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Characterization of Maximizers via Mass Transportation Techniques

Hichem Hajaiej

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Abstract In this paper, we address the question of existence and uniqueness of maximizers of a class of functionals under constraints, via mass transportation theory. We also determine suitable assumptions ensuring that balls are the unique maximizers. In both cases, we show that our hypotheses are optimal.

1 Introduction

The investigation of the properties of the extremals of integral functionals has received a huge interest from mathematicians over the past centuries. Numerous valuable articles were dedicated to these problems, which arise in most cases in physics and engineering [1]; and are crucial to handle ; the variational problems in which they are involved. It is sometimes sufficient to derive qualitative properties of optimizers to reach this goal. In other cases, we need very precise and subtle information about maximizers/minimizers. Moreover, in some situations, we have to determine these functions explicitly. In this paper, we discuss such a case. More precisely, variational problems for steady axisymmetric vortex-rings in which kinetic energy is maximized subject to a prescribed impulse involve Riesz-type functionals under constraint [2,3]. In [2] G.R. Burton proved the existence of maximizers for integrands $F(r) = rs$ in a natural constraint set, and then showed that optimizers are unique (up to translations). This was the gist of his approach to solve the corresponding variational problem. In [3], we have extended his result to general Riesz-type functionals, which enabled us to treat a much larger class of functionals. Our method is based on various approximation techniques. A key step is the establishment of maximizers of the Hardy-Littlewood type functionals [4, Lemma 2], which is very interesting in itself [5].

H. Hajaiej
Department of Mathematics
King Saud University
E-mail: hhajaiej@ksu.edu.sa

In this paper, we develop an innovative approach based on a powerful result in mass transportation theory [1, Theorem 1], which turns out to be very fruitful for our purpose. We are convinced that our method applies to other optimization problems. First, we have established suitable conditions to prove that balls are maximizers of the Hardy-Littlewood type functionals [Part 1, Theorem 3.1]. Note that this result improves [5, Proposition 3.1], in which C. Draghici and myself needed the superfluous supermodularity of the integrand. We have then focused our attention to prove the optimality of our hypotheses. It is natural to try to answer a question very frequently found in the literature: Are balls the unique maximizers inheriting the symmetry properties of the integrand of functionals ? Parts 2 of Theorems 3.1 and 3.2 are addressing this challenging problem. We also prove the optimality of these results. Characterization and uniqueness of optimizers of Riesz-type functionals were established in [3] ; our new approach also allows us to weaken $(\Psi 3)$ of [3] thanks to a subtle reduction of the Riesz-type functionals to the Hardy-Littlewood ones [Theorem 3.2]. This reduction is always possible; we will then give detailed proofs for Hardy-Littlewood functionals, and results concerning Riesz-type functionals are immediate consequences.

2 Notations and Preliminaries

- For $n \in \mathbb{N}^*$, μ is Lebesgue measure in $\mathbb{R}^n$.
- For a Lebesgue measurable set A in $\mathbb{R}^n$, $\mu(A)$ denotes its measure.
- $\mathcal{B}_0 = \{x \in \mathbb{R}^n : |x| < 1\}$ is the ball centered in the origin with radius 1.
- For $a > 0$, $a\mathcal{B}_0 = \{x \in \mathbb{R}^n : |x| < a\}$.
- $M(\mathbb{R}^n)$ is the set of measureable functions in $\mathbb{R}^n$.

