Orbital stability of standing waves of some ℓ-coupled nonlinear Schrödinger equations

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Abstract

We extend the notion of orbital stability to systems of nonlinear Schrödinger equations, then we prove this property under suitable assumptions on the local nonlinearity involved.

1 Introduction

For $\ell \in \mathbb{N}^*$, the author has studied the following Cauchy problem:

$$\begin{cases} i\partial_t\Phi_1 + \Delta\Phi_1 + h_1(x, |\Phi_1|^2, ..., |\Phi_\ell|^2)\Phi_1 & = 0 \\ \vdots & \vdots \\ i\partial_t\Phi_\ell + \Delta\Phi_\ell + h_\ell(x, |\Phi_1|^2, ..., |\Phi_\ell|^2)\Phi_\ell & = 0 \\ \Phi_j(0, x) = \Phi_j^0(x) & \text{for } 1 \leq j \leq \ell, \end{cases} \tag{1.1}$$

$\Phi_j^0 : \mathbb{R}^N \to \mathbb{C}, h_j : \mathbb{R} \times \mathbb{R}_+^\ell \to \mathbb{R}$ and $\Phi_j : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, [4]. (1.1) has numerous applications in engineering and physics. It appears in the study of spatial solitons in nonlinear wave guides, the theory of Bose-Einstein condensates, optical pulse propagation in birefrigent fibers, interactions of ℓ-wave packets, wavelength division multiplexed optical systems, see [3] and references therein. Physically Φ_j is the jth component of the beam in Kerr-like photorefractive media. In these contexts, it is always possible to write (1.1) in a compact vectorial form:

$$\begin{cases} i\frac{\partial\vec{\Phi}}{\partial t} & = \hat{E}'(\vec{\Phi}) \\ \vec{\Phi}(0, x) & = \vec{\Phi}^0 = (\Phi_1^0, ..., \Phi_\ell^0), \end{cases} \tag{1.2}$$

where $\vec{\Phi} = (\Phi_1, ..., \Phi_\ell)$.
Under appropriate growth assumptions on H, we can define the energy functional:

$$\hat{E}(\vec{\Phi}) = \frac{1}{2}\left\{ |\nabla\vec{\Phi}|_2^2 - \int H(x, \Phi_1, .., \Phi_\ell) \right\}. \tag{1.3}$$

H is such that:

$$\frac{\partial H}{\partial s_j}(x, s_1, ..., s_\ell) = 2h_j(x, s_1^2, ..., s_\ell^2)s_j, \tag{1.4}$$

and

$$H(x, s_1, ..., s_\ell) = H(x, |s_1|, ..., |s_\ell|). \tag{1.5}$$

Note that when $\ell = 1$, we have $H(x, s) = \int_0^{s^2} h(x, t)dt$.
A soliton or a standing wave of (1.1) is a solution of (1.1) with the particular form: $\vec{\Phi}(t, x) = (\Phi_1(t, x), ..., \Phi_\ell(t, x))$ where $\Phi_j(t, x) = u_j(x)e^{-i\lambda_j t}$; λ_j are real numbers.
Hence $\vec{u} = (u_1, ..., u_\ell)$ is a solution of the following $\ell \times \ell$ elliptic eigenvalue problem:

$$\begin{cases} \Delta u_1 + h_1(x, u_1^2, ..., u_\ell^2)u_1 + \lambda_1 u_1 & = 0 \\ \vdots & \vdots \\ \Delta u_\ell + h_\ell(x, u_1^2, ..., u_\ell^2)u_\ell + \lambda_\ell u_\ell & = 0. \end{cases} \tag{1.6}$$

When $\ell = 1$, (1.6) becomes:

$$\Delta w + h(x, w^2)w + \lambda w = 0, \tag{1.7}$$

where $w \in H^1(\mathbb{R}^N, \mathbb{C})$. (1.7) can be written as a 2×2 real elliptic system for (u, v) where $w = (u, v) = u + iv$, namely:

$$\begin{cases} \Delta u + h(x, u^2 + v^2)u + \lambda u & = 0 \\ \Delta v + h(x, u^2 + v^2)v + \lambda v & = 0. \end{cases} \tag{1.8}$$

