ON THE SPIN-1 BOSE EINSTEIN CONDENSATES IN THE PRESENCE OF IOFFE-PRITCHARD MAGNETIC FIELD

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ABSTRACT. We study the Cauchy problem of an antiferromagnetic spin-1 Bose Einstein condensates under Ioffe-Pritchard magnetic field B. We then address the existence of ground state solutions and characterize the orbit of standing waves of the associated variational problem.

1. INTRODUCTION

In 1995, it was proved that massive non-interacting bosons at low temperature can occupy the same lowest energy (ground state solution) single particle state and form the Bose-Einstein condensate. The total spin number F corresponding to the lowest energy has to be an integer with $2F+1$ hyperfine states $(m_F = -F, -F+1, ..., F)$. They are called spin-F (BEC).

The recent developments of optical trapping techniques enabled to confine atoms independently of their spin orientation to obtain the spinor condensates. The main motivation of such condensates lies essentially in their numerous applications in quantum mechanics. This cannot be covered by the classical single (BEC), which do not possess the appropriate properties.

In this paper, we will focus our study on the 1-spin (BEC) which is described by the following Gross-Pitaevskii system:

$$(1.1) \quad \begin{cases} i\partial_t\psi_1 = \left(H + \beta_s\left(|\psi_1|^2 + |\psi_0|^2 - |\psi_{-1}|^2\right)\right)\psi_1 + \beta_s\psi_0^2\bar{\psi}_{-1} + B\psi_0, \\ i\partial_t\psi_0 = \left(H + \beta_s\left(|\psi_1|^2 + |\psi_{-1}|^2\right)\right)\psi_0 + 2\beta_s\psi_1\bar{\psi}_0\psi_{-1} + \bar{B}\psi_1 + B\psi_{-1}, \\ i\partial_t\psi_{-1} = \left(H + \beta_s\left(|\psi_{-1}|^2 + |\psi_0|^2 - |\psi_1|^2\right)\right)\psi_{-1} + \beta_s\psi_0^2\bar{\psi}_1 + \bar{B}\psi_0, \\ \psi_j(0,x) = \Phi_j(x) \quad \forall\ -1 \leqslant j \leqslant 1, \end{cases}$$

where $x \in \mathbb{R}^d$, $1 \leqslant d \leqslant 3$, and the (nonlinear) operator H is defined by

$$H = -\frac{1}{2}\Delta + V_{\text{ext}}(x) + \beta_n \sum_{j=-1}^{1} |\psi_j|^2.$$

The external potential is typically a harmonic potential, possibly anisotropic,

$$(1.2) \quad V_{\text{ext}}(x) = \sum_{j=1}^{d} \omega_j^2 \frac{x_j^2}{2}.$$

We consider (1.1) along with initial data

$$(1.3) \quad \psi_{j|t=0} = \varphi_j, \quad j = -1, 0, 1.$$

$\psi_j(t,x)$ is the complex valued wave function of the $j^{th}(j = -1, 0, 1)$ component of $\psi = (\psi_{-1}, \psi_0, \psi_1)$. The parameters β_n and β_s describe the spin independent interaction and spin exchange interaction, respectively. They can be positive or negative. They are proportional to the total number of atoms N. They describe the collisions between atoms (see [1] and references therein). The function $B \in L^\infty(\mathbb{R}^d;\mathbb{C})$ denotes the external Ioffe-Pritchard magnetic field. For a more detailed account of the physical model, the reader can refer to [9, 11, 13]

As noticed in [1], the following quantities are formally conserved by the flow:

$$\text{Total mass: } \frac{d}{dt}\left(\sum_{j=-1}^{1}\|\psi_j(t)\|_{L^2}^2\right)=0, \tag{1.4}$$

$$\text{Total energy: } \frac{d}{dt}\left(\widehat{E}_0\left(\Psi(t)\right)+2\operatorname{Re}\left(\int_{\mathbb{R}^d}B\left(\bar{\psi}_1\psi_0+\bar{\psi}_0\psi_{-1}\right)dx\right)\right)=0, \tag{1.5}$$

where the magnetic-free energy is defined by

$$\begin{aligned}\widehat{E}_0\left(\Psi\right)=\int_{\mathbb{R}^d}&\left(\sum_{j=-1}^{1}\left(\frac{1}{2}|\nabla\psi_j|^2+V_{\text{ext}}(x)|\psi_j|^2\right)+\frac{\beta_n}{2}\left(\sum_{j=-1}^{1}|\psi_j|^2\right)^2\right.\\&\left.+\frac{\beta_s}{2}\left(|\psi_1|^2-|\psi_{-1}|^2\right)^2+\beta_s|\psi_0|^2\left(|\psi_1|^2+|\psi_{-1}|^2\right)+2\beta_s\operatorname{Re}\left(\bar{\psi}_1\psi_0^2\bar{\psi}_{-1}\right)\right)dx.\end{aligned}$$

We denote by $\widehat{E}(\Psi)$ the total energy, which is formally time-independent,

$$\widehat{E}(\Psi)=\widehat{E}_0\left(\Psi(t)\right)+2\operatorname{Re}\left(\int_{\mathbb{R}^d}B\left(\bar{\psi}_1\psi_0+\bar{\psi}_0\psi_{-1}\right)dx\right). \tag{1.6}$$

In the present study, we will consider the case where the spin interaction is repulsive, $\beta_n>0$, and the spin exchange interaction is anti-ferromagnetic $\beta_s>0$. The only observable solutions of (1.1) in experiments are the so-called ground state solutions. They are the minimizers of the following constrained variational problem.

For two prescribed positive numbers $M,N>0$, we set

$$\widehat{I}_{M,N}=\inf_{\psi=(\psi_{-1},\psi_0,\psi_1)\in\widehat{C}_{M,N}}\widehat{E}(\psi), \tag{1.7}$$

where

$$\begin{aligned}\widehat{C}_{M,N}=\Big\{&\psi=(\psi_{-1},\psi_0,\psi_1)\in\Sigma_{\mathbb{C}}(\mathbb{R}^d)\times\Sigma_{\mathbb{C}}(\mathbb{R}^d)\times\Sigma_{\mathbb{C}}(\mathbb{R}^d)\\&\int_{\mathbb{R}^d}\left(|\psi_{-1}|^2+|\psi_0|^2+|\psi_1|^2\right)dx=N\text{ and }\int_{\mathbb{R}^d}\left(|\psi_1|^2-|\psi_{-1}|^2\right)dx=M\Big\},\end{aligned} \tag{1.8}$$

with

$$\Sigma(\mathbb{R}^d)=\Sigma=\left\{u\in H^1(\mathbb{R}^d;\mathbb{R}):|u|_{V_d}^2:=\int V_d(x)u(x)^2dx<\infty\right\},$$

and

$$|u|_\Sigma^2=\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+|u|_{V_d}^2.$$

