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# COMPLEMENTARY STUDY OF THE STANDING WAVE SOLUTIONS OF THE GROSS-PITAEVSKII EQUATION IN DIPOLAR QUANTUM GASES

RÉMI CARLES AND HICHEM HAJAJEJ

ABSTRACT. We study the stability of the standing wave solutions of a Gross-Pitaevskii equation describing Bose-Einstein condensation of dipolar quantum gases and characterize their orbit. As an intermediate step, we consider the corresponding constrained minimization problem and establish existence, symmetry and uniqueness of the ground state solutions.

## 1. INTRODUCTION

Since the experimental realization of the first Bose-Einstein condensate (BEC) by Eric Cornell and Carl Wieman in 1995, tremendous efforts have been undertaken by mathematicians to exploit this achievement especially in atomic physics and optics. In the last years, a new kind of quantum gases with dipolar interaction, which acts between particles as a permanent magnetic or electric dipole moment has attracted the attention of a lot of scientists. The interactions between particles are both long-range and non-isotropic. Describing the corresponding BEC via Gross Pitaevskii approximation, one gets the following nonlinear Schrödinger equation

$$(1.1) \quad i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + g|\psi|^2\psi + d^2(K * |\psi|^2)\psi + V(x)\psi, \quad t \in \mathbb{R}, x \in \mathbb{R}^3,$$

where  $|g| = \frac{4\pi\hbar^2 N|a|}{m}$ ,  $N \in \mathbb{N}$  is the number of particles,  $m$  denotes the mass of individual particles and  $a$  its corresponding scattering length. The external potential  $V(x)$  describes the electromagnetic trap and has the following harmonic confinement

$$V(x) = \frac{|x|^2}{2}.$$

The factor  $d^2$  denotes the strength of the dipole moment in Gaussian units and

$$(1.2) \quad K(x) = \frac{1 - 3\cos^2\theta}{|x|^3},$$

where  $\theta = \theta(x)$  is the angle between  $x \in \mathbb{R}^3$  and the dipole axis  $n \in \mathbb{R}^3$ . The local term  $g|\psi|^2\psi$  describes the short-range interaction forces between particles, while the non-local potential  $K * |\psi|^2$  describes their long-range dipolar interactions.

For the mathematical analysis, it is more convenient to rescale (1.1) into the following dimensionless form

$$(1.3) \quad i\partial_t\psi + \frac{1}{2}\Delta\psi = \frac{|x|^2}{2}\psi + \lambda_1|\psi|^2\psi + \lambda_2(K * |\psi|^2)\psi,$$

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where

$$\lambda_1 = \frac{4\pi aN}{a_0}, \quad \lambda_2 = \frac{d^2}{\hbar w_0 a_0^3}, \quad \text{and} \quad a_0 = \sqrt{\frac{\hbar}{m}}.$$

In the following, we assume that  $\lambda_1$  and  $\lambda_2$  are two given real-valued parameters.

In [5], the authors have studied the existence and uniqueness of the equation (1.3) with initial condition  $\psi_0 \in H^1(\mathbb{R}^3)$ ,

$$(1.4) \quad i\partial_t \psi + \frac{1}{2}\Delta \psi = \frac{|x|^2}{2}\psi + \lambda_1 |\psi|^2 \psi + \lambda_2 (K * |\psi|^2)\psi; \quad \psi(0, x) = \psi_0(x).$$

They have established that (1.4) has a unique, global solution if  $\lambda_1 \geq \frac{4}{3}\pi\lambda_2 \geq 0$ . They called this situation *stable regime*, referring to the fact that no singularity is formed in finite time. In this paper, we study another notion of stability, that is, the stability of standing waves. They have also showed that in the *unstable regime* ( $\lambda_1 < \frac{4}{3}\pi\lambda_2$ ), finite time blow up may occur, hence the denomination.

The evidence of blow-up relies on a function for which the corresponding energy is strictly negative ([5, Lemma 5.1]). They concluded using the virial approach of Zakharov and Glassey. Some refinements of the above result have been discussed in [5, Proposition 5.4].

The most important issue in view of the applications of (1.4) in atomic physics and quantum optics seems to be the study of ground state solutions of (1.4). These solutions are the “only” observable states in experiments. A standing wave solution of (1.4) is a wave function having the particular form  $\psi(t, x) = e^{i\mu t}u(x)$ ,  $\mu \in \mathbb{R}$ . Therefore  $\psi(t, x)$  is a solution of (1.1) if and only if  $u$  solves the following elliptic partial differential equation:

$$(1.5) \quad -\frac{1}{2}\Delta u + \lambda_1 |u|^2 u + \lambda_2 (K * u^2)u + \frac{|x|^2}{2}u + \mu u = 0.$$

Ground state solutions are the solutions of (1.5) obtained by minimizing an associated energy functional. The most common way to get such minimizers is to consider the corresponding Weinstein functional or the constrained energy functional.