When $v \equiv 0$, (1.8) leads to the scalar equation:

$$\Delta u + h(x, u^2)u + \lambda u = 0. \tag{1.9}$$

(1.9) alone forms an important chapter of nonlinear analysis in which many mathematicians have intensively contributed. Special attention was given to the case $h(x, s^2) = |s|^{p-1}$. The famous concentration-compactness principle perfectly applies to study the orbital stability of standing waves of (1.9) [1],[8].
In the scalar setting, there are two approaches to determine the orbital stability of standing waves of (1.1). The first approach boils this question down

to checking the strict inequality $\frac{d}{d\lambda}\int u_\lambda^2 < 0$ for certain solutions u_λ of (1.9). For non-autonomous equations, it is hard to establish conditions on the nonlinearity h ensuring the latter monotonicity property ([7] and references therein). In the vectorial setting, it does not seem possible to extend this approach for (1.6). The second alternative exploits the hamiltonian structure of (1.1) when $\ell = 1$ via the characterization of standing waves as constrained minimizers. We will adapt this approach to generalize the notion of orbital stability of standing waves of (1.1). We will then establish the stability of the latter particular solutions under general assumptions on H including the most relevant physical situations where $H(x,\vec{s})$ converges as $|x|$ goes to infinity to a function $H^\infty(x,\vec{s})$ that depends periodically on x.

2 Definition and Notation

Before formulating the notion of orbital stability of the standing waves of (1.1), let us first introduce some useful notation:
$H^1(\mathbb{R}^N,\mathbb{R})$ is the standard Sobolev space of real valued functions. $H^{-1}(\mathbb{R}^N,\mathbb{R})$ is its dual space.
$H^1_\ell(\mathbb{R}^N,\mathbb{R})$ is the ℓ cartesian product of $H^1(\mathbb{R}^N,\mathbb{R})$. Its dual space is denoted by $H^{-1}_\ell(\mathbb{R}^N,\mathbb{R})$.
$H^1(\mathbb{R}^N,\mathbb{C})$ is the standard Sobolev space of complex valued functions. $H^1_\ell(\mathbb{R}^N,\mathbb{C})$ is the ℓ cartesian product of $H^1(\mathbb{R}^N,\mathbb{C})$. Its dual space is denoted by $H^{-1}_\ell(\mathbb{R}^N,\mathbb{C})$.

Definition: $$\begin{aligned} \text{For } z &= (u,v)\ ; |z|^2_{H^1(\mathbb{R}^N,\mathbb{C})} = |z|_2^2 + |\nabla z|_2^2 \\ |z|_2^2 &= |u|_2^2 + |v|_2^2\ ; \quad |\nabla z|_2^2 = |\nabla u|_2^2 + |\nabla v|_2^2. \end{aligned}$$

$|\ |_p$ denotes the usual norm on $L^p(\mathbb{R}^N,\mathbb{R}) = L^p$.

Notation: $\vec{z} = (z_1,...,z_\ell) = ((u_1,v_1),...,(u_\ell,v_\ell)) = (\vec{u},\vec{v})$,

where
$$z_j = u_j + iv_j = (u_j,v_j).$$
The modulus of the vector $\vec{z}$, denoted by $|\vec{z}|$ is the vector
$$|\vec{z}| = (|\vec{z}_1|,...,|\vec{z}_\ell|)\ ,\quad |z_j| = (u_j^2+v_j^2)^{1/2}.$$

Under (H_0) that we state in Theorem 3.1 below, we can define the functionals:

$\hat{E} : H^1_\ell(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ and $E : H^1_\ell(\mathbb{R}^N, \mathbb{R}) \to \mathbb{R}$.

$$\begin{aligned}\hat{E}(\vec{z}) = \hat{E}(\vec{u}, \vec{v}) &= \frac{1}{2}\{|\nabla \vec{z}|_2^2 - \int H(x, |\vec{z}|)\} \\ &= \frac{1}{2}\{\sum_{j=1}^{\ell}\{|\nabla u_j|_2^2 + |\nabla v_j|_2^2\} - \int H(x, |\vec{z}|)\} \\ &= \frac{1}{2}\{\sum_{j=1}^{\ell}|\nabla u_j|_2^2 + |\nabla v_j|_2^2 - \int H(x, (u_1^2+v_1^2)^{1/2}, ..., (u_\ell^2+v_\ell^2)^{1/2}\}\end{aligned}$$

$$E(u) = \hat{E}(u, 0) = \frac{1}{2}\{|\nabla \vec{u}|_2^2 - \int H(x, |\vec{u}|)\}$$