From now on:

$$F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \text{ is a Borel measurable function,}$$

$$j : \mathbb{R}^+ \to \mathbb{R} \text{ is a non-increasing function.}$$

- For $f, g \in M_+(\mathbb{R}^n)$:

$$I(f,g) := \int_{\mathbb{R}^n} F\left(f(x), g(x)\right)\, dx \tag{1}$$

  is the Hardy-Littlewood type functional and

$$J(f,g) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F\left(f(x), g(x)\right) j\left(|x-y|\right)\, dx\, dy \tag{2}$$

  is the Riesz type functional.
- For $\ell_1, k_1, \ell_2, k_2 > 0$:
- $\mathcal{B}_1$ is the ball centered in the origin such that $\mu(\mathcal{B}_1) = \ell_1/k_1$.
- $\mathcal{B}_2$ is the ball centered in the origin such that $\mu(\mathcal{B}_2) = \ell_2/k_2$.
- $C(k_1, \ell_1) = \{f \in M(\mathbb{R}^n) : 0 \le f \le k_1 \text{ and } \int f \le \ell_1\}$

– $C(k_1, \ell_1, k_2, \ell_2) = \left\{ (f,g) \in M(\mathbb{R}^n) \times M(\mathbb{R}^n) : \begin{array}{lll} 0 \leq f \leq \ell_1 & \text{and} & \int f \leq \ell_1, \\ 0 \leq g \leq k_\ell & \text{and} & \int g \leq \ell_2. \end{array} \right\}$
– A function $u : \mathbb{R}^n \to \mathbb{R}^+$ is Schwarz-symmetric iff it is radial and radially decreasing. We say that it is strictly Schwarz-symmetric iff it is radial and strictly radially decreasing. For the properties related to Schwarz symmetrization [4].

In this paper, we are interested in the following maximization problems:
For u Schwarz symmetric, $k_1, \ell_1, k_2, \ell_2 > 0$;

$$\sup\{I(u,v) : v \in C(k_1, \ell_1)\} \tag{I1}$$

and

$$\sup\{J(f,g) : (f,g) \in C(k_1, \ell_1, k_2, \ell_2)\}. \tag{J1}$$

For the convenience of the reader, let us recall the Brenier Theorem, which we will use further in the proof of part A of Theorem 3.1 : "Given two random measures μ_1 and μ_2 on $\mathbb{R}^n$ absolutely continuous with respect to the Lebesgue measure such that $\mu_1(\mathbb{R}^n) = \mu_2(\mathbb{R}^n)$, there exists a convex function $\varphi : \mathbb{R}^n \to [0, \infty]$ and a Borel measurable map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(x) = \nabla\varphi(x)$ at almost every $x \in \mathbb{R}^n$ and T pushes forward μ_1 into μ_2, that is to say $\int_{\mathbb{R}^n} H(y)\, d\mu_2(y) = \int_{\mathbb{R}^n} H\left(T(x)\right)\, d\mu_1(x)$ for every Borel function $H : \mathbb{R}^n \to [0, \infty]$."

3 Main Results

In next theorem, we will suppose without any loss of generality that $F(x, 0) = 0$ for $x \geq 0$.

Theorem 3.1
Part 1 (Balls are maximizers of the Hardy-Littlewood type functionals)
The following assertions are equivalent :
[H1] For any $\ell_1, k_1 > 0$, u Schwarz symmetric, (I1) admits the function $k_1 \mathbf{1}_{\mathcal{B}_1}$ as a maximizer.
[H2] The function F satisfies the monotonicity properties:

$$F(x_1, y) \leq F(x_2, y), \forall x_1, x_2, y \geq 0 : x_1 < x_2 \tag{3}$$

$$F(x, ty) \leq tF(x, y), \forall x, y \geq 0, t \in]0, 1[. \tag{4}$$

Part 2 (Balls are the unique maximizers of the Hardy-Littlewood type functionals)

(A) For every $\ell_1, k_1 > 0, u$, Schwarz symmetric, if (I1) admits $k_1 \mathbf{1}_{\mathcal{B}_1}$ as a unique maximizer, then
(B) –
  – $F(x_1, y) < F(x_2, y) : \forall x_1, x_2, y \geq 0, x_1 < x_2$.
  – $F(x, ty) < tF(x, y) : \forall x, y \geq 0, t \in]0, 1[$.