$\Sigma_{\mathbb{C}}$ denotes the set of complex-valued functions with a Σ regularity:

$$\Sigma_{\mathbb{C}}(\mathbb{R}^d)=\widehat{\Sigma}=\{z=(u,v)\simeq u+iv;\ (u,v)\in\Sigma\times\Sigma\}.$$

In terms of the polar coordinates $\psi_j = |\psi_j|e^{i\theta_j}$, $\hat{E}_0$ can be rewritten in the following way:

$$\begin{aligned}\widehat{E}_0(\Psi) = \int_{\mathbb{R}^d} \Bigg(& \sum_{j=-1}^{1} \left(\frac{1}{2}|\nabla\psi_j|^2 + V_{\text{ext}}(x)|\psi_j|^2\right) + \frac{\beta_n}{2}\left(\sum_{j=-1}^{1}|\psi_j|^2\right)^2 \\ & + \frac{\beta_s}{2}\left(|\psi_1|^2 - |\psi_{-1}|^2\right)^2 + \beta_s|\psi_0|^2\left(|\psi_1|^2 + |\psi_{-1}|^2\right) \\ & + 2\beta_s|\psi_1||\psi_0|^2|\psi_{-1}|\cos(2\theta_0 - \theta_1 - \theta_{-1})\Bigg)dx. \end{aligned} \tag{1.9}$$

Let θ_B be the angle function such that $B = |B|e^{i\theta_B}$.

$$\begin{aligned}\widehat{E}(\psi) = \int_{\mathbb{R}^d} \Bigg(& \sum_{j=-1}^{1} \left(\frac{1}{2}|\nabla\psi_j|^2 + V_{\text{ext}}(x)|\psi_j|^2\right) + \frac{\beta_n}{2}\left(\sum_{j=-1}^{1}|\psi_j|^2\right)^2 \\ & + \frac{\beta_s}{2}\left(|\psi_1|^2 - |\psi_{-1}|^2\right)^2 + \beta_s|\psi_0|^2\left(|\psi_1|^2 + |\psi_{-1}|^2\right) \\ & + 2\beta_s|\psi_1||\psi_0|^2|\psi_{-1}|\cos(2\theta_0 - \theta_1 - \theta_{-1}) \\ & + 2|B||\psi_0|\left(|\psi_1|\cos(\theta_B + \theta_0 - \theta_1) + |\psi_{-1}|\cos(\theta_B + \theta_{-1} - \theta_0)\right)\Bigg)dx. \end{aligned} \tag{1.10}$$

To (1.10) (i.e (1.6)), we will associate the following constrained minimization problem

$$I_{M,N} = \inf_{u=(u_{-1},u_0,u_1)\in C_{M,N}} E(u), \tag{1.11}$$

where

$$C_{M,N} = \left\{ u = (u_{-1}, u_0, u_1) \in \Sigma^3 : \int_{\mathbb{R}^d}\left(u_{-1}^2 + u_0^2 + u_1^2\right) = N \text{ and } \int_{\mathbb{R}^d}\left(u_1^2 - u_{-1}^2\right) = M \right\},$$

$$\begin{aligned} E_0(u) = \frac{1}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^d} \Bigg(& \sum_{j=-1}^{1}\int_{\mathbb{R}^d} V_{\text{ext}}(x)u_j^2 + \frac{\beta_n}{2}\left(\sum_{j=-1}^{1}u_j^2\right)^2 \\ & + \int_{\mathbb{R}^d}\frac{\beta_s}{2}(u_1^2 - u_{-1}^2)^2 + \beta_s u_0^2(u_1^2 + u_{-1}^2) - 2\beta_s u_1 u_0^2 u_{-1}\Bigg)dx, \end{aligned} \tag{1.12}$$

and

$$E(u) = E_0(u) - 2\int_{\mathbb{R}^d}|B(x)||u_0||u_1| - 2\int_{\mathbb{R}^d}|B(x)||u_0||u_{-1}|. \tag{1.13}$$

Note that (1.12) can be rewritten in the following way :

$$\begin{aligned} E_0(u) = & \frac{1}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^d}\sum_{j=-1}^{1}V_{\text{ext}}(x)u_j^2(x) \\ & + \frac{\beta_n + \beta_s}{2}\int_{\mathbb{R}^d}(u_1^4 + u_{-1}^4)dx + \frac{\beta_n}{2}\int_{\mathbb{R}^d}u_0^4 \\ & + (\beta_n + \beta_s)\int_{\mathbb{R}^d}u_0^2(u_1^2 + u_{-1}^2)dx \\ & + (\beta_n - \beta_s)\int_{\mathbb{R}^d}u_1^2u_{-1}^2dx - 2\beta_s\int u_1u_{-1}u_0^2dx. \end{aligned} \tag{1.14}$$

Recall that (1.1) has been studied numerically quite recently, [1]. The authors have been able to show the existence of ground state solutions of (1.6) via some readapted and innovative numerical methods. Stationary states with lowest energy are very important in Physics, since they enable us to understand the properties of BEC. In the literature, there exist many results, based on experiments, proving the existence of ground states of 1-spin condensates, see e.g. [1, 3, 14] and references therein.

To our knowledge, the first theoretical result established in this context, has been established in [3]. However the authors only considered the case $d = 1, V_{\text{ext}} \equiv 0$ and $B \equiv 0$. This significantly limits the scope of applications in physics. They proved the existence of ground state solutions when $\beta_n < 0$ and $\beta_s < 0$. When $V_{\text{ext}} \neq 0$, we will prove that (1.11) has a minimizer when both β_n and β_s are non-negative. Note that in this case, both spin independent interaction and spin-exchange interaction are repulsive, and if there is no trapping potential V_{ext}, the atoms cannot be confined, this is why when β_n and $\beta_s > 0$ and $V_{\text{ext}} \equiv 0$, there are no ground state solutions. Technically, (1.11) is a problem which is completly different in the presence or in the absence of the trapping potential V_{ext}.

When $V_{\text{ext}} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ and $\lim\limits_{R\to\infty}(\text{ess}\inf\limits_{|x|\geqslant R} V(x)) = \infty$, L. Lin and L. Chern [14] have studied (1.7) with $B \equiv 0$. They were essentially concerned with the bifurcation problem of the ground states. More precisely, they have studied the bifurcation problem:

$$\begin{cases} (\mu+\lambda)u_1 = Hu_1 + 2\beta_s\left(u_0^2(u_1 - u_{-1}) + u_1(u_1^2 - u_{-1}^2)\right) \\ \qquad \mu u_0 = Hu_0 + 2\beta_s u_0(u_1 - u_{-1})^2, \\ (\mu-\lambda)u_1 = Hu_1 + 2\beta_s\left(u_0^2(u_{-1} - u_1) + u_{-1}(u_{-1}^2 - u_1^2)\right), \end{cases}$$

whereλ and μ are the Lagrange multipliers arising from the constraint $C_{M,N}$.