For $c_1, .., c_\ell > 0$, we set $c^2 = \sum_{i=1}^{\ell} c_i^2$ and:

$$\hat{S}_{c_1,...,c_\ell} = \{\vec{z} \in H^1_\ell(\mathbb{R}^N, \mathbb{C}) : |z_i|_2^2 = c_i^2 \quad 1 \le i \le \ell\},$$

$$S_{c_1,...,c_\ell} = \{\vec{u} \in H^1_\ell(\mathbb{R}^N, \mathbb{R}) : |u_i|_2^2 = c_i^2 \quad 1 \le i \le \ell\},$$

$$\hat{I}_{c_1,...,c_\ell} = \inf\{\hat{E}(\vec{z}) : \vec{z} \in \hat{S}_{c_1,...,c_\ell}\},$$

and

$$I_{c_1,...,c_\ell} = \inf\{E(\vec{u}) : \vec{u} \in S_{c_1,...,c_\ell}\},$$

$$\hat{O}_{c_1,...,c_\ell} = \{\vec{z} \in \hat{S}_{c_1,...,c_\ell} : \hat{E}(\vec{z}) = \hat{I}_{c_1,...,c_\ell}\}.$$

From now on we fix $c_1, ..., c_\ell > 0$ and $c^2 = \sum_{i=1}^{\ell} c_i^2$.
Following the definition in the scalar setting, we will say that $\hat{O}_{c_1,...,c_\ell}$ is stable if it is not empty and:

$$\begin{cases} \forall\, \varepsilon > 0, \exists\, \delta > 0 \text{ such that for any } \vec{\Phi}_0 \in H^1_\ell(\mathbb{R}^N, \mathbb{C}), \\ \text{such that } \inf_{\vec{z} \in \hat{O}_{c_1,...,c_\ell}} |\vec{\Phi}_0 - \vec{z}|_{H^1_\ell(\mathbb{R}^N,\mathbb{C})} < \delta, \\ \text{it follows that } \inf_{\vec{z} \in \hat{O}_{c_1,...,c_\ell}} |\vec{\Phi}(t,.) - \vec{z}|_{H^1_\ell(\mathbb{R}^N,\mathbb{C})} < \varepsilon \quad \forall\, t \in \mathbb{R}. \end{cases} \tag{2.1}$$

$\vec{\Phi}(t,.)$ denotes the global solution of (1.1) corresponding to the initial condition $\vec{\Phi}_0$. Under suitable conditions on H, (see Theorem 3.1 below), this solution is unique.

3 Preliminaries

In this section, we state crucial results obtained recently by the author which will be very helpful to establish the orbital stability. We first take advantage of the recent result established in [4], in which the author has determined assumptions on h_j ensuring the existence and uniqueness of global solutions of (1.1). Under slight modifications of Theorem 2.11 and Theorem 3.1 of [4], we have the following result:

Theorem 3.1: Let $H : \mathbb{R} \times \mathbb{R}_+^\ell \to \mathbb{R}$ be a Carathéodory function such that:
(H_0) There exist $K > 0$ and $0 < \ell_1 < \frac{4}{N}$ such that

$$0 \leq H(x, \vec{s}) \leq K(|\vec{s}|^2 + |\vec{s}|^{\ell_1+2})$$

for any $x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}_+^\ell$.

(H_1)

- There exist constants $c' > 0$ and $\alpha \in [0, \frac{4}{N-2})$ for $N \geq 3, \alpha \in [0, \infty)$ for $N = 2$ such that

  $$|h_j(x, |\vec{s}|^2)s_j - h_j(x, |\vec{r}|^2)r_j| \leq c'\{1 + |\vec{s}|^\alpha + |\vec{r}|^\alpha\}|\vec{s} - \vec{r}|$$

  for all $1 \leq j \leq \ell, \vec{r}, \vec{s} \in \mathbb{R}_+^\ell$.

