in A_2$. Moreover any $w_\alpha : x \mapsto (1 + x^2)^{\alpha/2}$ is in $A_2(\mathbb{R}^n)$ for $\alpha \in (-n, n)$. Similarly

$$d_j = 1\!\!1_{|x|\leq 1} + \sum_{j\geq 1} 2^{j/2} 1\!\!1_{2^{j-1}\leq |x| \leq 2^j}.$$

is in A_2 as $d_j \leq \sqrt{2} w_{1/2} \leq 2d_j$.

We now can state the result from [29] we need.

Theorem 5 ([29, Theorem 2.1])**.** *There exists a constant c so that for all $w \in A_2$ the Riesz transforms R_k as operators in weighted space $R_k : L^2_w(\mathbb{R}^n) \to L^2_w(\mathbb{R}^n)$ have norm $\|R_k\| \leq cC_2(w)$ and this result is sharp.*

We thus have the following corollary.

Corollary 6.4. *There exists a constant c such that for any $f \in L^2_{\text{loc}}(\mathbb{R}^n)$*

$$\sup_{j\geq 0} 2^{-\frac{j}{2}} \|\nabla f\|_{L^2(D_j)} \leq cC_2(d_j) \sup_{j\geq 0} 2^{-\frac{j}{2}} \||\nabla| f\|_{L^2(D_j)}$$

Lemma 6.5 (Absence of 0-resonances)**.** *Let $A \in L^2_{loc}(\mathbb{R}^n)$ such that $\mathrm{div} A \in L^2_{loc}(\mathbb{R}^n)$. Let $u \in H^1_{loc}(\mathbb{R}^n)$ such that $\Delta_A u = 0$ then $u = 0$ in $L^2(\mathbb{R}^n)$.*

Proof. Let $u \in H^1_{loc}(\mathbb{R}^n)$ such that $\Delta_A u = 0$. Notice that

$$0 = \Delta_A u = -\Delta u - 2iA \cdot \nabla u - i\mathrm{div} A u + |A|^2.$$

Let ϕ a smooth function with compact support then ϕu is a tempered distribution and

$$\Delta \phi u \in L^2(\mathbb{R}^n).$$

From elliptic regularity this gives $u \in H^2_{\text{loc}}(\mathbb{R}^n)$. Bootstrapping the above argument provides $u \in C^\infty(\mathbb{R}^n)$.

Then

$$\begin{aligned}
0 &= \langle \Delta_A u, \phi u \rangle \\
&= \langle \mathrm{div}\, \nabla_{\mathrm{A}}\, u, \phi u \rangle + i \langle A \cdot \nabla_{\mathrm{A}}\, u, \phi u \rangle \\
&= \langle \nabla_{\mathrm{A}}\, u, \nabla(\phi u) \rangle + \langle \nabla_{\mathrm{A}}\, u, -iA\phi u \rangle \\
&= \langle \nabla_{\mathrm{A}}\, u, \nabla_{\mathrm{A}}(\phi u) \rangle \\
&= \langle \nabla_{\mathrm{A}}\, u, \phi \nabla_{\mathrm{A}}\, u \rangle + \langle \nabla_{\mathrm{A}}\, u, \nabla(\phi) u \rangle.
\end{aligned}$$

Let $\phi = \psi_n$ where $\psi_n(x) := \psi(2^{-n}x)$ and ψ has support in the ball of center 0 and radius 2 is positive smaller than 1 and equal to one in the ball of center 0 and radius 1. Then

$$\langle \nabla_{\mathrm{A}}\, u, \psi(2^{-n} \cdot) \nabla_{\mathrm{A}}\, u \rangle = -2^{-n} \langle \nabla_{\mathrm{A}}\, u, \nabla(\psi)(2^{-n} \cdot)) u \rangle.$$