– Conversely, if

(B') –
 – $F(x_1, y) < F(x_2, y) : \forall x_1, x_2, y \geq 0, x_1 < x_2$.
 – $F(x, ty) < tF(x, y) : \forall x, y \geq 0, t \in]0, 1[$.
 – u is strictly Schwarz symmetric, then

(A') For any $\ell_1, k_1 > 0, u$, $k_1 \mathbf{1}_{\mathcal{B}_1}$ is the unique maximizer of (I1).

Proof :

Part 1: First note that $I(u, v)$ is well-defined for any $f \in C(k_1, \ell_1)$ [5, Proposition 3.1].

Let us prove that (H2) implies (H1). Let u be a Schwarz symmetric function, $f \in C(k_1, \ell_1)$, we want to prove that $I(u, f) \leq I(u, k_1 \mathbf{1}_{\mathcal{B}_1})$.

For any $\nu \in S = \partial B$ set

$$\begin{aligned} g_\nu : \mathbb{R}^+ &\to [0, k_1] \\ t &\mapsto f(t\nu). \end{aligned}$$

The monotonicity of u ensures the existence of a non-increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $u(t\nu) = h(t)$.

Using the Appendix (see below), we can write

$$I(u, f) = \int_S d\mathcal{H}^{n-1}(\nu) \int_{\mathbb{R}^+} F\left(h(t), g_\nu(t)\right) t^{n-1} \, dt. \tag{5}$$

Now, let us set $\ell_\nu = \int_{\mathbb{R}^+} g_\nu(t) t^{n-1} \, dt$.

$$\ell_1 \geq \int_{\mathbb{R}^n} f = \int_S d\mathcal{H}^{n-1}(\nu) \int_{\mathbb{R}^+} g_\nu(t) t^{n-1} \, dt \geq \int_S \ell_\nu d\mathcal{H}^{n-1}(\nu).$$

From this it follows easily that $\ell_\nu < \infty$ for $\mathcal{H}^{n-1}$ a.e. $\nu \in S$. For such ν, we set

$$i_\nu : \mathbb{R}^+ \to [0, k_1], t \mapsto k_1 \mathbf{1}_{[0, \rho_\nu]},$$

where ρ_ν is chosen such that $\displaystyle\int_{\mathbb{R}^+} i_\nu(t) t^{n-1} \, dt = \int_{\mathbb{R}^+} g_\nu(t) t^{n-1} \, dt$. Then

$$\ell_\nu = \int_{\mathbb{R}^+} i_\nu(t) t^{n-1} \, dt = \frac{k_1 \rho_\nu^n}{n}. \tag{6}$$

The map $T_\nu : \mathbb{R}^+ \to [0, \rho_\nu]$ defined for any $t \in \mathbb{R}^+$ by

$$\int_0^t g_\nu(s) s^{n-1} \, ds = \int_0^{T_\nu(t)} i_\nu(s) s^{n-1} \, ds \tag{7}$$

has the push forward property :

$$\int_{\mathbb{R}^+} H(s) i_\nu(s) s^{n-1} \, ds = \int_{\mathbb{R}^+} H\left(T_\nu(t)\right) g_\nu(t) t^{n-1} dt \tag{8}$$

for every Borel measurable function $H : \mathbb{R}^+ \to \mathbb{R}^+$ by [1, Theorem 1]. Since $0 \le g_\nu \le k_1$, it is obvious that:

$$T_\nu(t) \le t \quad \forall t \in \mathbb{R}^+, \nu \in S. \tag{9}$$

In the last preparatory step, let us define $z(x) = i_\nu(t)$, $\nu = x/|x|$ and $t = |x|$ for $x \in \mathbb{R}^n \setminus \{0\}$; $z(0) = k_1$.
Applying again the Coarea formula (see Appendix), it follows by the properties of z,

$$\begin{aligned} I(u,z) &= \int_S d\mathcal{H}^{n-1}(\nu) \int_{\mathbb{R}} F\left(h(s), i_\nu(s)\right) s^{n-1}\, ds \\ &= \int_S d\mathcal{H}^{n-1}(\nu) \int_0^{\rho_\nu} F\left(h(s), k_1\right) s^{n-1}\, ds \\ &= \frac{1}{k_1} \int_S d\mathcal{H}^{n-1}(\nu) \int_{\mathbb{R}} F\left(h(s), k_1\right) i_\nu(s) s^{n-1}\, ds. \end{aligned}$$