- For $N = 1$ and any $R > 0$, there exists a constant $L(R) > 0$ such that $|h_j(x, |\vec{s}|^2)s_j - h_j(x, |\vec{r}|^2)r_j| \leq L(R)|\vec{s} - \vec{r}|$ for all $\vec{s}, \vec{r} \in \mathbb{R}_+^\ell$ such that $|\vec{r}| + |\vec{s}| \leq R$.

Then for every $\vec{\Phi}_0 \in H_\ell^1(\mathbb{R}^N, \mathbb{C})$, the initial value, the Cauchy problem (1.1) has a unique solution $\vec{\Phi} \in C(\mathbb{R}, H_\ell^1(\mathbb{R}^N, \mathbb{C})) \cap C^1(\mathbb{R}, H_\ell^{-1}(\mathbb{R}^N, \mathbb{C}))$. Furthermore $\sup_{t\in\mathbb{R}} |\vec{\Phi}(t, .)|_{H_\ell^1(\mathbb{R}^N,\mathbb{C})} < \infty$.
We have also the conservation of charges and energy:

$$(C1) \qquad |\Phi_j(t, .)|_2 = |\Phi_0^j|_2 \quad \forall\, 1 \leq j \leq \ell \text{ and } \forall\, t \in \mathbb{R},$$

$$(C2) \qquad \hat{E}(\vec{\Phi}(t, .)) = \hat{E}(\vec{\Phi}_0) \quad \forall\, t \in \mathbb{R}.$$

Theorem 3.2: Suppose that H satisfies (H_0),

(H_2) There exists $B > 0$ such that

$$|\partial_j H(x, \vec{s})| \leq B(|\vec{s}| + |\vec{s}|^{\ell_1+1}) \quad \text{for all } x \in \mathbb{R}^N$$

and $\vec{s} \in \mathbb{R}_+^\ell; 1 \leq j \leq \ell$.

(H_3) $\exists\, \Delta > 0,\ R > 0,\ s > 0,\ \alpha_1, ..., \alpha_\ell \geq 0,\ t \in [0, 2)$
such that $H(x, \vec{s}) > \Delta |x|^{-t} |s_1|^{\alpha_1} ... |s_\ell|^{\alpha_\ell}$ for all $|x| \geq R$ and $|\vec{s}| < S$, where $N + 2 > \frac{N}{2}\alpha + t\,; \alpha = \sum_{j=1}^{\ell} \alpha_j$.

(H_4) $H(x, \theta_1 s_1, ..., \theta_\ell s_\ell) \geq \theta_{max}^2 H(x, s_1, ..., s_\ell)$ for all $x \in \mathbb{R}^N$, $s_i \in \mathbb{R}$, $\theta_i \geq 1$ where $\theta_{max} = \max_{1 \leq j \leq \ell} \theta_j$.

There exists a periodic function $H^\infty(x, \vec{s})$ (i.e, $\exists\, T \in \mathbb{Z}^N$ such that $H^\infty(x, +T, \vec{s}) = H^\infty(x, \vec{s}), \forall\, x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}_+^\ell$) satisfying (H_3) and such that :

(H_5) There exists $0 < \Gamma < \frac{4}{N}$ such that

$$\lim_{|x| \to \infty} \frac{H(x, \vec{s}) - H^\infty(x, \vec{s})}{|\vec{s}|^2 + |\vec{s}|^{\Gamma+2}} = 0 \quad \text{uniformly for any } \vec{s}.$$

(H_6) There exist $A', B' > 0$ and $0 < \beta < \ell_1 < \frac{4}{N}$ such that

$$0 \leq H^\infty(x, \vec{s}) \leq A'(|\vec{s}|^{\beta+2} + |\vec{s}|^{\ell_1+2})$$

and $\forall\, 1 \leq j \leq \ell$:

$$\partial_j H^\infty(x, \vec{s}) \leq B'(|\vec{s}|^{\beta+1} + |\vec{s}|^{\ell_1+1}) \quad \forall\, x \in \mathbb{R}^N\ , \vec{s} \in \mathbb{R}_+^\ell\ .$$

(H_7) There exists $\sigma \in (0, \frac{4}{N})$ such that:

$$H^\infty(x, \theta_1 s_1, ..., \theta_\ell s_\ell) \geq \theta_{max}^{\sigma+2} H^\infty(x, s_1, ..., s_\ell)$$

for any $\theta_i \geq 1, x \in \mathbb{R}^N, \vec{s} \in \mathbb{R}_+^\ell$, where $\theta_{max} = \max_{1 \leq j \leq \ell} \theta_i$.