As $w_{-1} u$ is in $L^2(\mathbb{R}^n)$ and

$$\sup 2^{-\frac{n}{2}} \|\psi_n \nabla u\| < \infty$$

and $w_1 A \in L^\infty(\mathbb{R}^n)$ then

$$\langle \nabla_{\mathrm{A}}\, u, \psi(2^{-n} \cdot) \nabla_{\mathrm{A}}\, u \rangle = i2^{-n} \langle Au, \nabla(\psi)(2^{-n} \cdot)) u \rangle - 2^{-n} \langle \nabla u, \nabla(\psi)(2^{-n} \cdot)) u \rangle.$$

so that using Fatou Lemma for the left hand side and Lebesgues Theorem for the right hand one we obtain $\nabla_{\mathrm{A}}\, u = 0$ and thus $u = 0$. □

6.1.2. *The statement.* From [18], we deduce the following.

Proposition 6.6. *Let $n \geq 3$. Let A satisfy the conditions*

$$A \text{ is continuous, and } \|A\|_X < \infty$$

Then there exists $C > 0$ such that Strichartz inequalities

$$\forall u_0 \in L^2(\mathbb{R}^n),\ \|e^{-it\Delta_A} u_0\|_{L^p_t L^q_x} \leq C \|u_0\|_2, \qquad \frac{2}{p} + \frac{n}{q} = \frac{n}{2} \tag{6.1}$$

are valid for each exponent $p > 2$, as well as the Kato smoothing :

$$\|\langle x\rangle^{-1} e^{-it\Delta_A} u_0\|_{H_t^{1/4} L_x^2} \leq C' \|u_0\|_2, \tag{6.2}$$

$$\|\langle x\rangle g(x) \nabla e^{-it\Delta_A} u_0\|_{H_t^{-1/4} L_x^2} \leq C' \|g^2\|_X^{\frac{1}{2}} \|u_0\|_2,$$

$$\|g(x) |\nabla|^{1/2} e^{-it\Delta_A} u_0\|_{L_t^2 L_x^2} \leq C' \|g^2\|_X^{\frac{1}{2}} \|u_0\|_2.$$

for any $u_0 \in L^2(\mathbb{R}^n)$ and some $C' > 0$ independent of u_0 and g.

Proof. This is a consequence of [18, Theorem 1 & Lemma 7] in the case we are interested in. □

As an immediate consequence we have the following proposition.

Proposition 6.7. *Let $n \geq 3$. Let A satisfy the conditions*

$$A \text{ is continuous, and } \|A\|_X < \infty$$

Then there exists $C > 0$ such that

(1) the dual Strichartz inequalities

$$\forall F \in L_t^{p'}(\mathbb{R}, L_x^{q'}(\mathbb{R}^n)),\ \|\int_{\mathbb{R}} e^{-it\Delta_A} F(t)\, dt\|_2 \leq \|F\|_{L_t^{p'} L_x^{q'}} \tag{6.3}$$

(2) the inhomogeneous Strichartz inequalities

$$\forall F \in L_t^{p'}(\mathbb{R}, L_x^{q'}(\mathbb{R}^n)),\ \|\int_{\mathbb{R}} e^{-i(t-s)\Delta_A} F(t)\, dt\|_{L_t^p L_x^q} \leq \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \tag{6.4}$$

are valid for each exponent $p > 2$, $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ and $(\tilde{p}, \tilde{q})$ satisfy the same condition as (p, q).

Proof. Inequality (6.3) is the dual of (6.1) and Inequality (6.4) follows from (6.3) and (6.1) by [9, Theorem 1.2]. □

We then extend the Strichartz estimates to the non-autonomous case.