In [1], the authors have studied (1.5) (without the term  $V(x)$ ) by using the first approach. More precisely, they introduced the following minimization problem:

$$(1.6) \quad \inf_{v \in H^1(\mathbb{R}^3)} J(v), \quad \text{where} \quad J(v) = \frac{\|\nabla v\|_{L^2}^3 \|v\|_{L^2}}{-\lambda_1 \|v\|_{L^4}^4 - \lambda_2 \langle K * |v|^2, |v|^2 \rangle},$$

and  $\langle \cdot, \cdot \rangle$  is the scalar product in  $L^2(\mathbb{R}^3)$ .

Using various tricks, they were able to show that (1.6) is achieved when  $\lambda_1 < \frac{4}{3}\pi\lambda_2$  if  $\lambda_2 > 0$  and  $\lambda_1 < -\frac{8\pi}{3}\lambda_2$  if  $\lambda_2 < 0$ . They then deduced the main result of their paper ([1, Theorem 1.1]), which we recall for the convenience of the reader.

**Theorem 1.1** (Antonelli–Sparber [1]). *Let  $\lambda_1, \lambda_2 \in \mathbb{R}$  be such the following condition holds*

$$(1.7) \quad \lambda_1 < \begin{cases} \frac{4\pi}{3}\lambda_2 & \text{if } \lambda_2 > 0, \\ -\frac{8\pi}{3}\lambda_2 & \text{if } \lambda_2 < 0. \end{cases}$$

*Then there exists a non-negative function  $u \in H^1(\mathbb{R}^3)$  solution to*

$$-\frac{1}{2}\Delta u + \lambda_1 |u|^2 u + \lambda_2 (K * u^2)u + \mu u = 0, \quad \mu > 0.$$

Note that there is no contradiction with Proposition 4.1 and Lemma 5.1 of [5], since the solitary wave constructed in [1] corresponds to an initial data with a positive energy, while finite time blow up is established in [5] only for negative energy solutions. A complete analysis of such situations has been done in [10] and [11]. The second variational formulation associated to (1.5) is the following constrained minimization problem

$$(1.8) \quad I_c = \inf \{E(u) ; u \in S_c\},$$

where

$$(1.9) \quad E(u) = \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \int_{\mathbb{R}^3} |x|^2 |u|^2 + \frac{\lambda_1}{2} \|u\|_{L^4(\mathbb{R}^3)}^4 + \frac{\lambda_2}{2} \int_{\mathbb{R}^3} (K * |u|^2)^2 |u|^2,$$

and

$$(1.10) \quad S_c = \left\{ u \in \Sigma : \int_{\mathbb{R}^3} u^2 = c^2 \right\},$$

with

$$\Sigma = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |x|^2 |u(x)|^2 dx < \infty \right\}.$$

According to the breakthrough paper of Grillakis, Shatah and Strauss [8], the stable solutions of (1.5) are the ones obtained via the variational problem (1.8). In [1], the authors seem to be very skeptical concerning the use of such approach in this context. In [1, p. 427], after the introduction of the energy functional they stated ‘‘At this point it might be tempting to study (1.5) via minimization of the energy  $E(u)$ . However, it is well known, that even without the dipole nonlinearity, i.e.  $\lambda_2 = 0$ , this approach fails (...)’’. Note however that in [1], (1.1) is considered in the absence of an external potential,  $V = 0$ . A key aspect in our approach consists in using a balance between both nonlinear terms, the cubic one and the dipolar one. Also, the presence of the confining potential  $V$  (not necessarily quadratic, see below) seems to be extremely helpful in the proof, although it is not clear whether it is necessary or not.

Their feelings have been reinforced by the approach of Bao et al. in [2], which, however, contains some flaws, which we fix in the present paper. Note also that our method is simpler and applies to any potential  $V(|x|)$  which increases to infinity when  $|x|$  tends to infinity (the radial symmetry of the potential is needed in order to ensure that the minimizer is Steiner symmetric). Their main result, which we revisit here, can be stated as follows:

**Theorem 1.2.** *A) If*

$$(1.11) \quad \begin{cases} \lambda_2 > 0 \text{ and } \lambda_1 \geq \frac{4\pi}{3} \lambda_2, \\ \text{or} \\ \lambda_2 < 0 \text{ and } \lambda_1 \geq -\frac{8\pi}{3} \lambda_2, \end{cases}$$

*then (1.8) has a unique non-negative minimizer, which is Steiner symmetric.*

*B) If*

$$(1.12) \quad \begin{cases} \lambda_2 > 0 \text{ and } \lambda_1 < \frac{4\pi}{3} \lambda_2, \\ \text{or} \\ \lambda_2 < 0 \text{ and } \lambda_1 < -\frac{8\pi}{3} \lambda_2, \end{cases}$$

then  $I_c = -\infty$ .