Then any sequence $\{\vec{u}_n\} \subset H_\ell^1(\mathbb{R}^N, \mathbb{R})$ such that $|u_{n,j}|_2^2 \to c_j$ and $E(\vec{u}_n) \to I_{c_1,...,c_\ell}$ admits a subsequence converging to $\vec{u} \in S_{c_1,...,c_\ell}$.

Following the proof of Lemma 3.1 of [5], we can easily derive the following proposition.

Proposition 3.3: Under the hypothesis (H_0), the functionals $\hat{E}$ and E are differentiable and we have the following properties:

1. There exists a constant $C > 0$ such that

$$\hat{E}(\vec{z}) \geq \frac{1}{4}|\nabla \vec{z}|_2^2 - C(c^2 + c^\gamma)$$

for all $\vec{z} \in \hat{S}_{c_1,\ldots c_\ell}$ and all $c_1, ..., c_\ell > 0$ where

$$\gamma = \frac{2(2\ell_1 + 4 - N\ell_1)}{4 - N\ell_1} > 2.$$

2. For all $c_1, ..., c_\ell > 0$, $I_{c_1,\ldots,c_\ell} \geq \hat{I}_{c_1,\ldots,c_\ell} > -\infty$ and any minimizing sequences for $I_{c_1,\ldots,c_\ell}$ and $\hat{I}_{c_1,\ldots,c_\ell}$ are bounded in $H^1_\ell(\mathbb{R}^N, \mathbb{R})$ (resp $H^1_\ell(\mathbb{R}^N, \mathbb{C})$).

3. $(c_1, ..., c_\ell) \to I_{c_1,\ldots,c_\ell}$ is continuous on $(0, \infty)^\ell$.

   Now for the convenience of the reader, we recall the following classical result [9] .

Proposition 3.4:
Let $u, v \in H^1(\mathbb{R}^N, \mathbb{R})$, then $(u^2 + v^2)^{1/2} \in H^1(\mathbb{R}^N, \mathbb{R})$ and for $1 \leq i \leq N$

$$\partial_i (u^2 + v^2)^{1/2} = \begin{cases} \frac{u\partial_i u + v\partial_i v}{(u^2+v^2)^{1/2}} & \text{if } u^2 + v^2 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

4 Main Result

Theorem 4.1: Suppose that (H_0) to (H_7) are satisfied, then for any $c_1, ..., c_\ell > 0$, the orbit $\hat{O}_{c_1,\ldots,c_\ell}$ is stable.

Example:

Below we give a class of nonlinearities satisfying all the hypotheses of theorem 4.1:
$H(x, \vec{s}) = p(x)|\vec{s}|^2 + q(x) \sum_{i,j=1, i<j}^{\ell} |s_i|^{1+\frac{\delta}{2}} |s_j|^{1+\frac{\delta}{2}}$; $0 < \delta < \frac{4}{N}$, where $0 < A \leq p(x) \leq B$; $0 < C \leq q(x) \leq D$; $A, B, C, D > 0$ and $\lim_{|x|\to\infty} p(x) = 0$; $\lim_{|x|\to\infty} q(x) = q(\infty) < \infty$.
Until now, only very special cases have been treated in the literature for $\ell = 2$, p and q are constants, see [10], [11].

Proof:
By Theorem 3.1, we now know that (1.1) admits a unique solution. Thus we

can prove (2.1) by contradiction.
Suppose that $\hat{O}_{c_1,\dots,c_\ell}$ is not stable, then either $\hat{O}_{c_1,\dots,c_\ell}$ is empty or there exist $\vec{w} \in \hat{O}_{c_1,\dots,c_\ell}, \varepsilon_0 > 0$ and a sequence $\{\vec{\Phi}_0^n\} \in H_\ell^1(\mathbb{R}^N, \mathbb{C})$ such that:

$$|\vec{\Phi}_0^n - \vec{w}|_{H_\ell^1(\mathbb{R}^N,\mathbb{C})} \to 0 \text{ as } n \to \infty \text{ but } \inf_{\vec{z} \in \hat{O}_{c_1,\dots c_\ell}} |\vec{\Phi}^n(t_n, .) - \vec{z}|_{H_\ell^1(\mathbb{R}^N,\mathbb{C})} \geq \varepsilon_0 \tag{4.1}$$