Proposition 6.8. *Assume $g(t) = \sup_{x \in \mathbb{R}^d} \|\partial_t A(t,x)\|_X$ is in $L_t^1(\mathbb{R})$. Then there exists $C > 0$ such that*

(1) the Strichartz inequalities

$$\forall u_0 \in L^2(\mathbb{R}^n),\ \|U(t,s) u_0\|_{L_t^p L_x^q} \leq C \|u_0\|_2$$

(2) the dual Strichartz inequalities

$$\forall F \in L_t^{p'}(\mathbb{R}, L_x^{q'}(\mathbb{R}^n)),\ \|\int_{\mathbb{R}} U(t,s) F(t)\, dt\|_2 \leq C \|F\|_{L_t^{p'} L_x^{q'}}$$

(3) the inhomogeneous Strichartz inequalities

$$\forall F \in L_t^{p'}(\mathbb{R}, L_x^{q'}(\mathbb{R}^n)),\ \|\int_{\mathbb{R}} U(t,s) F(t)\, dt\|_{L_t^p L_x^q} \leq C |F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}$$

are valid for each exponent $p > 2$, $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ and $(\tilde{p}, \tilde{q})$ satisfy the same condition as (p, q).

Proof. Notice that we can focus only in an interval of time $[T,\infty)$ as the local in time Strichartz estimates are provided by Theorem 1.

Let us recall (2.2)

$$\Delta_{A(t')}=\Delta_{A(t)}+W(t,t')(i\nabla_x-A(t))+(i\nabla_x-A(t))W(t,t')+W(t,t')^2$$

with $x\mapsto W(t,t',x)=\int_t^{t'}\partial_s A(s,x)ds$. Notice that due to the assumption

$$A_\infty:=\lim_{t\to\infty}A(t')\text{ exists in }X.$$

and

$$W_\infty(t'):=\lim_{t\to\infty}W(t,t')\text{ exists in }X.$$

as well with

$$\lim_{t'\to\infty}W_\infty(t')=0\text{ in }X.$$

We have for $A=A_\infty$

$$U_0(t,s)\varphi=e^{-i(t-s)\Delta_A}\varphi-i\int_s^t e^{-i(t-r)\Delta_A}B(r)U_0(r,s)\varphi\,dr \tag{6.5}$$

$$=e^{-i(t-s)\Delta_A}\left(\varphi-i\int_s^t e^{-i(s-r)\Delta_A}B(r)U_0(r,s)\varphi\,dr\right) \tag{6.6}$$

where $U_0(t,s)$ is defined in Proposition 2.6 and

$$B(t):=W_\infty(t)(i\nabla_x-A_\infty)+(i\nabla_x-A_\infty)W_\infty(t)+W_\infty(t)^2.$$

From Kato estimates (6.2) and the smallness of $W_\infty(t)$ for t large enough we deduce from (6.5) that

$$\|\langle x\rangle^{-1}U(t,s)u_0\|_{H_t^{1/4}L_x^2}\le\|u_0\|_2,$$

$$\|\langle x\rangle g(x)\nabla U(t,s)u_0\|_{H_t^{-1/4}L_x^2}\le\|g^2\|_X^{\frac12}\|u_0\|_2,$$

$$\|g(x)|\nabla|^{1/2}U(t,s)u_0\|_{L_t^2L_x^2}\le\|g^2\|_X^{\frac12}\|u_0\|_2.$$

and then using this estimates on from (6.6) gives the aimed estimates. □

6.2. **Global wellposdness in mass space for mass subcritical nonlinearities.** The idea here is to follow Tsutsumi's proof of global wellposedness in $L^2(\mathbb{R}^n)$. Let us make our assumption (6.7) stronger

$$\sup_{r>0}\frac{|f(x,r)|}{r^{\alpha+1}}\le M_f \tag{6.7}$$

and in the same spirit strengthen 2 into

$$\exists M\ge0,\alpha\in[0,\frac{4}{n-2}[,\ |f(x,z_1)-f(x,z_2)[\le M(|z_1|^\alpha+|z_2|^\alpha)|z_1-z_2|,$$

recall that here $n\ge3$.

Proposition 6.9. *Suppose that* $0<\alpha<\frac4n$, *then the equation* (mNLS) *with initial condition in* $L^2(\mathbb{R}^n)$ *has a unique solution* u *in* $C^0(\mathbb{R},L^2)$.