However, the proof of B) in [2] contains a flaw, which we fix here.

From now on, we suppose that  $\lambda_2 > 0$ . The case  $\lambda_2 < 0$  can be treated in the same fashion.

Note that the range of  $\lambda_1$  and  $\lambda_2$  ensuring the existence of minimizers via Weinstein function does not intersect at all with the one enabling us to get minimizers of (1.8).

Our paper is organized as follows. In the next section, we fix some notations and state some preliminary results. In Section 3, we prove Theorem 1.2. Finally, in the last section, we prove the orbital stability of standing waves when (1.11) holds true. We also characterize the orbit of standing waves.

## 2. PRELIMINARIES

**2.1. Notations.** The space  $L^p(\mathbb{R}^3)$ , denoted by  $L^p$  for shorthand, is equipped with the norm  $|\cdot|_p$ . For  $w = (u, v) \in L^p \times L^p$ , we set  $\|w\|_p^p = |u|_p^p + |v|_p^p$ . Similarly if

$$w = (u, v) \in H^1 \times H^1, \|w\|_{H^1}^2 := \|w\|_{H^1}^2 + \|\nabla w\|_{H^1}^2,$$

with

$$\|\nabla w\|_{H^1}^2 = \|\nabla u\|_{H^1}^2 + \|\nabla v\|_{H^1}^2.$$

Recall that

$$\Sigma = \{u \in H^1, |u|_\Sigma^2 := |xu|_{L^2}^2 + \|\nabla u\|_2^2 + |u|_2^2 < \infty\}.$$

We set  $\tilde{\Sigma} = \Sigma \times \Sigma$ , equipped with the norm given by

$$\|w\|_{\tilde{\Sigma}}^2 = \|(u, v)\|_{\tilde{\Sigma}}^2 := |u|_\Sigma^2 + |v|_\Sigma^2.$$

For  $w \in \tilde{\Sigma}$ , we define

$$(2.1) \quad \tilde{E}(w) = \frac{1}{2} \|\nabla w\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} |x|^2 |w|^2 dx + \frac{\lambda_1}{2} \|w\|_4^4 + \frac{\lambda_2}{2} \int_{\mathbb{R}^3} (K * |w|^2) |w|^2 dx.$$

Equivalently for all  $c > 0$ , we set

$$(2.2) \quad \begin{aligned} \tilde{I}_c &= \inf\{\tilde{E}(w) : w \in \tilde{\Sigma}, \|w\|_2^2 = c^2\}, \\ \tilde{S}_c &= \{w \in \tilde{\Sigma}, \|w\|_2^2 = c^2\}, \\ Z_c &= \{w \in \tilde{\Sigma} : \|w\|_2^2 = c^2 \text{ and } \tilde{E}(w) = \tilde{I}_c\}, \\ W_c &= \{u \in \Sigma \cap C^1(\mathbb{R}^3) : E(u) = I_c, |u|_2^2 = c^2 \text{ and } u > 0\}. \end{aligned}$$

**2.2. Technical results.** We first recall two important properties of the dipole established in [5].

**Lemma 2.1** (Lemma 2.1 from [5]). *The operator  $\mathcal{K} : u \mapsto K * u$  can be extended as a continuous operator on  $L^p(\mathbb{R}^3)$  for all  $1 < p < \infty$ .*

**Lemma 2.2** (Lemma 2.3 from [5]). *Define the Fourier transform on the Schwartz space as*

$$\mathcal{F}u(\xi) \equiv \hat{u}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} u(x) dx, \quad u \in \mathcal{S}(\mathbb{R}^3).$$

*Then the Fourier transform of  $K$  is given by*

$$(2.3) \quad \hat{K}(\xi) = \frac{4\pi}{3} \left( 3 \frac{\xi_3^2}{|\xi|^2} - 1 \right) = \frac{4\pi}{3} \left( \frac{2\xi_3^2 - \xi_1^2 - \xi_2^2}{|\xi|^2} \right) \in \left[ -\frac{4\pi}{3}, \frac{8\pi}{3} \right].$$

Using Fourier transform and Plancherel's Theorem, we can rewrite the energy functional as

$$(2.4) \quad E(u) = \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2}|xu|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} \left( \lambda_1 + \lambda_2 \widehat{K}(\xi) \right) |\widehat{\rho}(\xi)|^2 d\xi,$$

where  $\rho(x) = |u(x)|^2$ .