for some sequence $\{t_n\} \subset \mathbb{R}$, where $\Phi^n(t_n, .)$ is the solution of (1.1) corresponding to the initial condition $\vec{\Phi}_0^n$.
Let $\vec{w}_n = \vec{\Phi}^n(t_n, .)$. Since $\vec{w} \in \hat{S}_{c_1,\dots,c_\ell}$ and $\hat{E}(\vec{w}) = \hat{I}_{c_1,\dots,c_\ell}$. It follows from the continuity of $|\ |_2$ and $\hat{E}$ on $H_\ell^1(\mathbb{R}^N, \mathbb{C})$ (Proposition 3.3) that: $|\Phi_{0,j}^n|_2 \to c_j$ $\forall\ 1 \leq j \leq \ell$ and $\hat{E}(\vec{w}_n) = \hat{E}(\Phi_0^n) = \hat{I}_{c_1,\dots,c_\ell}$. Thus we deduce from Theorem 3.1 that

$$|w_{n,j}|_2 = |\Phi_{0,j}^n|_2 \to c_j \quad \forall\ 1 \leq j \leq \ell$$

and

$$\hat{E}(\vec{w}_n) = \hat{E}(\vec{\Phi}_0^n) \to \hat{I}_{c_1,\dots,c_\ell}.$$

If $\{\vec{w}_n\}$ admits a subsequence converging to an element $\vec{w} \in H_\ell^1(\mathbb{R}^N, \mathbb{C})$, $\vec{w} = (w_1, ..., w_\ell)$ then $|w_j|_2 \to c_j$ and $\hat{E}(\vec{w}) = \hat{I}_{c_1,\dots,c_\ell}$. This proves that $\vec{w} \in \hat{O}_{c_1,\dots,c_\ell}$ but $\inf_{\vec{z} \in \hat{O}_{c_1,\dots,c_\ell}} |\vec{\Phi}^n(t_n, .) - \vec{z}|_{H_\ell^1(\mathbb{R}^N,\mathbb{C})} \leq |\vec{w}_n - \vec{w}|_{H_\ell^1(\mathbb{R}^N,\mathbb{C})}$ contradicting (4.1).
Hence to show the orbital stability of $\hat{O}_{c_1,\dots,c_\ell}$, one has to prove that $\hat{O}_{c_1,\dots,c_\ell}$ is not empty and any sequence $\{\vec{w}_n\}$ satisfying:

$$\begin{cases} \{\vec{w}_n\} \subset H_\ell^1(\mathbb{R}^N, \mathbb{C}),\ |w_{n,j}|_2 \to c_j \\ \text{for } 1 \leq j \leq \ell \text{ and } \hat{E}(\vec{w}_n) \to \hat{I}_{c_1,\dots,c_\ell} \end{cases} \tag{4.2}$$

is relatively compact in $H_\ell^1(\mathbb{R}^N, \mathbb{C})$.

In the following, $\{\vec{w}_n\}$ denotes a sequence satisfying (4.2). Our objective is to prove that $\{\vec{w}_n\}$ admits a subsequence converging to an element $\vec{w} \in H_\ell^1(\mathbb{R}^N, \mathbb{C})$. Our line of attack consists of the following steps:

Step 1: If $\{\vec{w}_n\}$ satisfies (4.2) then the sequence $|\vec{w}_n| = (|\vec{w}_{n,1}|, ..., |\vec{w}_{n,\ell}|)$ is such that $E(|\vec{w}_n|) \to I_{c_1,\dots,c_\ell}$ and $|w_{n,j}|_2^2 \to c_j$. By Theorem 3.2, such a sequence is relatively compact in $H_\ell^1(\mathbb{R}^N, \mathbb{R})$.