Applying (8) with $H(s) = F(h(s), k_1)$, we find

$$I(u,z) = \int_S d\mathcal{H}^{n-1}(\nu) \int_{\mathbb{R}} F\left(h\left(T_\nu(s)\right), k_1\right) \frac{g_\nu(s)}{k_1} s^{n-1}\, ds. \tag{10}$$

By (9) and the monotonicity of u, we have $f(T_\nu(s)) \ge f(s)$ for any non-negative s.
Therefore, (3) and (9) imply that:

$$F\left(h\left(T_\nu(s)\right), k_1\right) \ge F\left(h(s), k_1\right) \quad \forall s \ge 0. \tag{11}$$

Recalling that $0 \le g_\nu(s) \le k_1$, using (4) we obtain

$$F\left(h(s), k_1\right) \frac{g_\nu(s)}{k_1} \ge F\left(h(s), g_\nu(s)\right) \quad \forall s \ge 0. \tag{12}$$

Combining (10) with (12), we deduce

$$\begin{aligned} I(u,z) &= \int_S d\mathcal{H}^{n-1}(\nu) \int_{\mathbb{R}^+} F\Big(h\left(T_\nu(s)\right), k_1\Big) \frac{g_\nu(s)}{k_1} s^{n-1}\, ds \\ &\ge \int_S d\mathcal{H}^{n-1}(\nu) \int_{\mathbb{R}^+} F\Big(h(s), g_\nu(s)\Big) s^{n-1}\, ds = I(u,f) \\ I(u,z) &\ge I(u,f). \end{aligned} \tag{13}$$

It remains to prove that

$$I(u, k_1 \mathbf{1}_{\mathcal{B}_1}) \ge I(u,z). \tag{14}$$

Let us first define the set

$$A := \left\{ x \in \mathbb{R}^n : |x| \in [0, \rho_\nu] \text{ for } \nu = \frac{x}{|x|} \right\},$$

$$z(x) = k_1 \mathbf{1}_A(x).$$

$$\mu_A = \int_S d\mathcal{H}^{n-1}(\nu) \int_0^{\rho_\nu} s^{n-1}\, ds = \int_S \frac{\rho_\nu{}^n}{n}\, d\mathcal{H}^{n-1}(\nu) = \frac{1}{k_1}\int_S \ell_\nu\, d\mathcal{H}^{n-1}(\nu) = \frac{\ell_1}{k_1}.$$

$$I(u, k_1\mathbf{1}_{\mathcal{B}_1}) = \int_{\mathcal{B}_1} F\left(u(x), k_1\right)\, dx.$$

$$I(u, z) = \int_A F\left(u(x), k_1\right)\, dx.$$

By the definition of $\mathcal{B}_1$, note that $\mu(A) = \mu(\mathcal{B}_1)$. Now consider the transport map $T_0 : A \setminus \mathcal{B}_1 \to \mathcal{B}_1 \setminus A$ satisfying $\det \nabla T_0 = 1$ on $A \setminus \mathcal{B}_1$.
Since $A \setminus \mathcal{B}_1$ and $\mathcal{B}_1 \setminus A$ have the same measure, it follows by the change of variables $x = T_0(y)$ that

$$\begin{aligned} I(u, k_1\mathbf{1}_{\mathcal{B}_1}) &= \int_{\mathcal{B}_1\cap A} F\Big(u(x), k_1\Big)\, dx + \int_{\mathcal{B}_1\setminus A} F\Big(u(x), k_1\Big)\, dx \\ &= \int_{\mathcal{B}_1\cap A} F\Big(u(x), k_1\Big)\, dx + \int_{A\setminus B_1} F\Big(u\Big(T_0(y)\Big), k_1\Big) \det\nabla T_0(y)\, dy \\ &= \int_{\mathcal{B}_1\cap A} F\Big(u(x), k_1\Big)\, dx + \int_{A\setminus B_1} F\Big(u\Big(T_0(y)\Big), k_1\Big)\, dy. \end{aligned} \tag{15}$$