Since  $V(x) \rightarrow +\infty$  as  $|x| \rightarrow \infty$ , we have the standard result:

**Lemma 2.3.** *For all  $p \in [2, 6)$ , the embedding  $\Sigma \hookrightarrow L^p(\mathbb{R}^3)$  is compact.*

Proceeding as in [9], we have:

**Lemma 2.4.** (1) *The energy functional  $E$  and  $\widetilde{E}$  are  $C^1$  on  $\Sigma$  and  $\widetilde{\Sigma}$ , respectively.*  
(2) *The mapping  $c \mapsto I_c$  is continuous.*

**2.3. Cauchy problem.** We shall consider the initial value problem (1.4) in two situations: either  $\psi$  is a scalar function, or  $\psi = (\psi_1 \psi_2)$  is a vector function. In the second case, (1.4) means

$$i\partial_t \psi_j + \frac{1}{2}\Delta \psi_j = \frac{|x|^2}{2} \psi_j + \lambda_1 (|\psi_1|^2 + |\psi_2|^2) \psi_j + \lambda_2 (K * (|\psi_1|^2 + |\psi_2|^2)) \psi_j, \quad j = 1, 2,$$

along with the initial condition  $\psi(0, x) = \psi_0(x)$ . The main technical remark concerning the Cauchy problem for (1.4), made in [5], is that in view of Lemma 2.1, the operator  $u \mapsto (K * |u|^2)u$  is continuous from  $L^4(\mathbb{R}^3)$  to  $L^{4/3}(\mathbb{R}^3)$ . Therefore, on a technical level, it is not really different from considering a cubic nonlinearity, for which the local existence theory at the level of  $\Sigma$  follows from Strichartz inequalities and a fixed point argument (see e.g. [6]). Note that because of the presence of the harmonic potential, working in  $H^1(\mathbb{R}^3)$  is not enough to ensure local well-posedness: working in  $\Sigma$  is necessary if one wants to consider a solution which remains in  $H^1(\mathbb{R}^3)$  ([4]). Standard arguments (which can also be found in [6]) imply the conservations of mass and energy.

**Proposition 2.5.** *Let  $\lambda_1, \lambda_2 \in \mathbb{R}$ , and  $\psi_0 \in \Sigma$ . There exists  $T$ , depending on  $\|\psi_0\|_\Sigma$  and a unique solution*

$$\psi \in C([-T, T]; \Sigma), \quad \text{with } \psi, x\psi, \nabla \psi \in L^{8/3}([-T, T]; L^4(\mathbb{R}^3))$$

to (1.4). *The following quantities are conserved by the flow:*

$$\begin{aligned} \text{Mass:} & \quad |\psi(t)|_2 = |\psi_0|_2, \quad \forall t \in [-T, T]. \\ \text{Energy:} & \quad E(\psi(t)) = E(\psi_0), \quad \forall t \in [-T, T]. \end{aligned}$$

*In particular, if  $\lambda_1 \geq \frac{4\pi}{3}\lambda_2 \geq 0$  or if  $\lambda_1 \geq -\frac{8\pi}{3}\lambda_2 > 0$ , then  $T$  can be chosen arbitrarily large, and the solution is defined for all time.*

*If  $\psi_0 \in \widetilde{\Sigma}$ , the above conclusions remain true, up to replacing  $E$  with  $\widetilde{E}$ , along with other obvious modifications.*

**2.4. Stability.** For a fixed  $c > 0$ , we use the following definition of stability introduced by Cazenave and Lions [7].

**Definition 2.6.** *The set  $Z_c$  is said to be stable if  $Z_c \neq \emptyset$  and:*

*For all  $w \in Z_c$  and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $\psi_0 \in \widetilde{\Sigma}$ , we have*

$$\|\psi_0 - w\|_{\widetilde{\Sigma}} < \delta \Rightarrow \inf_{w \in Z_c} \|\psi(t, x) - w\|_{\widetilde{\Sigma}} < \varepsilon,$$

*where  $\psi(t, x)$  is the unique solution of (1.4), corresponding to the initial data  $\psi_0$ .*

Notice that if  $w \in Z_c$ , then there exists a Lagrange multiplier  $\lambda \in \mathbb{R}$  such that

$$-\frac{1}{2}\Delta w + \frac{|x|^2}{2}w + \lambda_1|w|^2w + \lambda_2(K * |w|)^2w + \lambda w = 0.$$