In this paper, we give a complete study of the spin-1 (BEC) problem. First, we prove in Section 3 that the Cauchy problem (1.1)–(1.3) has a unique solution in a suitable function space. In Section 4, we characterize the orbit of ground state solutions. To achieve this goal, we will need the results established in Section 2, where we prove the existence of these stationary solutions. Our method hinges on the approach built up by Cazenave and Lions in [8], and by Hajaiej-Stuart in [10]. In [8], the authors gave a precise line of attack to establish the orbital stability. In fact, they showed that it is sufficient to show that all minimizing sequences of the variational problem (1.7) are relatively compact in $\hat{\Sigma}^3$. To characterize the orbit of standing waves, a more subtle and in-depth study must be undertaken. This has been developed in [10]. In Section 4, we give a detailed account for the current framework.

2. Notations

For fixed positive numbers M and N, we define

$$Z_{M,N} = \left\{z = (z_{-1}, z_0, z_1) \in \widehat{C}_{M,N} : \widehat{E}(z) = \widehat{I}_{M,N}\right\}, \tag{2.1}$$

and

$$W_{M,N} = \{u = (u_{-1}, u_0, u_1) \in C_{M,N} : u_{-1}, u_0 \text{ and } u_1 > 0 \text{ and } E(u) = I_{M,N}\}.$$

Definition 2.1. *We say that $Z_{M,N}$ is stable if $Z_{M,N} \neq \emptyset$, and for all $w \in Z_{M,N}$,*

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \textit{ such that for any } \Psi^0 \in \widehat{\Sigma}^3 \textit{ satisfying } \|\Psi^0 - w\|_{\widehat{\Sigma}^3} < \delta,$$

then for all $t \in \mathbb{R}$,

$$\inf_{z \in Z_{M,N}} \|\Psi(t,.) - z\|_{\widehat{\Sigma}^3} < \varepsilon, \tag{2.2}$$

where $\Psi(t,.)$ *is the unique solution of* (1.1) *initial condition* Ψ^0.

The following result can be found in e.g. [12].

Lemma 2.2. Σ *is compactly embedded in* $L^q(\mathbb{R}^d)$ *for any* $2 \leqslant q < \frac{2d}{d-2}$.

Lemma 2.3. (1) *The minimization problem* (1.11) *is well-posed and any minimizing sequence of* (1.11) *is relatively compact in* Σ^3.

(2) *Any minimizing sequence of* (1.11) *is relatively compact in* Σ^3, *i.e for all sequences* $u_n \in C_{M,N}$ *such that* $E(u_n) \to I_{M,N}$, *there exists* $u \in \Sigma^3$ *such that, up to a subsequence,* $u_n \to u$ *in* Σ^3.

(3) *The functionals* E *and* $\widehat{E}$ *are* C^1 *on* Σ^3 *and* $\widehat{\Sigma}^3$, *respectively.*

(4) *The function:*

$$\begin{array}{rcl} (0,\infty) \times (0,\infty) & \longrightarrow & \mathbb{R} \\ (M,N) & \longmapsto & I_{M,N} \end{array}$$

is continuous with respect to both variables.

Proof. (1) Since we are supposing that β_n and $\beta_s \geqslant 0$, we can easily deduce that

$$\begin{aligned} E(u) &\geqslant -2\int_{\mathbb{R}^d} |B(x)||u_0||u_1| + |B(x)||u_0||u_{-1}|dx \\ &\geqslant -2\|B\|_\infty \|u_0\|_{L^2} \left(\|u_1\|_{L^2} + \|u_{-1}\|_{L^2}\right) \\ &\geqslant -\|B\|_{L^\infty} \left(\|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \|u_0\|_{L^2}^2 + \|u_{-1}\|_{L^2}^2\right) \geqslant -2\|B\|_\infty N. \end{aligned}$$

This proves that the minimization problem $I_{M,N}$ is well-posed and that every minimizing sequence is bounded in Σ^3.
(2) Let $u_n = (u_{n,-1}, u_{n,0}, u_{n,1})$ be a minimizing sequence of (1.11). From (1), we know that (u_n) is bounded in Σ^3, therefore (up to a subsequence), there exists $u = (u_{-1}, u_0, u_1) \in \Sigma^3$ such that $u_{n,j} \rightharpoonup u_j$ in Σ, $-1 \leqslant j \leqslant 1$.
First note that by the lower semi-continuity of the norm Σ we have

$$|u|_{\Sigma^3}^2 \leqslant \liminf |u_n|_{\Sigma^3}^2. \tag{2.3}$$

By Lemma 2.2, we have $u_{n,j} \to u_j$ in $L^p(\mathbb{R}^d)$ for any $2 \leqslant p < \dfrac{2d}{d-2}$ and $-1 \leqslant j \leqslant 1$.

In particular with $p = 4$, we infer

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} (u_{n,1}^2 - u_{n,-1}^2)^2 = \int_{\mathbb{R}^d} (u_1^2 - u_{-1}^2)^2, \tag{2.4}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} u_{n,0}^2(u_{n,1}^2 + u_{n,-1}^2) = \int_{\mathbb{R}^d} u_0^2(u_1^2 + u_{-1}^2), \tag{2.5}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} u_{n,0}^2 u_{n,-1} u_{n,1} = \int_{\mathbb{R}^d} u_1 u_0^2 u_{-1}. \tag{2.6}$$

On the other hand, using the fact that $B \in L^\infty(\mathbb{R}^d)$, and that $u_{n,j} \to u_j$ in $L^2(\mathbb{R}^d)$, it follows that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} |B|(|u_{n,0}||u_{n,1}| + |u_{n,0}||u_{n,-1}|)dx = \int_{\mathbb{R}^d} |B|(|u_0||u_1| + |u_0||u_{-1}|)dx. \tag{2.7}$$

Combining (2.3) to (2.7), we certainly have that

$$E(u) \leqslant \liminf_{n\to\infty} E(u_n) = I_{M,N}.$$

But since $u = (u_{-1}, u_0, u_1) \in C_{M,N}$, we can conclude that

$$E(u) = I_{M,N}.$$

(3) The proof is identical to [**?**, Proposition 3.2.9].
(4) The proof also goes exactly in the some way as in [**?**, Proposition 3.2.9]. Note also that in Section 3 of [14] the continuity and monotonicity of $I_{M,N}$ have been discussed in details. □