Proof. Let us consider the map $\mathcal{F}$ defined by

$$(\mathcal{F}u)(t) = U(t,0)\varphi - i\mu^\gamma \int_0^t U(t,s)f(x,u(s))\,ds.$$

Let $T > 0$ For any $u \in L^p_t([0,T], L^q_x(\mathbb{R}^n))$ with $p > 2$, $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$, we have for any $r < p$

$$\begin{aligned}
\|\mathcal{F}u\|_{L^r_t([0,T],L^q_x)} &\le T^{1/r-1/p}\|\mathcal{F}u\|_{L^p_t([0,T],L^q_x)} \\
&\le T^{1/r-1/p}\|U(t,0)\varphi\|_{L^p_t([0,T],L^q_x)} \\
&+ T^{1/r-1/p}|b|^\gamma \| \int_0^t U(t,s)f(x,u(s))\,ds\|_{L^p_t([0,T],L^q_x)} \\
&\le CT^{1/r-1/p}\|\varphi\|_2 + CT^{1/r-1/p}|b|^\gamma \|f(\cdot,u)\|_{L^{p'}_t([0,T],L^{q'}_x)} \\
&\le CT^{1/r-1/p}\|\varphi\|_2 + CT^{1/r-1/p}|b|^\gamma M_f \|u\|^{\alpha+1}_{L^{p'(\alpha+1)}_t([0,T],L^{q'(\alpha+1)}_x)}.
\end{aligned}$$

Now as $0 < \alpha < 1 + \frac{4}{n}$ we consider r, p and q such that $q'(\alpha+1) = q$ that is $q = \alpha + 2$ ($\alpha > 0$) and $p'(\alpha+1) < r < p$ that is $p < \alpha + 2$ so that

$$\|\mathcal{F}u\|_{L^r_t([0,T],L^q_x)} \le CT^{1/r-1/p}\|\varphi\|_2 + CT^{1/(p'(\alpha+1))-1/p}|b|^\gamma \|u\|^{\alpha+1}_{L^r_t([0,T],L^q_x)}.$$

So that we can choose both T and m such that the set

$$\{\|u\|_{L^r_t([0,T],L^q_x)} \le m\}$$

is invariant by $\mathcal{F}$.

Observe that a similar calculation provides

$$\begin{aligned}
\|\mathcal{F}u - \mathcal{F}v\|_{L^r_t([0,T],L^q_x)} &\le CT^{1/r-1/p}|b|^\gamma \|f(\cdot,u) - f(\cdot,v)\|_{L^{p'}_t([0,T],L^{q'}_x)} \\
&\le CT^{1/r-1/p}|b|^\gamma M \left(\||u|^\alpha |u-v|\|_{L^{p'}_t([0,T],L^{q'}_x)} + \||v|^\alpha |u-v|\|_{L^{p'}_t([0,T],L^{q'}_x)} \right) \\
&\le CT^{1/(p'(\alpha+2))-1/p}|b|^\gamma M 2m^{\alpha-1}\||u-v|\|_{L^r_t([0,T],L^q_x)}
\end{aligned}$$

So that $\mathcal{F}$ is a contraction on

$$\{\|u\|_{L^r_t([0,T],L^q_x)} \le m\}.$$

Now standard arguments provides the results due the charge conservation and the fact that m and t only depend on the L^2 norm of the initial condition. ☐

Appendix A. A counterexample to the negative energy criterion for blow-up

Our aim in this section is to show that the usual negative energy criterion for blow-up may fail when B is not 0. In the following we consider the case $n = 3$ and B constant. Up to rotation and rescalling we set $B = (0,0,2)^T$. Then $A = (-x_2, x_1, 0)$.