Therefore  $w = (w_1, w_2)$  solves the following elliptic system

$$(2.5) \quad \begin{cases} -\frac{1}{2}\Delta w_1 + \frac{1}{2}|x|^2w_1 + \lambda_1|w|^2w_1 + \lambda_2(K * |w|^2)w_1 + \lambda w_1 = 0, \\ -\frac{1}{2}\Delta w_2 + \frac{1}{2}|x|^2w_2 + \lambda_1|w|^2w_2 + \lambda_2(K * |w|^2)w_2 + \lambda w_2 = 0. \end{cases}$$

### 3. PROOF OF THEOREM 1.2

Let us first prove part A). Thanks to (2.4), the minimization problem (1.8) can be rewritten in the following manner

$$I_c = \inf \left\{ \frac{1}{2}(|\nabla u|_2^2 + |xu|_2^2) + \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2 \widehat{K}(\xi)) |\widehat{\rho}(\xi)|^2 d\xi; u \in S_c \right\}.$$

Now in view of (2.3) and (1.11),  $E(u) \geq 0$  for any  $u \in S_c$ .

Let  $\{u_n\} \subset \Sigma$  be such that  $|u_n|_2^2 \rightarrow c^2$  and  $\lim_{n \rightarrow \infty} E(u_n) = I_c$ . the above property implies that  $(u_n)$  is bounded in  $\Sigma$ , therefore, we can suppose (up to a subsequence) that  $u_n \rightharpoonup u$  in  $\Sigma$ . On the other hand, by the lower semi-continuity of the norm, we certainly have

$$(3.1) \quad |xu|_2^2 + |\nabla u|_2^2 \leq \liminf |\nabla u_n|_2^2 + |xu_n|_2^2,$$

$$(3.2) \quad \int_{\mathbb{R}^3} |u_n|^4 \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^3} |u|^4.$$

Finally using Lemma 2.1, we obtain that

$$(3.3) \quad \int_{\mathbb{R}^3} (K * |u_n|^2) u_n^2 \rightarrow \int_{\mathbb{R}^3} (K * |u|^2) u^2.$$

Relations (3.1), (3.2) and (3.3) imply that

$$E(u) \leq \liminf E(u_n) \rightarrow I_c.$$

We conclude that  $E(u) = I_c$ , since  $|u|_2 = c$ .

Now, taking into account the fact that  $|\nabla|u||_2 \leq |\nabla u|_2$ , we have  $E(|u|) \leq E(u)$  for any  $u \in H^1$ . Finally, using rearrangement inequalities established by F. Brock [3], we certainly get that  $E(|u|^\#) \leq E(|u|) \leq E(u)$ , where  $u^\#$  stands for the Steiner symmetrization with respect to the  $x_3$ -axis.

As proved in ([2, Lemma 2.1]), the energy  $E$  is strictly convex, and therefore the minimizer constructed above is unique.

*Remark 3.1.* All minimizing sequences of (1.8) are relatively compact in  $\Sigma$ .

Now, let us prove part B) of Theorem 1.2. To reach this goal, we need to construct an appropriate sequence of functions ensuring that  $I_c = -\infty$ . In doing so, we fix a flaw in the proof of [2]. Let  $f_1 \in C_0^\infty(\mathbb{R}^2)$  and  $f_2 \in C_0^\infty(\mathbb{R})$  be such that

$$\int_{\mathbb{R}^3} f_1(x_1, x_2)^2 f_2(x_3)^2 dx = \left( \int_{\mathbb{R}^2} f_1^2 \right) \left( \int_{\mathbb{R}} f_2^2 \right) = c^2.$$

At this stage, the idea is to use anisotropy. For  $\varepsilon, h > 0$  to be made precise later, let

$$u(x) = \frac{1}{\varepsilon} f_1\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \frac{1}{\sqrt{h}} f_2\left(\frac{x_3}{h}\right), \quad x \in \mathbb{R}^3.$$

Then  $u \in S_c$ . For  $\rho = |u|^2$ , we have

$$\widehat{\rho}(\xi) = \mathcal{F}(|f_1|^2)(\varepsilon\xi_1, \varepsilon\xi_2)\mathcal{F}(|f_2|^2)(h\xi_3), \quad \xi \in \mathbb{R}^3.$$

We now measure the order of magnitude, as  $\varepsilon, h \rightarrow 0$  of each term in the energy, leaving out the precise value of positive multiplicative constants. We obviously have

$$\int |\nabla u|^2 \approx \frac{1}{\varepsilon^2} + \frac{1}{h^2} \quad \text{and} \quad \int |x|^2 |u|^2 \approx \varepsilon^2 + h^2.$$