Remark 2.4. Contrary to the case $V_{\text{ext}} \equiv 0$ studied in [3], it seems that it is not reasonable to expect to have a symmetric minimizer of (1.11), even when V_{ext} is isotropic. This is essentially due to the conflicts between the different terms of the energy functional. More precisely, note that $E(|u|) \leqslant E(u)$, therefore without loss of generality, we can suppose that the minimizer of (1.11) is non-negative. Let u^* be its Schwarz symmetrization. Then using rearrangement inequalities established in [2], we certainly have that

$$\|\nabla u_j^*\|_{L^2} \leqslant \|\nabla u_j\|_{L^2} \quad \forall\ -1 \leqslant j \leqslant 1, \tag{2.8}$$

$$\int_{\mathbb{R}^d} V_{\text{ext}}(x)(u_j^*)^2 dx \leqslant \int_{\mathbb{R}^d} V_{\text{ext}}(x)(u_j)^2 dx \quad \forall\ -1 \leqslant j \leqslant 1, \tag{2.9}$$

$$(\beta_n+\beta_s)\int_{\mathbb{R}^d} \left(u_1^4 + u_{-1}^4\right) dx = (\beta_n+\beta_s)\int_{\mathbb{R}^d} \left((u_1^*)^4 + (u_{-1}^*)^4\right) dx, \tag{2.10}$$

$$\beta_n \int_{\mathbb{R}^d} u_0^4 dx = \beta_n \int_{\mathbb{R}^d} (u_0^*)^4 dx, \tag{2.11}$$

$$(\beta_n+\beta_s)\int_{\mathbb{R}^d} u_0^2(u_1^2 + u_{-1}^2) dx \leqslant (\beta_n+\beta_s)\int_{\mathbb{R}^d} (u_0^*)^2((u_{-1}^*)^2 + (u_1^*)^2)^2 dx. \tag{2.12}$$

If in addition, $\beta_n < \beta_s$, then

$$(\beta_n-\beta_s)\int_{\mathbb{R}^d} u_1^2 u_{-1}^2 \geqslant (\beta_n-\beta_s)\int_{\mathbb{R}^d} (u_1^*)^2(u_{-1}^*)^2, \tag{2.13}$$

$$\beta_s \int u_1 u_{-1} u_0^2 \leqslant \beta_s \int u_1^* u_{-1}^* (u_0^*)^2. \tag{2.14}$$

A classical tool to prove that (1.11) has a Schwarz symmetric minimizer is to show that $E(u^*) \leqslant E(u)$. Here (2.8), (2.9) and (2.13) are in conflict with (2.12) and (2.14). We have strong reasons to believe that there is no Schwarz symmetric minimizer for (1.11). This issue will be investigated.

From now on in an integral where no domain of integration is indicated, it is to be understrod that it runs over $\mathbb{R}^d$. Sometimes we will omit the measure element dx.

3. Cauchy problem

Denote by

$$H_0 = -\frac{1}{2}\Delta + V_{\text{ext}}(x)$$

the linear part of H.

Definition 3.1. *A pair* (p,q) *is admissible if* $2 \leqslant q < \frac{2d}{d-2}$ *(*$2 \leqslant q \leqslant \infty$ *if* $d=1$*,* $2 \leqslant q < \infty$ *if* $d=2$*) and*

$$\frac{2}{p} = \delta(q) := d\left(\frac{1}{2} - \frac{1}{q}\right).$$

We have the standard result (see e.g. [7]):

Lemma 3.2. *Let* $d \geqslant 1$*, and suppose that the external potential is smooth and at most quadratic:*

$$\partial^\alpha V_{\rm ext} \in L^\infty(\mathbb{R}^d), \quad \forall |\alpha| \geqslant 2.$$

Denote by $U(t) = e^{-itH_0}$ *the linear flow map, and let* $T>0$*.*
(1) *For any admissible pair* (p,q)*, there exists* C_q *such that*

$$\|U(t)\varphi\|_{L^p([0,T];L^q)} \leqslant C_q \|\varphi\|_{L^2}, \quad \forall \varphi \in L^2(\mathbb{R}^d).$$

(2) *Denote*

$$D(F)(t,x) = \int_0^t U(t-\tau)F(\tau,x)d\tau.$$

For all admissible pairs (p_1,q_1) *and* (p_2,q_2)*, there exists* $C = C_{q_1,q_2}$ *such that*

$$\|D(F)\|_{L^{p_1}([0,t];L^{q_1})} \leqslant C\, \|F\|_{L^{p_2'}\left([0,t];L^{q_2'}\right)}, \tag{3.1}$$

for all $F \in L^{p_2'}([0,T];L^{q_2'})$ *and* $0 \leqslant t \leqslant T$*.*

Remark 3.3. In general, the above constant C_q and C_{q_1,q_2} must be expected to depend on T. See e.g. [4].

Duhamel's formulation for (1.1)-(1.3) takes the form

$$\begin{aligned} \psi_j(t) = U(t)\varphi_j - i\int_0^t U(t-\tau)\mathcal{C}_j(\psi_{-1},\psi_0,\psi_1)(\tau)d\tau \\ - i\int_0^t U(t-\tau)\mathcal{B}_j(\psi_{-1},\psi_0,\psi_1)(\tau)d\tau, \quad j=-1,0,1, \end{aligned} \tag{3.2}$$

where the $\mathcal{C}_j$'s are $\mathbb{R}$-trilinear operators in their argument, and the $\mathcal{B}_j$ are $\mathbb{R}$-linear operators in their argument. Therefore, (3.2) is the system version of a cubic nonlinear equation, with a linear source term stemming from B. The classical theory for scalar nonlinear Schrödinger equations, as presented in e.g. [4, 7], yields:

Proposition 3.4. *Suppose that* B *is bounded,* $B \in L^\infty(\mathbb{R}^d)$*, and that the external potential is smooth and at most quadratic:*

$$\partial^\alpha V_{\rm ext} \in L^\infty(\mathbb{R}^d), \quad \forall |\alpha| \geqslant 2.$$

1. *If* $d=1$ *and* $\varphi_j \in L^2(\mathbb{R})$*, then there exists* T *depending only on the* $\|\varphi_j\|_{L^2}$*'s, such that* (3.2) *has a unique solution*

$$\psi_j \in C([-T,T];L^2(\mathbb{R})) \cap L^8([-T,T];L^4(\mathbb{R})), \quad j=-1,0,1.$$

If in addition $B \in W^{1,\infty}(\mathbb{R})$ *and* $\varphi_j \in \widehat{\Sigma}$*, then*

$$\psi_j, \partial_x\psi_j, x\psi_j \in C([-T,T];L^2(\mathbb{R})) \cap L^8([-T,T];L^4(\mathbb{R})), \quad j=-1,0,1.$$

2. *If* $2 \leqslant d \leqslant 3$*, suppose that* $B \in W^{1,\infty}(\mathbb{R}^d)$*. If* $\varphi_j \in \widehat{\Sigma}$*, then there exists* T *depending only on the* $\|\varphi_j\|_{\widehat{\Sigma}}$*'s, such that* (3.2) *has a unique solution*

$$\psi_j, \nabla\psi_j, x\psi_j \in C([-T,T];L^2(\mathbb{R}^d)) \cap L^{8/d}([-T,T];L^4(\mathbb{R}^d)), \quad j=-1,0,1.$$

3. *In all the above cases, the conservation* (1.4) *holds, as well as* (1.5) *provided that* $\varphi_j \in \widehat{\Sigma}$.