In this case we have

$$2E(u) = |\nabla u|_2^2 + |Au|_2^2 - 2\,\mathrm{Im}\langle \nabla u, Au\rangle - \frac{2}{p+1}|u|^{p+1}_{p+1}$$

and

$$\frac{1}{8}K(u) = |\nabla u|_2^2 + |Au|_2^2 - \frac{n(p-1)}{2(p+1)}|u|^{p+1}_{p+1}.$$

We build u such that $K(u) = 0$ and $E(u) < 0$. Hence we aim to

$$\frac{1}{n(p-1)}|\nabla u|_2^2 + \frac{1}{n(p-1)}|Au|_2^2 = \frac{2}{(p+1)}|u|_{p+1}^{p+1}.$$

and

$$(1 - \frac{1}{n(p-1)})|\nabla u|_2^2 + (1 - \frac{1}{n(p-1)})|Au|_2^2 - 2\,\mathrm{Im}\langle \nabla u, Au\rangle < 0$$

It is clear that we have to make $\mathrm{Im}\langle \nabla u, Au\rangle$ as big as possible and thus have as far as possible a solution of

$$\nabla u = icAu$$

with $c > 0$. Actually we only have to consider the first two components and take the gradient in the (x_1, x_2) variables. For the moment x_3 is fixed

Notice that due to the transversal gauge, $x\cdot\nabla u(x)$ should vanish so that u is radially constant. Hence u only depends on the angle θ for the cylindrical coordinates around z and

$$u'(\theta) = ic(r, \theta)r^2 u(\theta)$$

notice that the z dependence is implicit. As everything is r-independent it appears that

$$c(r, \theta)r^2 = k(\theta).$$

So that if K is the antederivative of k vanishing at 0

$$u(r, \theta, z) = e^{iK(\theta,z)}u(0, z)$$

Then notice that in order to have a negative energy at the level of the density we have to consider values of $x = |\nabla u|$ and $y = |iAu|$ such that

$$(1 - \frac{1}{n(p-1)})x^2 + (1 - \frac{1}{n(p-1)})y^2 - 2xy < 0$$

which is equivalent to

$$|\frac{x}{y} - \frac{1}{1 - \frac{1}{n(p-1)}}| < \sqrt{1 - (1 - \frac{1}{n(p-1)})^2}.$$

so that we have to localize

$$|k(\theta, z) - \frac{n(p-1)}{n(p-1)-1}| < \sqrt{1 - (1 - \frac{1}{n(p-1)})^2}$$

and instead of u we consider a slight modification

$$u_\epsilon(r, \theta, z) = e^{iK(\theta,z)}u(0, z)\chi(\frac{1}{\epsilon}[(n(p-1)-1)\frac{k(\theta, z)}{r^2} - n(p-1)])e^{-n\frac{1}{r}}$$

actually we make some other choices

$$u_{\epsilon,d,\delta,\alpha}(r, \theta, z) = \alpha e^{id\theta}e^{-\delta\sqrt{1+z^2}}\chi(\frac{1}{\epsilon}[(n(p-1)-1)\frac{d}{r^2} - n(p-1)]).$$

The parameter δ will control all the first derivative in z, it will be big to allow some control by the other terms. The α will be adjusted to make

$K(u) = 0$[2] and the localization ensures as much as possible that $\nabla u = cAu$ with c between the roots of the polynomial introduced previously. The parameter $d \in \mathbb{N}$ is analyzed below in order to treat the modification as a small perturbation. Indeed

$$\begin{aligned}
\partial_{x_1} u_{\epsilon,d,\delta,\alpha}(r,\theta,z) &= \frac{id}{r^2} Au_{\epsilon,x,\delta,\alpha,n}(r,\theta,z) \\
&\quad + \alpha e^{id\theta} e^{-\delta\sqrt{1+z^2}} \chi'(\frac{1}{\epsilon}[(n(p-1)-1)\frac{d}{r^2} - n(p-1)])\times \\
&\quad \times \frac{1}{\epsilon}(n(p-1)-1)\frac{d}{r^3} \\
\partial_{x_2} u_{\epsilon,d,\delta,\alpha}(r,\theta,z) &= \frac{id}{r^2} Au_{\epsilon,d,\delta,\alpha}(r,\theta,z) \\
&\quad - \alpha e^{d\theta} e^{-\delta\sqrt{1+z^2}} \chi'(\frac{1}{\epsilon}[(n(p-1)-1)\frac{d}{r^2} - n(p-1)])\times \\
&\quad \times \frac{1}{\epsilon}(n(p-1)-1)3\frac{d}{r^3} \\
\partial_z u_{\epsilon,d,\delta,\alpha}(r,\theta,z) &= -\delta\frac{2z}{\sqrt{1+z^2}} u_{\epsilon,d,\delta,\alpha}(r,\theta,z)
\end{aligned}$$