Let  $w(\xi) = \lambda_1 + \lambda_2 \widehat{K}(\xi)$ ,  $\varphi = |\mathcal{F}(|f_1|^2)|^2$  and  $\psi = |\mathcal{F}(|f_2|^2)|^2$ . Then

$$\begin{aligned} \int_{\mathbb{R}^3} w(\xi) |\widehat{\rho}(\xi)|^2 d\xi &= \int_{\mathbb{R}^3} w(\xi) \varphi(\varepsilon\xi_1, \varepsilon\xi_2) \psi(h\xi_3) d\xi \\ &= \frac{1}{\varepsilon^2 h} \int w\left(\frac{\eta_1}{\varepsilon}, \frac{\eta_2}{\varepsilon}, \frac{\eta_3}{\varepsilon}\right) \varphi(\eta_1, \eta_2) \psi(\eta_3) d\eta. \end{aligned}$$

Now since  $w$  is homogeneous of degree 0,

$$w\left(\frac{\eta_1}{\varepsilon}, \frac{\eta_2}{\varepsilon}, \frac{\eta_3}{\varepsilon}\right) = \lambda_1 + \frac{4\pi}{3} \lambda_2 \frac{2\varepsilon^2 \eta_3^2 - h^2 \eta_1^2 - h^2 \eta_2^2}{h^2 \eta_1^2 + h^2 \eta_2^2 + \varepsilon^2 \eta_3^2}.$$

If  $h/\varepsilon \rightarrow +\infty$ , then

$$w\left(\frac{\eta_1}{\varepsilon}, \frac{\eta_2}{\varepsilon}, \frac{\eta_3}{\varepsilon}\right) \xrightarrow{\varepsilon, h \rightarrow 0} \lambda_1 - \frac{4}{3} \pi \lambda_2.$$

Now using the fact that  $\lambda_1 < \frac{4}{3} \pi \lambda_2$ ,  $\varphi$  and  $\psi$  are non-negative functions, we certainly have that

$$\int w(\xi) |\widehat{\rho}(\xi)|^2 d\xi \approx -\frac{1}{\varepsilon^2 h}.$$

Finally, taking  $h = \sqrt{\varepsilon}$  and letting  $\varepsilon$  tend to zero, we get that  $I_c = -\infty$ .

#### 4. STABILITY OF STANDING WAVES

In this section, we assume that

$$\lambda_1 \geq \frac{4\pi}{3} \lambda_2 > 0.$$

**Theorem 4.1.** *The following properties hold:*

- i) For any  $c > 0$ ,  $I_c = \widetilde{I}_c$ ,  $Z_c \neq \emptyset$  and  $Z_c$  is orbitally stable.
- ii) For any  $z \in Z_c$ ,  $|z| \in W_c$ .
- iii)  $Z_c = \{e^{i\theta} w, \theta \in \mathbb{R}\}$  where  $w$  is the unique minimize of (1.9).

*Proof.* We follow the approach presented in [7] and resumed in [9]. In fact to prove the stability, it suffices to show that  $Z_c \neq \emptyset$  and any minimizing sequence  $\{z_n\} \subset \widetilde{\Sigma}$  such that  $\|z_n\|_2 \rightarrow c$  and  $\widetilde{E}(z_n) \rightarrow \widetilde{I}_c$  is relatively compact in  $\widetilde{\Sigma}$ .

Let  $z_n = (u_n, v_n) \subset \widetilde{\Sigma}$  be a sequence such that  $\|z_n\|_2 \rightarrow c$  and  $\widetilde{E}(z_n) \rightarrow \widetilde{I}_c$ .

The first step consists in proving that  $\{z_n\}$  has a subsequence which is convergent in  $\widetilde{\Sigma}$ .

By the fact that  $\widetilde{E}$  is a non-negative functional, we can easily deduce that  $\{z_n\}$  is bounded in  $\widetilde{\Sigma}$ , therefore passing to a subsequence, one can suppose that

$$z_n \rightharpoonup z = (u, v) \quad \text{in } \widetilde{\Sigma},$$

hence

$$u_n \rightharpoonup u \quad \text{in } \Sigma \quad \text{and} \quad v_n \rightharpoonup v \quad \text{in } \Sigma,$$



and

$$(4.1) \quad \lim_{n \rightarrow \infty} \int |\nabla u_n|^2 + |\nabla v_n|^2 \quad \text{exists.}$$