Remark 3.5. If $V_{\rm ext}$ is at most linear in x ($\nabla V_{\rm ext} \in L^\infty(\mathbb{R}^d)$), then the assumption $\varphi_j \in \widehat{\Sigma}$ can be relaxed to $\varphi_j \in H^1(\mathbb{R}^d)$, and the solution satisfies

$$\psi_j \in C([-T,T];H^1(\mathbb{R}^d)) \cap L^{8/d}([-T,T];W^{1,4}(\mathbb{R}^d)), \quad j=-1,0,1.$$

On the other hand, if $V_{\rm ext}$ is quadratic as in (1.2), working in $\widehat{\Sigma}$ (and not only in $H^1(\mathbb{R}^d)$) is necessary. See [5].

Corollary 3.6. *Suppose that* B *is bounded,* $B \in L^\infty(\mathbb{R}^d)$*, and that the external potential is smooth and at most quadratic.*
1. *If* $d=1$ *and* $\varphi_j \in L^2(\mathbb{R})$*, then* (3.2) *has a unique global solution*

$$\psi_j \in C(\mathbb{R};L^2(\mathbb{R})) \cap L^8_{\rm loc}(\mathbb{R};L^4(\mathbb{R})), \quad j=-1,0,1.$$

If in addition $B \in W^{1,\infty}(\mathbb{R})$*, and* $\varphi_j \in \widehat{\Sigma}$*, then*

$$\psi_j, \partial_x \psi_j, x\psi_j \in C(\mathbb{R};L^2(\mathbb{R})) \cap L^8_{\rm loc}(\mathbb{R};L^4(\mathbb{R})), \quad j=-1,0,1.$$

2. *If* $2 \leqslant d \leqslant 3$*, suppose that* $B \in W^{1,\infty}(\mathbb{R}^d)$*. If* $\varphi_j \in \widehat{\Sigma}$*, and* $\beta_s, \beta_n \geqslant 0$*, then*

$$\psi_j, \nabla \psi_j, x\psi_j \in C(\mathbb{R};L^2(\mathbb{R}^d)) \cap L^{8/d}_{\rm loc}(\mathbb{R};L^4(\mathbb{R}^d)), \quad j=-1,0,1.$$

Proof. Only the second point requires an explanation. From Proposition 3.4, we infer that (3.2) has a unique maximal solution. It is maximal in the future in the sense that either it is defined for all positive time, or there exists a finite T_+ such that

$$\|\psi_j(t)\|_{\widehat{\Sigma}} \underset{t\to T_+}{\longrightarrow} \infty,$$

for at least one j. We then proceed as in [6], and introduce a modified energy functional

$$\begin{aligned}\mathcal{E}(\Psi) = \int_{\mathbb{R}^d} \Bigg(&\sum_{j=-1}^{1} \left(\frac{1}{2}|\nabla\psi_j|^2 + \frac{|x|^2}{2}|\psi_j|^2\right) + \frac{\beta_n}{2}\left(\sum_{j=-1}^{1}|\psi_j|^2\right)^2 \\ &+ \frac{\beta_s}{2}\left(|\psi_1|^2 - |\psi_{-1}|^2\right)^2 + \beta_s|\psi_0|^2\left(|\psi_1|^2+|\psi_{-1}|^2\right) \\ &+ 2\beta_s \operatorname{Re}\left(\bar\psi_1\psi_0^2\bar\psi_{-1}\right) + 2\operatorname{Re}\left(B\left(\bar\psi_1\psi_0 + \bar\psi_0\psi_1\right)\right)\Bigg)dx.\end{aligned}$$

Since

$$2\left|\operatorname{Re}\left(\bar\psi_1\psi_0^2\bar\psi_{-1}\right)\right| \leqslant |\psi_0|^2\left(|\psi_1|^2+|\psi_{-1}|^2\right),$$

and since the total mass is conserved, (1.4), in the case $\beta_s \geqslant 0$, we have

$$\begin{aligned}\mathcal{E}(\Psi) \geqslant \int_{\mathbb{R}^d} \Bigg(&\sum_{j=-1}^{1} \left(\frac{1}{2}|\nabla\psi_j|^2 + \frac{|x|^2}{2}|\psi_j|^2\right) + \frac{\beta_n}{2}\left(\sum_{j=-1}^{1}|\psi_j|^2\right)^2 \\ &+ 2\operatorname{Re}\left(B\left(\bar\psi_1\psi_0 + \bar\psi_0\psi_1\right)\right)\Bigg)dx.\end{aligned}$$

Since B is bounded, (1.4) yields

$$2\operatorname{Re}\int_{\mathbb{R}^d}\left(B\left(\bar\psi_1\psi_0 + \bar\psi_0\psi_1\right)\right)dx \leqslant C,$$

for some uniform constant C. Therefore, in the case $\beta_n \geqslant 0$, there exists C such that

$$\mathcal{E}\left(\Psi\right) \geqslant \frac{1}{2}\sum_{j=-1}^{1} \|\psi_j(t)\|_{\widehat{\Sigma}}^2 - C. \tag{3.3}$$

On the other hand, we compute, in view of (1.5):

$$\begin{aligned}\frac{d}{dt}\mathcal{E}\left(\Psi(t)\right) &= \frac{d}{dt}\left(\mathcal{E}\left(\Psi(t)\right) - E\left(\Psi(t)\right)\right)\\ &= \sum_{j=-1}^{1} \frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{|x|^2}{2} - V_{\rm ext}(x)\right)|\psi_j(t,x)|^2 dx.\end{aligned}$$

For a general potential $\mathtt{V}(x)$, let

$$\mathcal{V}(t) := \sum_{j=-1}^{1}\int_{\mathbb{R}^d} \mathtt{V}(x)|\psi_j(t,x)|^2 dx.$$

We compute

$$\frac{d}{dt}\mathcal{V}(t) = 2\sum_{j=-1}^{1} \operatorname{Im}\int_{\mathbb{R}^d} \mathtt{V}(x)\bar{\psi}_j(t,x) i\partial_t\psi_j(t,x)dx.$$

In the same fashion as the total mass is conserved, algebraic cancellations yield

$$\frac{d}{dt}\mathcal{V}(t) = \sum_{j=-1}^{1} \operatorname{Im}\int_{\mathbb{R}^d} \bar{\psi}_j(t,x)\nabla\mathtt{V}(x)\cdot\nabla\psi_j(t,x)dx.$$

Since derivatives of order at least two of $V_{\rm ext}$ are bounded, there exists $C > 0$ such that

$$|\nabla V_{\rm ext}(x)| \leqslant C\left(1 + |x|\right), \quad \forall x \in \mathbb{R}^d.$$

We infer

$$\frac{d}{dt}\mathcal{E}\left(\Psi(t)\right) \leqslant C\sum_{j=-1}^{1}\int_{\mathbb{R}^d}(1+|x|)\left|\bar{\psi}_j(t,x)\nabla\psi_j(t,x)\right| dx \leqslant C\sum_{j=-1}^{1} \|\psi_j(t)\|_{\widehat{\Sigma}}^2,$$

where the last estimate stems from Cauchy-Schwarz inequality. In view of (3.3), this rules out the existence of such a T_+, and the solution is global. $\square$