But, if $y \in A \setminus \mathcal{B}_1$, $T_0(y) \in \mathcal{B}_1 \setminus \mathcal{B}_0$, then we obtain, using (3) :

$$F\left(u\left(T_0(y), k_1\right)\right) \geq F\left(u\left(y\right), k_1\right)\right) \quad \forall y \in A \setminus \mathcal{B}_1. \tag{16}$$

Combining (15) and (16), we deduce that

$$I(u, k_1\mathbf{1}_{\mathcal{B}_1}) \geq I(u, z).$$

□

(H1)⇒(H2):
Let us assume that $I(u, k_1\mathbf{1}_{\mathcal{B}_1}) \geq I(u, f)$ for any $k_1, \ell_1 > 0$ and $f \in C(k_1, \ell_1)$.
We want to prove (3) and(4).
Let $x_1, x_2, y \geq 0$ with $x_1 < x_2$ fixed. Since $\mu(2\mathcal{B}_0 \setminus \mathcal{B}_0) = (2^n - 1)\mu(\mathcal{B}_0)$, we can chose $E \subset 2\mathcal{B}_0 \setminus \mathcal{B}_0$ such that $\mu(E) = \mu(\mathcal{B}_0)$. Let us set $u = x_2\mathbf{1}_{\mathcal{B}_0} + x_1\mathbf{1}_{2\mathcal{B}_0\setminus\ell_1\mathcal{B}_0}$, $f_1 = y\mathbf{1}_E$, $k_1 = y$, $f_1 \in C(k_1, \ell_1)$. Choose ℓ_1 such that $\mathcal{B}_1 \equiv \mathcal{B}_0$.
Therefore, $I(u, k_1, \mathbf{1}_{\mathcal{B}_1}) = \mu(\mathcal{B}_0)F(x_2, y)$.

$$I(u, f_1) = \mu(\mathcal{B}_0)F(x_1, y).$$

But

$$I(u, k_1\mathbf{1}_{\mathcal{B}_1}) \geq I(u, f_1), \tag{17}$$

implying that

$$\mu(\mathcal{B}_0)F(x_2, y) \geq \mu(\mathcal{B}_0)F(x_1, y) \tag{18}$$

and (3) follows immediately.
Now we would like to prove (4).
For any $x, y \in \mathbb{R}^+$ and $t \in]0, 1[$ fixed, define $u = x\mathbf{1}_{\mathcal{B}_0}$, $k_1 = y$. Select ℓ_1 such that $\ell_1 = tk_1\mu(\mathcal{B}_0)$. Therefore, $k_1\mathbf{1}_{\mathcal{B}_1} = y\mathbf{1}_{t\mathcal{B}_1}$. Set $f_1 = ty\mathbf{1}_{\mathcal{B}_0}$, $f_1 \in C(k_1, \ell_1)$ and by our assumption $I(u, k_1\mathbf{1}_{\mathcal{B}_1}) \geq I(u, f_1)$.

But :

$$I(u,k_1\mathbf{1}_{\mathcal{B}_1}) = t\mu(\mathcal{B}_0)F(x,y) \geq I(u,f_1) = \mu(\mathcal{B}_0)F(x,ty), \tag{19}$$