Notice that if χ has support in a ball or radius 1 centered at 0, then χ' has support in the annulus of radius $\sqrt{\frac{(n(p-1)-1)d}{n(p-1)+\epsilon}}$ and $\sqrt{\frac{(n(p-1)-1)d}{n(p-1)-\epsilon}}$ in the plane (x_1, x_2), its area is $\pi(n(p-1)-1)d\frac{2n(p-1)}{n^2(p-1)^2-\epsilon^2}$ and thus of order d. So that compared to our original u we have an error term of order $\frac{d^{3/2}}{\epsilon\delta}$ as r is of the order of $\frac{1}{\sqrt{d}}$ so that will consider $d = \circ(\epsilon^{\frac{2}{3}(1-\ell)})$ say $d = \epsilon^2$ if $\delta \sim \epsilon^{-\ell}$.

Notice that a slightly smaller α makes K positive and E negative as $K(\alpha' u_{\epsilon,x,\delta,\alpha,n})$ positive without changing the sign of $E(\alpha' u_{\epsilon,x,\delta,\alpha,n})$ for $\alpha' < \alpha$.

References

[1] J. E. Avron, I. W. Herbst, and B. Simon, *Separation of center of mass in homogeneous magnetic fields*, Ann. Physics **114** (1978), no. 1-2, 431–451. MR 507741 (80a:81050)

[2] ______, *Schrödinger operators with magnetic fields. III. Atoms in homogeneous magnetic field*, Comm. Math. Phys. **79** (1981), no. 4, 529–572.

[3] Anne Berthier and Vladimir Georgescu, *On the point spectrum of Dirac operators*, J. Funct. Anal. **71** (1987), no. 2, 309–338.

[4] Nabile Boussaid, Piero D'Ancona, and Luca Fanelli, *Virial identity and weak dispersion for the magnetic Dirac equation*, J. Math. Pures Appl. (9) **95** (2011), no. 2, 137–150. MR 2763073 (2012e:81068)

[5] Federico Cacciafesta, *Virial identity and dispersive estimates for the n-dimensional Dirac equation*, J. Math. Sci. Univ. Tokyo **18** (2011), no. 4, 441–463 (2012). MR 2952314

[6] Rémi Carles, *Critical nonlinear Schrödinger equations with and without harmonic potential*, Math. Models Methods Appl. Sci. **12** (2002), no. 10, 1513–1523. MR 1933935 (2004b:35298)

[2]The existence of α is clear if $|\nabla u|_2 > |Au|_2$ which will follow from $\nabla u = cAu$ with $c > 1$. α is thus a function of ϵ, δ and d but won't have any influence in the sign of $E(u_{\epsilon,x,\delta,\alpha,n})$.

[7] Thierry Cazenave and Maria J. Esteban, *On the stability of stationary states for nonlinear Schrödinger equations with an external magnetic field*, Mat. Apl. Comput. **7** (1988), no. 3, 155–168.

[8] Thierry Cazenave and Fred B. Weissler, *The Cauchy problem for the nonlinear Schrödinger equation in H^1*, Manuscripta Math. **61** (1988), no. 4, 477–494.

[9] Michael Christ and Alexander Kiselev, *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001), no. 2, 409–425. MR 1809116 (2001i:47054)

[10] James Colliander, Magdalena Czubak, and Jeonghun Lee, *Interaction Morawetz estimate for the magnetic Schr*, arXiv preprint arXiv:1112.4452 (2011).