3.1. **Virial.** In this section, we show that without the assumption $\beta_s, \beta_n \geqslant 0$, finite time blow-up may occur. Set

$$\mathcal{V}(t) := \sum_{j=-1}^{1}\int_{\mathbb{R}^d} |x|^2|\psi_j(t,x)|^2 dx. \tag{3.4}$$

From the above computations, we have

$$\dot{\mathcal{V}}(t) = 2\sum_{j=-1}^{1} \operatorname{Im}\int_{\mathbb{R}^d} \bar{\psi}_j(t,x) x\cdot\nabla\psi_j(t,x)dx.$$

Therefore,

$$\ddot{\mathcal{V}}(t) = 2\sum_{j=-1}^{1} \operatorname{Im}\int_{\mathbb{R}^d}\left(\partial_t\bar{\psi}_j(t,x)x\cdot\nabla\psi_j(t,x) + \bar{\psi}_j(t,x)x\cdot\nabla\partial_t\psi_j(t,x)\right)dx.$$

Integrating by parts, we have

$$\begin{aligned}\operatorname{Im}\int_{\mathbb{R}^d}\bar\psi_j(t,x)x\cdot\nabla\partial_t\psi_j(t,x)dx &= \sum_{k=1}^d \operatorname{Im}\int_{\mathbb{R}^d}\bar\psi_j(t,x)x_k\partial_k\partial_t\psi_j(t,x)dx\\ &= -\sum_{k=1}^d \operatorname{Im}\int_{\mathbb{R}^d}\partial_t\psi_j(t,x)\partial_k\left(x_k\bar\psi_j(t,x)\right)dx\\ &= -\operatorname{Im}\int_{\mathbb{R}^d}\partial_t\psi_j(t,x)x\cdot\nabla\bar\psi_j(t,x)dx\\ &\quad - d\operatorname{Im}\int_{\mathbb{R}^d}\bar\psi_j(t,x)\partial_t\psi_j(t,x)dx\end{aligned}$$

Summing over j, we infer

$$\ddot{\mathcal{V}}(t) = -2\sum_{j=-1}^1 \operatorname{Im}\int_{\mathbb{R}^d}\partial_t\psi_j(t,x)\left(2x\cdot\nabla\bar\psi_j(t,x)+d\bar\psi_j(t,x)\right)dx.$$

Lemma 3.7. *Let $\Psi\in C(I;\widehat{\Sigma})^3$ given by Proposition 3.4, for some time interval $I\ni 0$. The virial $\mathcal{V}$, defined by* (3.4)*, satisfies*

$$\begin{aligned}\ddot{\mathcal{V}}(t) &= 2\sum_{j=-1}^1\|\nabla\psi_j\|_{L^2}^2 - 2\sum_{j=-1}^1\int|\psi_j|^2x\cdot\nabla V_{\text{ext}}(x)dx + d\beta_n\int\left(\sum_{j=-1}^1|\psi_j|^2\right)^2dx\\ &\quad + d\beta_s\int\left(\left(|\psi_{-1}|^2-|\psi_1|^2\right)^2+2|\psi_0|^2\left(|\psi_{-1}|^2+|\psi_1|^2\right)+4\operatorname{Re}\left(\bar\psi_{-1}\psi_0^2\bar\psi_1\right)\right)dx\\ &\quad - 4\operatorname{Re}\int x\cdot\nabla B\left(\psi_0\bar\psi_1+\psi_{-1}\bar\psi_0\right)dx,\end{aligned}$$

which can also be written, in terms of the total (conserved) energy,

$$\begin{aligned}\ddot{\mathcal{V}}(t) &= 4E(\Psi) - \sum_{j=-1}^1\int|\psi_j|^2\left(V_{\text{ext}}+2x\cdot\nabla V_{\text{ext}}(x)\right)dx\\ &\quad + \beta_n(d-2)\int\left(\sum_{j=-1}^1|\psi_j|^2\right)^2dx\\ &\quad + \beta_s(d-2)\int\left(\left(|\psi_{-1}|^2-|\psi_1|^2\right)^2+2|\psi_0|^2\left(|\psi_{-1}|^2+|\psi_1|^2\right)+4\operatorname{Re}\left(\bar\psi_{-1}\psi_0^2\bar\psi_1\right)\right)dx\\ &\quad - 2\operatorname{Re}\int\left(B+2x\cdot\nabla B\right)\left(\psi_0\bar\psi_1+\psi_{-1}\bar\psi_0\right)dx.\end{aligned}$$

We readily infer that if $\beta_n,\beta_s<0$, and $d=2$ or 3, one may have $E(\Psi)<0$ and, in the case where V_{ext} is harmonic, (1.2),

$$\ddot{\mathcal{V}}(t)\leqslant 4E(\Psi)-2\operatorname{Re}\int\left(B+2x\cdot\nabla B\right)\left(\psi_0\bar\psi_1+\psi_{-1}\bar\psi_0\right)dx.$$

With $B\in W^{1,\infty}$,

$$\begin{aligned}\ddot{\mathcal{V}}(t) &\leqslant 4E(\Psi)+\int\left(\|B\|_{L^\infty}+|x|\|\nabla B\|_{L^\infty}\right)|\psi_0\bar\psi_1+\psi_{-1}\bar\psi_0|dx\\ &\leqslant 4E(\Psi)+\left(\|B\|_{L^\infty}\|\psi_0\|_{L^2}+\|\nabla B\|_{L^\infty}\|x\psi_0\|_{L^2}\right)\left(\|\psi_1\|_{L^2}+\|\psi_{-1}\|_{L^2}\right)\\ &\leqslant 4E(\Psi)+\left(\|B\|_{L^\infty}\|\psi_0\|_{L^2}+\|\nabla B\|_{L^\infty}\sqrt{\mathcal{V}(t)}\right)\left(\|\psi_1\|_{L^2}+\|\psi_{-1}\|_{L^2}\right).\end{aligned}$$

From this relation, it is clear that this inequality may lead to the cancellation of $\mathcal{V}$, which is not possible in the case of a solution with $\widehat{\Sigma}$ regularity, for which the total mass is conserved. Since the main goal of this paper is to study the orbital stability of standing waves in the case $\beta_s, \beta_n \geqslant 0$, we shall not pursue on the possibility of finite time blow-up.

4. ORBITAL STABILITY

As in the previous sections, we suppose that $\beta_s, \beta_n > 0$ but $B \equiv 0$.