from which we deduce that

$$F(x,ty) \leq tF(x,y). \tag{20}$$

We now turn to the proof of Part 2 of our result:
(A)⇒(B)
We are assuming that $I(u,k_1\mathbf{1}_{\mathcal{B}_1}) > I(u,f)$ for every Schwarz symmetric function u and $k_1,\ell_1 > 0$, $f \in C(k_1,\ell_1)$. Then (18) and(20) hold true with strict sign, it follows that (3) and (4) are strict.
(B')⇒(A')
For this case, we are supposing that (3) and (4) are strict, u is strictly Schwarz symmetric, we want to prove that: For any $\ell_1,k_1 > 0$, $I(u,k_1\mathbf{1}_{\mathcal{B}_1}) > I(u,f)$ for any $f \in C(k_1,\ell_1)$. This is equivalent to prove that $I(u,k_1\mathbf{1}_{\mathcal{B}_1}) = I(u,f)$ if and only if $f = k_1\mathbf{1}_{\mathcal{B}_1}$.
Assume that $I(u,k_1\mathbf{1}_{\mathcal{B}_1}) = I(u,f)$. It follows that $I(u,k_1\mathbf{1}_{\mathcal{B}_1}) = I(u,z) = I(u,f)$, where z is the function defined by $z(x) = k_1\mathbf{1}_A(x)$ in the proof of Part 1.
In particular, equality holds in (11) and (12) for a.e $\nu \in S$ and $s \in \mathrm{spt}g_\nu$ and (16) for a.e $y \in A\setminus\mathcal{B}_1$. That is, we have :

$$F\left(h(t_\nu(s),k_1\right) = F\left(h(s),k_1\right);\ \text{for a.e } \nu \in S, s \in \mathrm{spt}g_\nu, \tag{21}$$

$$F\left(h(s),k_1\right)\frac{g_\nu(s)}{k_1} = F\left(h(s),g_\nu(s)\right);\ \text{for a.e } \nu \in S, s \in \mathrm{spt}g_\nu, \tag{22}$$

$$F\left(u\left(T_0(y)\right),k_1\right) = F\left(u(y),k_1\right);\ \text{for a.e } y \in A\setminus\mathcal{B}_1. \tag{23}$$

h, g_ν, T_ν and T_0 are as in the quoted proof.
Now, if (3) holds with a strict sign and u (consequently h) is strictly decreasing then (21)-(23) imply that

$$\begin{aligned} T_\nu(s) = s &\quad \text{for a.e } \nu \in S, s \in \mathrm{spt}g_\nu, \\ T_0(x) = x &\quad \text{for a.e } x \in A\setminus\mathcal{B}_1. \end{aligned}$$

The second condition forces $A\setminus\mathcal{B}_1 = \mathcal{B}_1\setminus A$ implying that $A \equiv \mathcal{B}_1$ (up to null sets).
Therefore, $\rho_\nu = r$ for a.e $\nu \in S$ (r is defined by: $\mu(\mathcal{B}_0)r^n = \frac{\ell_1}{k_1}$). On the other hand, the first condition implies that $\rho_\nu \geq T_\nu(s) = s$ for a.e $s \in \mathrm{spt}g_\nu$. In conclusion, $\mathrm{spt}g_\nu \subset [0,r] = \mathrm{spt}i_\nu$. Finally applying (8) to $H(s) = \mathbf{1}_{\mathrm{spt}g_\nu}$, we can easily conclude that $g_\nu = k_1$ on its support and then $\mathrm{spt}g_\nu = [0,r]$, consequently $f = k_1\mathbf{1}_{\mathcal{B}_1}$.
Remark 3.1: Note that (A)⇒(B) cannot be done if u is strictly Schwarz symmetric since in our construction, we need u to be constant on some domains. However for any u Schwarz symmetric function the condition $F(x_1,y) < F(x_2,y)$ for $x_1,x_2,y \geq 0$ with $x_1 < x_2$ is necessary for the uniqueness of the maximizer. More precisely consider $F(x,y) = y^2$. For any $f = k_1\mathbf{1}_A$ where $\mu(A) = \mu(\mathcal{B}_1)$ f is a maximizer of (I1) independently of the choice of u since
$I(u,f) = k_1^2\mu(A) = k_1^2\mu(\mathcal{B}_1) = I(u,k_1\mathbf{1}_{\mathcal{B}_1})$. □