[11] Piero D'Ancona, Luca Fanelli, Luis Vega, and Nicola Visciglia, *Endpoint Strichartz estimates for the magnetic Schrödinger equation*, J. Funct. Anal. **258** (2010), no. 10, 3227–3240. MR 2601614 (2011e:35325)

[12] Maria J. Esteban and Pierre-Louis Lions, *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field*, Partial differential equations and the calculus of variations, Vol. I, Progr. Nonlinear Differential Equations Appl., vol. 1, Birkhäuser Boston, Boston, MA, 1989, pp. 401–449.

[13] Luca Fanelli, *Non-trapping magnetic fields and Morrey-Campanato estimates for Schrödinger operators*, J. Math. Anal. Appl. **357** (2009), no. 1, 1–14. MR 2526801 (2010c:81093)

[14] Luca Fanelli and Luis Vega, *Magnetic virial identities, weak dispersion and Strichartz inequalities*, Math. Ann. **344** (2009), no. 2, 249–278. MR 2495769 (2010j:35427)

[15] Søren Fournais and Bernard Helffer, *Spectral methods in surface superconductivity*, Progress in Nonlinear Differential Equations and their Applications, 77, Birkhäuser Boston Inc., Boston, MA, 2010. MR 2662319 (2011j:35003)

[16] Andoni Garcia, *Magnetic virial identities and applications to blow-up for Schrödinger and wave equations*, J. Phys. A **45** (2012), no. 1, 015202, 16. MR 2871382

[17] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1977), no. 9, 1794–1797. MR 0460850 (57 #842)

[18] Michael Goldberg, *Strichartz estimates for Schrödinger operators with a non-smooth magnetic potential*, Discrete Contin. Dyn. Syst. **31** (2011), no. 1, 109–118. MR 2805802 (2012f:35070)

[19] M Griesemer and D Wellig, *The strong coupling polaron in electromagnetic field*, Preprint.

[20] Alexandru D. Ionescu and Wilhelm Schlag, *Agmon-Kato-Kuroda theorems for a large class of perturbations*, Duke Math. J. **131** (2006), no. 3, 397–440. MR 2219246 (2007b:35073)

[21] Tosio Kato, *Integration of the equation of evolution in a Banach space*, J. Math. Soc. Japan **5** (1953), 208–234. MR 0058861 (15,437b)

[22] Herbert Koch and Daniel Tataru, *Carleman estimates and absence of embedded eigenvalues*, Comm. Math. Phys. **267** (2006), no. 2, 419–449. MR 2252331 (2007k:35154)

[23] Herbert Leinfelder and Christian G. Simader, *Schrödinger operators with singular magnetic vector potentials*, Math. Z. **176** (1981), no. 1, 1–19. MR 606167 (82d:35073)

[24] Elliott H. Lieb and Michael Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001. MR 1817225 (2001i:00001)

[25] Michael Loss and Bernd Thaller, *Scattering of particles by long-range magnetic fields*, Ann. Physics **176** (1987), no. 1, 159–180.

[26] Laurent Michel, *Remarks on non-linear Schrödinger equation with magnetic fields*, Comm. Partial Differential Equations **33** (2008), no. 7-9, 1198–1215.

[27] Benjamin Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226. MR 0293384 (45 #2461)

[28] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[29] Stefanie Petermichl, *The sharp weighted bound for the Riesz transforms*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1237–1249. MR 2367098 (2009c:42034)

[30] Claude-Alain Pillet and C. Eugene Wayne, *Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations*, J. Differential Equations **141** (1997), no. 2, 310–326. MR 1488355 (99b:35193)
[31] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I*, second ed., Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Functional analysis.
[32] Kenji Yajima, *Schrödinger evolution equations with magnetic fields*, J. Analyse Math. **56** (1991), 29–76. MR 1243098 (94k:35270)
[33] ______, *Schrödinger equations with time-dependent unbounded singular potentials*, Rev. Math. Phys. **23** (2011), no. 8, 823–838. MR 2836569 (2012i:81096)