Theorem 4.1. *Let $M, N > 0$.*

(1) *$I_{M,N} = \widehat{I}_{M,N}, Z_{M,N} \neq \emptyset$ and $Z_{M,N}$ is stable.*
(2) *For any $z = (z_{-1}, z_0, z_1) \in Z_{M,N}$, we have $|z| = (|z_{-1}|, |z_0|, |z_1|) \in W_{M,N}$ and*

$$Z_{M,N} = \{e^{i\theta_1} w_{-1}, e^{i\theta_2} w_0, e^{i\theta_3} w_1; \quad w \in W_{M,N},\ (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3\}$$

(w_{-1}, w_0, w_1) is a solution of (1.11) and if

$$w_{-1} = |w_{-1}|e^{i\theta_{-1}}, w_0 = |w_0|e^{i\theta_0} \textit{ and } w_1 = |w_1|e^{i\theta_1}$$

then

$$\begin{cases} 2\theta_0 - \theta_1 - \theta_{-1} = \pi & \textit{mod } 2\pi, \\ \theta_0 - \theta_1 = \pi & \textit{mod } 2\pi, \\ -\theta_0 + \theta_{-1} = \pi & \textit{mod } 2\pi. \end{cases} \tag{4.1}$$

Proof. (1) As suggested in [8], to prove the orbital stability of standing waves of (1.1), it suffices to prove that $Z_{M,N} \neq \emptyset$ and any minimizing sequence,

$$z_n = (z_{n,-1}, z_{n,0}, z_{n,1}) \in \widehat{\Sigma}^3 \text{ such that } z_n \in \widehat{C}_{M,N} \text{ and } \widehat{E}(z_n) = \widehat{I}_{M,N} \tag{4.2}$$

is relatively compact in $\widehat{\Sigma}^3$. Consider a sequence such that $z_n \in \widehat{C}_{M,N}$ and $\widehat{E}(z_n) \to \widehat{I}_{M,N}$. The first goal is to show that $\{z_n\}$ has a subsequence which is convergent in $\widehat{\Sigma}^3$. By Lemma 2.3, we see that z_n is bounded in $\widehat{\Sigma}^3$. Therefore passing to a subsequence, we can suppose that:

$$u_{n,j} \rightharpoonup u_j \text{ and } v_{n,j} \rightharpoonup v_j \text{ in } \Sigma, -1 \leqslant j \leqslant 1. \tag{4.3}$$

Now let $\rho_{n,j} = |z_{n,j}| = (u_{n,j}^2 + v_{n,j}^2)^{1/2}$, $z_{n,j} = \rho_{n,j} e^{i\theta_{n,j}}$, $-1 \leqslant j \leqslant 1$. It follows that $\rho_{n,j} \in \Sigma$ and that for all $n \in \mathbb{N}$ and $-1 \leqslant j \leqslant 1, 1 \leqslant k \leqslant d$,

$$\partial_k \rho_{n,j}(x) = \begin{cases} \dfrac{u_{n,j}(x)\partial_k u_{n,j}(x) + v_{n,j}(x)\partial_k v_{n,j}(x)}{(u_{n,j}(x)^2 + v_{n,j}(x)^2)^{1/2}} \text{ if } u_{n,j}(x)^2 + v_{n,i}(x)^2 > 0 \\ \qquad 0 \text{ otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \widehat{E}(z_n) - E(\rho_n) = &\frac{1}{2} \left(\|\nabla z_n\|_2^2 - \|\nabla |z_n|\|_2^2 \right) \\ &+ 2\beta_s \int_{\mathbb{R}^d} |z_{n,1}||z_{n,0}|^2 |z_{n,-1}| (\cos(2\theta_{n,0} - \theta_{n,-1} - \theta_{n,1}) + 1) \\ &+ 2 \int_{\mathbb{R}^d} |z_{n,1}||z_{n,0}| \left(\cos(\theta_{n,0} - \theta_{n,1}) + 1\right) \\ &+ 2 \int_{\mathbb{R}^d} |z_{n,-1}||z_{n,0}| \left(\cos(\theta_{n,-1} - \theta_{n,0}) + 1\right). \end{aligned} \tag{4.4}$$

The triangle inequality

$$\Big||z_n(t,x+h)|-|z_n(t,x)|\Big| \leqslant |z_n(t,x+h)-z_n(t,x)|$$

implies

$$\|\nabla z_n\|_2^2-\|\nabla|z_n|\|_2^2 \geqslant 0. \tag{4.5}$$

Therefore $\widehat{E}(z_n)-E(\rho_n)\geqslant 0$, hence $\widehat{I}_{M,N}=\lim_{n\to\infty}\widehat{E}(z_n)\geqslant \limsup E(\rho_n)$.
Taking into account that

$$\|z_{n,j}\|_2^2=|\rho_{n,j}|_2^2=:c_{n,j}^2,\quad \forall\ -1\leqslant j\leqslant 1,$$

and that $\widehat{I}_{M,N}\leqslant I_{M,N}$, it follows by using Lemma 2.3 that

$$\liminf E(\rho_n)\geqslant \liminf I_{M_n,N_n}\geqslant I_{M,N}\geqslant \widehat{I}_{M,N},$$
$$M_n=c_{n,1}^2+c_{n,0}^2+c_{n,-1}^2 \text{ and } N_n=c_{n,1}^2-c_{n,-1}^2.$$

Therefore

$$\lim_{n\to\infty}E(\rho_n)=\lim_{n\to\infty}\widehat{E}(z_n)=I_{M,N}=\widehat{I}_{M,N}. \tag{4.6}$$

Now on the other hand (4.5) implies that for any $-1\leqslant j\leqslant 1$ we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}|\nabla u_{n,j}|^2+|\nabla v_{n,j}|^2-|\nabla(u_{n,j}^2+v_{n,j}^2)^{1/2}|^2=0. \tag{4.7}$$

Thus:

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}|\nabla u_{n,j}|^2+|\nabla v_{n,j}|^2=\lim_{n\to\infty}\int_{\mathbb{R}^d}|\nabla(u_{n,j}^2+v_{n,j}^2)^{1/2}|^2, \tag{4.8}$$

which is equivalent to saying that

$$\lim_{n\to\infty}\|\nabla z_n\|_2^2=\lim_{n\to\infty}\|\nabla|z_n|\|_2^2. \tag{4.9}$$

Using (4.5)–(4.6) and Lemma 2.3, $\rho_n=(\rho_{n,-1},\rho_{n,0},\rho_{n,1})$ is relatively compact in Σ^3. Thus there exist $\rho_{-1},\rho_0,\rho_1\in\Sigma$ such that

$$\begin{cases}(u_{n,j}^2+v_{n,j}^2)^{1/2} \text{ converges to } \rho_j \text{ in } \Sigma, \quad -1\leqslant j\leqslant 1,\\ |\rho_i|_2=c_j,\ \text{with} \displaystyle\sum_{j=-1}^{1}c_j^2=M \text{ and } c_1^2-c_{-1}^2=N,\\ E(\rho_{-1},\rho_0,\rho_1)=I_{M,N}.\end{cases} \tag{4.10}$$