Remark 3.2: Let us recall that Theorem 3.1 (sufficient condition) was proved in [5] under the further assumption that F is super-modular, ie.

$$F(r_1, s_1) + F(r_2, s_2) \geq F(r_1, s_2) + F(r_2, s_1), \tag{24}$$

whenever $0 \leq r_1 < r_2$ and $0 \leq s_1 < s_2$. Super-modularity is a necessary and sufficient condition in many inequalities of rearrangement theory, but is not necessary for the validity of Theorem 3.1. We are now going to construct a Lipschitz function $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the hypothesis of Theorem 3.1 fand failing to be super-modular. We set $F(r, s) = rs$ on the region $E_0 = \{(r, s) : r > 1$ and failing to be super-modular. We set $F(r, s) = rs$ on the region $E_0 = \{(r, s) : r > 1\}$, and then extend F on the regions $E_k, k = 1, ..., 4$, as follows :

$$F(r,s) = \begin{cases} -rs + 2r + 4s - 4, & \text{on } E_1 = [0,1] \times [1,2]; \\ rs - 2r + 4, & \text{on } E_1 = [1,2] \times [2,\infty]; \\ rs + 2r, & \text{on } E_3 = [0,1] \times [2,\infty]; \\ 3rs - 2r, & \text{on } E_4 = [0,1] \times [1,2]. \end{cases}$$

We have $F(r, s) = 0$ on $\{rs = 0\}$. The partial derivatives F_r and F_s are piecewise affine, and their restriction to the and horizontal lines, respectively, are non negative and increasing. Therefore $F(r, s)$ is increasing $F \geq 0$ on $\mathbb{R}^n \times \mathbb{R}_+$ as $F \geq 0$ on $[0, 1] \times [0, 1]$: F is separately convex, i.e. $F(r, s)$ and $F(r, s)$ are both joining $(0, F(r, 0)) = (0, 0)$ and $(s, F(r, s))$: that is to say $F(r, \lambda s) \leq \lambda F(r, s)$ for every $\lambda \in [0, 1]$, and (2) holds true. In conclusion it is easily checked that F fails to be super-modular on E_1. □

Without any loss of generality, we suppose that $F(x, 0) = F(0, y) = 0$ for $x, y \geq 0$ in Theorem 3.2.

Theorem 3.2

Part 1 (Balls are maximizers of the Riesz type functionals)

Suppose that $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

$(F1)(i)$ $F(tx, y) \leq tF(x, y) \quad \forall x, y \geq 0, t \in]0, 1[$

(ii) $F(x, ty) \leq tF(x, y) \quad \forall x, y \geq 0, t \in]0, 1[$

$(F2)$ $F(b, d) - F(b, c) - F(a, d) + F(a, c) \geq 0 \quad \forall 0 \leq a < b, 0 \leq c < d.$

$[(j1)]$ $j : \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing, then for any $(f_1, f_2) \in C(k_1, \ell_1, k_2, \ell_2)$:

$$J(f_1, f_2) \leq J(k_1 \mathbf{1}_{\mathcal{B}_1}, k_2 \mathbf{1}_{\mathcal{B}_2}).$$

Part 2 Uniqueness of the maximizers (up to translations)

If in addition (F1), (F2) and (j1) hold true with strict sign then for any $(f, g) \in C(k_1, \ell_1, k_2, \ell_2)$:

$$J(f_1, f_2) < J(k_1 \mathbf{1}_{\mathcal{B}_1}, k_2 \mathbf{1}_{\mathcal{B}_2}) = J(h_1, h_2)$$

where h_1 and h_2 are translates by the same vector of $k_1 \mathbf{1}_{\mathcal{B}_1}$ and $k_2 \mathbf{1}_{\mathcal{B}_2}$ (respectively).

(F1) is weaker than (Ψ3) of [3].