Step 2: we now know that there exists $\vec{\varphi} \in H_\ell^1(\mathbb{R}^N, \mathbb{R})$ such that $(u_{n,j}^2 + v_{n,j}^2)^{1/2}$ converges to φ_j in $H^1(\mathbb{R}^N, \mathbb{R})$ for any $1 \leq j \leq \ell$.
On the other hand, it follows from Proposition 3.4 that $\vec{w}_n = ((u_{n,1}, v_{n,1}), ..., (u_{n,\ell}, v_{n,\ell}))$ is bounded in $H_\ell^1(\mathbb{R}^N, \mathbb{C})$. Hence, we may suppose that, up to a subsequence:

$$u_{n,j} \rightharpoonup u_j \quad \text{and} \quad v_{n,j} \rightharpoonup v_j \quad \forall\ 1 \leq j \leq \ell.$$

Let us set $w_j = (u_j, v_j)$.

Step 3 : In this step, we will prove that $\varphi_j = |w_j| = (u_j^2 + v_j^2)^{1/2}$ $\forall\, 1 \le j \le \ell$ via the establishement of some estimates on $|\nabla \vec{w}_n|_2^2 - |\nabla |\vec{w}_n||_2^2$, which will enable us also to prove that $w_{n,j} \to w_j$ $\forall\, 1 \le j \le \ell$. This concludes the proof.

Below are the complete details:

For $|\vec{w}_n| = (|w_{n,1}|, ..., |w_{n,\ell}|)$, it follows from Proposition 3.4 that $|w_{n,j}| = (u_{n,j}^2 + v_{n,j}^2)^{1/2} \in H^1(\mathbb{R}^N, \mathbb{R})$ and for any $1 \le j \le \ell$ and $1 \le i \le N$:

$$\partial_i |w_{n,j}| = \begin{cases} \frac{u_{n,j}\partial_i u_{n,j} + v_{n,j}\partial_i v_{n,j}}{(u_{n,j}^2 + v_{n,j}^2)^{1/2}} & \text{if } u_{n,j}^2 + v_{n,j}^2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by Proposition 3.4, the sequence $\{\vec{w}_n\}$ is bounded in $H_\ell^1(\mathbb{R}^N, \mathbb{C})$, and hence passing to a subsequence, there exists $\vec{w} = (w_1, ..., w_\ell) = ((u_1, v_1), ..., (u_\ell, v_\ell)) \in H_\ell^1(\mathbb{R}^N, \mathbb{C})$ such that

$$\begin{cases} \forall\, 1 \le j \le \ell \quad u_{n,j} \rightharpoonup u_j, v_{n,j} \rightharpoonup v_j \text{ and} \\ \lim\limits_{n\to\infty} \displaystyle\int |\nabla u_{n,j}|^2 + |\nabla v_{n,j}|^2 \quad \text{exists.} \end{cases} \tag{4.3}$$

Now

$$\begin{aligned} \hat{E}(\vec{w}_n) - E(|\vec{w}_n|) &= \tfrac{1}{2}\{|\nabla \vec{w}_n|_2^2 - |\nabla |\vec{w}_n||_2^2\} \\ &= \frac{1}{2}\sum_{j=1}^{\ell} |\nabla w_{n,j}|_2^2 - |\nabla (u_{n,j}^2 + v_{n,j}^2)^{1/2}|_2^2 \\ &= \frac{1}{2}\sum_{j=1}^{\ell}\sum_{i=1}^{N} \frac{(u_{n,j}\partial_i v_{n,j} - v_{n,j}\partial_i u_{n,j})^2}{u_{n,j}^2 + v_{n,j}^2} \ge 0. \end{aligned} \tag{4.4}$$

Thus we have:

$$\hat{I}_{c_1,\dots c_\ell} = \lim_{n\to\infty} \hat{E}(\vec{w}_n) \ge \limsup E(|\vec{w}_n|). \tag{4.5}$$

But:

$$|w_{n,j}|_2^2 = ||w_{n,j}||_2^2 = c_{n,j}^2 \to c_j^2. \tag{4.6}$$

It follows by the continuity of $I_{c_1,\dots,c_\ell}$, proved in Proposition 3.3, that we have:

$$\lim \hat{E}(\vec{w}_n) \ge \liminf I_{c_{n,1},\dots,c_{n,\ell}} = I_{c_1,\dots,c_\ell} \ge \hat{I}_{c_1,\dots,c_\ell}.$$

Hence

$$\lim_{n\to+\infty} \hat{E}(\vec{w}_n) = \lim_{n\to\infty} E(|\vec{w}_n|) = I_{c_1,\dots,c_\ell} = \hat{I}_{c_1,\dots,c_\ell}. \tag{4.7}$$