Closely following [10], we will prove that $\rho_j=|z_j|=(u_j^2+v_j^2)^{1/2}$. By (4.3), we know that $u_{n,j}\to u_j$ and $v_{n,j}\to v_j$ in $L^2(B(0,R))$ we can easily deduce that

$$[(u_{n,j}^2+v_{n,j}^2)^{1/2}-(u_j^2+v_j^2)^{1/2}]^2\leqslant |u_{n,j}-u_j|^2+|v_{n,j}-v_j|^2,$$

thus $(u_{n,j}^2+v_{n,j}^2)^{1/2}\to\rho_j$ in $L^2(B(0,R))$, this certainly implies that $|z_j|=\rho_j\ \forall\ -1\leqslant j\leqslant 1$. On the other hand

$$\|z_{n,j}\|_2=\||z_{n,j}|\|_2\to c_j=\|z_j\|_2=\||z_j|\|_2.$$

The proof of the first part of Theorem 4.1 is now complete if we show that

$$\lim_{n\to\infty}\|\nabla z_{n,j}\|_2^2=\|\nabla z_j\|_2^2,\quad -1\leqslant j\leqslant 1.$$

From (4.7), we have

$$\lim_{n\to\infty}\|\nabla z_{n,j}\|_2=\lim_{n\to\infty}\||\nabla z_{n,j}|\|_2,$$

and

$$\lim_{n\to\infty} |\nabla|z_{n,j}||_2 = |\nabla|z_j||_2.$$

Thus by the lower semi-continuity of $|\cdot|_2$, we have

$$\|\nabla z_i\|_2^2 \leqslant \lim \|\nabla z_i\|_2 = \lim |\nabla|z_{n,j}||_2^2 = |\nabla|z_j||_2^2. \tag{4.11}$$

Finally, replacing $z_{n,j}$ by z_j in (4.5), we see that

$$\|\nabla z_j\|_2^2 \geqslant \||\nabla|z_j|\|_2^2 \quad \forall\ -1 \leqslant j \leqslant 1.$$

Now using (4.3), we know that $z_{n,j} \to z_j$ in $\widehat{\Sigma}$.

(2) Let $z = (z_1, z_0, z_1) \in Z_{M,N}$ with

$$z_j = |z_j|e^{i\theta_j} = (u_j, v_j), \quad -1 \leqslant j \leqslant 1.$$

By (4.5), we know that for all $-1 \leqslant j \leqslant 1$, $1 \leqslant k \leqslant d$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{(u_j\partial_k v_j - v_j\partial_k u_j)^2}{u_j^2+v_j^2}dx = 0,\\
&2\theta_0 - \theta_1 - \theta_{-1} = \pi \mod 2\pi,\\
&\theta_0 - \theta_1 = \pi \mod 2\pi,\\
&-\theta_0 + \theta_{-1} = \pi \mod 2\pi.
\end{aligned}$$

On the other hand $\widehat{E}(z) = \widehat{I}_{M,N}$, which implies that there exists a Lagrange multiplier $\alpha \in \mathbb{C}$ such that

$$\widehat{E}'(z)\xi = \frac{\alpha}{2}\sum_{i=-1}^{1} z_i\bar{\xi}_i + \xi_i\bar{z}_i \quad \text{for all } \xi \in \mathbb{C}^3.$$

By elementary regularity theory and maximum principle, we can prove that u_j and $v_j \in C^1(\mathbb{R}^d) \cap \Sigma$ and $\rho > 0$ (see Section 4 of [14]).

Set $\Omega_j = \{x \in \mathbb{R}^2 : u_j(x) = 0\}$. Then Ω_j is closed since u_j is continuous. Let us prove that it is also open. Let $x_0 \in \Omega_j$. Using the fact that $v_j(x_0) > 0$, we can find a ball B centered in x_0 such that $v_j(x) \neq 0$ for any $x \in B$. Consequently for $-1 \leqslant j \leqslant 1$, and $1 \leqslant k \leqslant d$,

$$\forall x \in B, \quad \left(\frac{(u_j\partial_k v_j - v_j\partial_k v_j)^2}{u_j^2+v_j^2}\right) = \left(\left(\partial_k\left(\frac{u_j}{v_j}\right)\right)^2 \frac{v_j^4}{u_j^2+v_j^2}\right).$$

This implies that

$$\int_B \left|\nabla\left(\frac{u_j}{v_j}\right)\right|^2 \frac{v_j^4}{u_j^2+v_j^2} = 0. \tag{4.12}$$

Hence $\nabla(\frac{u_j}{v_j}) = 0$ on B, and $\exists\ c$ such that $\frac{u_j}{v_j} = c$ on B. Since $x_0 \in B$, we infer $c \equiv 0$. Therefore Ω_j is also an open set of $\mathbb{R}^d$. Hence we have proved that for $-1 \leqslant ij \leqslant 1$, we have these two alternatives

$$\begin{aligned}
&A1 : u_j \equiv 0 \text{ or } u_j > 0 \quad \text{on } \mathbb{R}^d,\\
&A2 : v_j \equiv 0 \text{ or } v_j > 0 \quad \text{on } \mathbb{R}^d.
\end{aligned}$$

Now let $z_j = e^{i\sigma_j}w_j, \sigma \in \mathbb{R}, w_j \in W_{M,N}$:

$$\sum_{j=-1}^{1} |z_j|_2^2 = N, \quad \text{and} \quad |z_1|_2^2 - |z_{-1}|_2^2 = M.$$

Obviously

$$\{e^{i\sigma_j} w_j \ ; \ \sigma_j \in \mathbb{R}, w \in W_{M,N}\} \subset Z_{M,N}.$$

Conversely for $z_j = (u_j, v_j)$ such that $(z_{-1}, z_0, z_1) \in Z_{M,N}$, $\widehat{E}(z_1, z_0, z_1) = \widehat{E}(z) = \widehat{I}_{M,N}$. with $(w_{-1}, w_0, w_1) \in W_{M,N}$. Now we have six possible alternatives. We discuss one them in details.

Suppose that v_1, v_0 and $v_{-1} \neq 0$ for all $x \in \mathbb{R}^d$. In this case, it follows that $\nabla(\frac{u_j}{v_j}) = 0$ on $\mathbb{R}^d$. Thus we can find three constants K_{-1}, K_0, K_1 such that

$$u_j = K_j v_j, \quad -1 \leqslant j \leqslant 1.$$

Therefore $w_j = (K_j + i)v_j$. Let $\theta_j \in \mathbb{R}$ such that $K_j + i = |K_j + i|e^{i\theta_j}$ and let $\eta_j = 0$ if $v_j > 0$ and $\eta_j = \pi$ if $v_j < 0$ on $\mathbb{R}^d$. Setting, $\sigma_j = \theta_j + \eta_j$, we infer

$$z_j = (K_j + i)v_j = |K_j + i|e^{i\theta_j}|v_j|e^{i\eta_j} = w_j e^{i\sigma_j}.$$

□

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