Now let  $\rho_n = |z_n| = (u_n^2 + v_n^2)^{1/2}$ . Clearly  $\{\rho_n\} \subset \Sigma$  and for all  $n \in \mathbb{N}$  and  $1 \leq j \leq 3$ ,

$$\partial_j \rho_n(x) = \begin{cases} \frac{u_n(x) \partial_j u_n(x) + v_n(x) \partial_j v_n(x)}{(u_n^2(x) + v_n^2(x))^{1/2}} & \text{if } u_n^2 + v_n^2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\tilde{E}(z_n) - E(\rho_n) = \frac{1}{2} \sum_{j=1}^3 \int_{\{u_n^2 + v_n^2 > 0\}} \frac{(u_n \partial_j v_n - v_n \partial_j u_n)^2}{u_n^2 + v_n^2} dx.$$

Therefore  $\tilde{I}_c = \lim \tilde{E}(z_n) \geq \limsup E(\rho_n)$ . Since

$$(4.2) \quad \|z_n\|_2^2 = |\rho_n|_2^2 = c_n^2 \rightarrow c^2,$$

we get by Lemma 2.4 that

$$\liminf E(\rho_n) \geq \liminf I_{c_n} \geq I_c \geq \tilde{I}_c,$$

and hence

$$(4.3) \quad \lim_{n \rightarrow \infty} E(\rho_n) = \lim_{n \rightarrow +\infty} \tilde{E}(z_n) = I_c = \tilde{I}_c.$$

On the other hand (4.1) implies that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + |\nabla v_n|^2 - |\nabla(u_n^2 + v_n^2)^{1/2}|^2) dx = 0.$$

Consequently

$$(4.4) \quad \lim_{n \rightarrow \infty} \int |\nabla u_n|^2 + |\nabla v_n|^2 dx = \lim_{n \rightarrow \infty} \int |\nabla(u_n^2 + v_n^2)^{1/2}|^2 dx = \lim_{n \rightarrow \infty} \int |\nabla \rho_n|^2 dx.$$

We infer from (4.2), (4.3) and Remark 3.1 that there exists  $\rho \in \Sigma$  such that  $\rho_n \rightarrow \rho$  in  $\Sigma$ .

Clearly  $\rho \in S_c$  and  $E(\rho) = I_c$ . Then  $\rho \geq 0$  and Steiner symmetric. Moreover  $\rho$  is a weak solution of (1.5). Thus  $\rho \in C^1(\mathbb{R}^3)$  and  $\rho > 0$ . Hence  $\rho \in W_c \subset Z_c$  since  $I_c = \tilde{I}_c$ . Next, we prove that  $\rho = (u^2 + v^2)^{1/2}$ . In fact  $u_n \rightarrow u$  and  $v_n \rightarrow v$  in  $L^2(B(0, R))$  for any  $R > 0$ . Since  $[(u_n^2 + v_n^2)^{1/2} - (u^2 + v^2)^{1/2}]^2 \leq |u_n - u|^2 + |v_n - v|^2$ , it follows that  $(u_n^2 + v_n^2)^{1/2} \rightarrow (u^2 + v^2)^{1/2}$  in  $L^2(B(0, R))$  for any  $R > 0$ . Combining this and the fact that  $(u_n^2 + v_n^2)^{1/2} = \rho_n \rightarrow \rho$  in  $L^2$ , imply that  $(u^2 + v^2)^{1/2} = \rho$  a.e in  $\mathbb{R}^3$ .

Now to end the proof of part i) of Theorem 4.1, it suffices to prove that

$$\lim_{n \rightarrow \infty} \|\nabla z_n\|_2^2 = \|\nabla z\|_2^2.$$

By invoking (4.4), we have that :

$$\lim_{n \rightarrow \infty} \|\nabla z_n\|_2^2 = \lim_{n \rightarrow \infty} |\nabla \rho_n|_2^2 = |\nabla \rho|_2^2.$$

Thus

$$\|\nabla z\|_2^2 \leq \liminf \|\nabla z_n\|_2^2 \leq |\nabla \rho|_2^2.$$

On the other hand, by replacing  $z_n$  by  $z$  in (4.1), we have that  $\|\nabla z\|_2^2 \geq |\nabla \rho|_2^2$ .

This, together with the weak convergence of  $z_n$  to  $z$  in  $\tilde{\Sigma}$ , enables us to conclude.