(4.4) and (4.7) imply that

$$\forall\, 1 \le j \le \ell \quad \lim_{n\to\infty} \int |\nabla u_{n,j}|^2 + |\nabla v_{n,j}|^2 - |\nabla (u_{n,j}^2 + v_{n,j}^2)^{1/2} = 0. \tag{4.8}$$

Thus it follows from (4.3) that :

$$\lim_{n\to\infty} \int |\nabla u_{n,j}|^2 + |\nabla v_{n,j}|^2 = \lim_{n\to+\infty} \int |\nabla (u_{n,j}^2 + v_{n,j}^2)^{1/2}|^2 \tag{4.9}$$

which is equivalent to say that:

$$\lim_{n\to\infty} |\nabla \vec{w}_n|_2^2 = \lim_{n\to\infty} |\nabla |\vec{w}_n||_2^2. \tag{4.10}$$

(4.6) and (4.7) imply, using Theorem 3.2, that $|\vec{w}_n|$ is relatively compact in $H^1_\ell(\mathbb{R}^N, \mathbb{R})$. Thus there exists $\varphi_j \in H^1(\mathbb{R}^N, \mathbb{R})$ such that

$$\begin{cases} (u_{n,j}^2 + v_{n,j}^2)^{1/2} \text{ converges to } \varphi_j \text{ in } H^1(\mathbb{R}^N, \mathbb{R}) \text{ and} \\ |\varphi_j|_2 = c_j \quad \forall\, 1 \le j \le \ell \end{cases} \tag{4.11}$$

and $E(\varphi_1, \dots, \varphi_\ell) = I_{c_1,\dots,c_\ell}$.
We will first prove that $\varphi_j = |w_j| = (u_j^2 + v_j^2)^{1/2}$ where $w_j = (u_j, v_j)$ (u_j and v_j are given in (4.3)).
Using (4.3), it follows that $u_{n,j} \to u_j$ and $v_{n,j} \to v_j$ in $L^2(B(0,R))$. Furthermore a straigthfoward computation enables us to prove that:

$$[(u_{n,j}^2 + v_{n,j}^2)^{1/2} - (u_j^2 + v_j^2)^{1/2}]^2 \le |u_{n,j} - u_j|^2 + |v_{n,j}^2 - v_j|^2,$$

from which we deduce that :

$$(u_{n,j}^2 + v_{n,j}^2)^{1/2} \longrightarrow (u_j^2 + v_j^2)^{1/2} \text{ in } L^2(B(0,R))$$

for all $R > 0$. But $(u_{n,j}^2 + v_{n,j}^2)^{1/2} \to \varphi_j$ in L^2, thus we certainly have $(u_j^2 + v_j^2)^{1/2} = |w_j| = \varphi_j \;\forall\, 1 \le j \le \ell$.
On the other hand $|w_{n,j}|_2 = ||w_{n,j}||_2 \to c_j = ||w_j||_2 = |w_j|_2$, hence we are done if we prove that

$$\lim_{n\to\infty} |\nabla w_{n,j}|_2^2 \to |\nabla w_j|_2^2 \quad \text{for any } 1 \le j \le \ell.$$

From (4.9), we have $\lim\limits_{n\to\infty} |\nabla w_{n,j}|_2^2 = \lim\limits_{n\to\infty} |\nabla |w_{n,j}||_2^2$ and

$$\lim_{n\to\infty} |\nabla |w_{n,j}||_2^2 = |\nabla |w_j||_2^2.$$

Hence by the lower semi-continuity of $|\ \ |_2^2$, we obtain:

$$|\nabla w_j|_2^2 \leq \lim |\nabla |w_{n,j}||_2^2 = |\nabla |w_j||_2^2. \tag{4.12}$$

Finally replacing $w_{n,j}$ by w_j in (4.4), we see that

$$|\nabla w_j|_2^2 \geq |\nabla |w_j||_2^2 \quad \forall\, 1 \leq j \leq \ell. \tag{4.13}$$

By (4.3), we know that $w_{n,j} \rightharpoonup w_j$ in $H^1(\mathbb{R}^N, \mathbb{C})$. Thus $w_{n,j} \to w_j$, which completes the proof of Theorem 4.1.

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