**Proof of ii).** Let  $z = (u, v) \in Z_c$  and set  $\rho = (u^2 + v^2)^{1/2}$ . By the previous proof, we know that  $\rho \in W_c$  and

$$\sum_{j=1}^3 \int_{\mathbb{R}^3} \left( \frac{u \partial_j v - v \partial_j u}{u^2 + v^2} \right)^2 dx = 0.$$

On the other hand,  $\tilde{E}(z) = \tilde{I}_c$  which implies that there exists a Lagrange multiplier  $\lambda \in \mathbb{C}$  such that :

$$E(z)\xi = \frac{\lambda}{2} \int_{\mathbb{R}^3} (z\bar{\xi} + \bar{z}\xi) dx \quad \text{for all } \xi \in \tilde{\Sigma}.$$

Letting  $\xi = z$ , it follows immediately that  $\lambda \in \mathbb{R}$  and

$$\begin{cases} -\frac{1}{2}\Delta u + \frac{|x|^2}{2}u + \lambda_1(u^2 + v^2)u + \lambda_2(K * (u^2 + v^2))u + \lambda u = 0, \\ -\frac{1}{2}\Delta v + \frac{|x|^2}{2}v + \lambda_1(u^2 + v^2)v + \lambda_2(K * (u^2 + v^2))v + \lambda v = 0. \end{cases}$$

Elliptic regularity theory implies that  $u, v \in C^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ .

Let  $\Omega = \{x \in \mathbb{R}^3 : u(x) = 0\}$ , then  $\Omega$  is closed since  $u$  is continuous. Let us prove that it is also open. Suppose that  $x_0 \in \Omega$ , using the fact that  $v(x_0) > 0$ , we can find a Ball  $B$  centered in  $x_0$  such that  $v(x) \neq 0$  for any  $x \in B$ . Thus for  $x \in B$

$$\frac{(u \partial_j v - v \partial_j u)^2}{u^2 + v^2} = \left( \partial_j \left( \frac{u}{v} \right) \right)^2 \frac{v^4}{u^2 + v^2} \quad \text{for } 1 \leq j \leq 3.$$

This implies that

$$\int_B \left| \nabla \left( \frac{u}{v} \right) \right|^2 \frac{v^4}{u^2 + v^2} dx = 0.$$

Hence  $\nabla \left( \frac{u}{v} \right) = 0$  on  $B$ . Thus there exists a constant  $K$  such that  $\frac{u}{v} = K$  on  $B$ . But  $x_0 \in B$ , then  $K \equiv 0$ . We have proved that only the two alternatives below are plausible:

- a)  $u \equiv 0$  or  $u \neq 0$  for all  $x \in \mathbb{R}^3$ .
- b)  $v \equiv 0$  or  $v \neq 0$  for all  $x \in \mathbb{R}^3$ .

Now let us find the relationship between  $u$  and  $v$ .

**Proof of iii).** Let  $z = (w \cos \sigma, w \sin \sigma)$ ,  $\sigma \in \mathbb{R}, w \in W_c$ . We denote  $z$  by  $z = e^{i\sigma} w$  by identifying  $\mathbb{C}$  with  $\mathbb{R}^2$ . Then  $z \in \tilde{S}_c$  and  $\tilde{E}(z) = E(w) = I_c = \tilde{I}_c$ . Thus  $\{e^{i\sigma} w, \sigma \in \mathbb{R}, w \in W_c\} \subset Z_c$ . Conversely, for  $z = (u, v) \in Z_c$ , set  $w = |z|$ . Then  $\tilde{E}(z) = E(w) = \tilde{I}_c = I_c$  and  $w \in W_c$ . If  $v \equiv 0$ ,  $w = |w| > 0$  on  $\mathbb{R}^3$  and so  $z = e^{i\sigma} w \in W_c$  where  $\sigma = 0$  if  $u > 0$  and  $\sigma = \pi$  if  $u < 0$  on  $\mathbb{R}^3$ . Otherwise  $v(x) \neq 0$  for all  $x \in \mathbb{R}^3$ . In this case, it follows that  $\nabla \left( \frac{u}{v} \right) = 0$  on  $\mathbb{R}^3$ . Therefore there exists a constant  $\alpha \in \mathbb{R}$  such that  $u = \alpha v$  on  $\mathbb{R}^3$ . Hence  $z = (\alpha + i)v$  and  $W = |\alpha + i||v|$ . Let  $\theta \in \mathbb{R}$  be such that  $(\alpha + i) = |\alpha + i|e^{i\theta}$  and let  $\varphi = 0$  if  $v > 0$  and  $\varphi = \pi$  if  $v < 0$  on  $\mathbb{R}^3$ . Setting  $\sigma = \theta + \varphi$ , we have  $z = (\alpha + i)v = |\alpha + i|e^{i\theta}|v|e^{i\varphi} = we^{i\sigma}$ , where  $w \in W_c$ .  $\square$

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