Proof: First note that (F2) together with the fact that $F(x, 0) = F(0, y) = 0$

imply that F is non-decreasing with respect to each variable, and consequently it is non-negative. Let $(f,g) \in C(k_1,\ell_1,k_2,\ell_2)$. (F2) together with (j1) imply that: $J(f,g) \le J(f^*,g^*)$ (f^* denotes the Schwarz symmetrization of f) by [4, Theorem 1]. Thanks to (F1)(i):

$$\begin{aligned} J(f^*,g^*) &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f^*(x)}{k_1} F\left(k_1, g^*(y)\right) j\left(|x-y|\right)\, dx\, dy \\ &= \frac{1}{k_1}\int_{\mathbb{R}^n} f^*(x) u(x)\, dx \end{aligned}$$

where $u(x) = \int_{\mathbb{R}^n} F\left(k_1, g^*(y)\right) j\left(|x-y|\right)\, dy$ (which is of course a Schwarz symmetric function by (j1)).
By Theorem 3.1

$$\int_{\mathbb{R}^n} f^*(x) u(x)\, dx \le \int_{\mathbb{R}^n} k_1 \mathbf{1}_{\mathcal{B}_1}(x) u(x)\, dx$$

$$\begin{aligned} J(f,g) \le J(f^*,g^*) &\le \frac{1}{k_1}\int_{\mathbb{R}^n} k_1 \mathbf{1}_{\mathcal{B}_1}(x) u(x)\, dx = \int_{\mathcal{B}_1} u(x)\, dx \\ &\le \int_{\mathcal{B}_1}\int_{\mathbb{R}^n} F\left(k_1, g^*(y)\right) j\left(|x-y|\right)\, dx\, dy \\ &\le J(k_1 \mathbf{1}_{\mathcal{B}_1}, g^*). \end{aligned}$$

Similarly, using (F1)(ii), we deduce that

$$J(k_1 \mathbf{1}_{\mathcal{B}_1}, g^*) \le J(k_1 \mathbf{1}_{\mathcal{B}_1}, k_2 \mathbf{1}_{\mathcal{B}_2}).$$

In conclusion, for any $(f,g) \in C(k_1,\ell_1,k_2,\ell_2)$, we obtain

$$J(f,g) \le J(f^*,g^*) \le J(k_1 \mathbf{1}_{\mathcal{B}_1}, g^*) \le J(k_1 \mathbf{1}_{\mathcal{B}_1}, k_2 \mathbf{1}_{\mathcal{B}_2}).$$

Using equality cases established in [4, Theorem 1] and Part 2 of Theorem 3.1, we can easily prove Part 2 of Theorem 3.2. □

4 Concluding Remarks

Our results are optimal and considerably improve the previous ones obtained by G.B. Burton [2], the author himself and C. Draghici. It turns out that mass transportation techniques are very fruitful to prove the sharpness of our assumptions. It has also enabled us to drop the supermodularity condition which was used in [3, 5].

Appendix (Coarea formula)

We are changing variables through $\Phi : S\times]0,\infty[\to \mathbb{R}^n\backslash\{0\}$ defined by $\Phi(\nu,t)=\nu t$. Let (ν,t) be fixed and let $\{\varepsilon_i\}_{i=1}^{n-1}$ denote an orthonormal basis of $T_\nu S$ and e a basis of $T_t]0,\infty[$. Then $\{(\varepsilon_i,0),(0,e) : i=1,\dots,n-1\}$ is an orthonormal basis of $T_{(\nu,t)}\,(S\times]0,\infty[)$ with

$$d\Phi_{(\nu,t)}(\varepsilon,0)=\lim_{h\to 0}\frac{\Phi(\nu+h\varepsilon_i,t)-\Phi(\nu,t)}{h}=t\varepsilon$$
$$d\Phi_{(\nu,t)}(0,e)=\lim_{h\to 0}\frac{\Phi(\nu,t+h\varepsilon_i)-\Phi(\nu,t)}{h}=\nu.$$

Therefore :

$$d\Phi_{(\nu,t)}(\varepsilon_1,0)\wedge\ldots\wedge d\Phi_{(\nu,t)}(0,e)=t^{n-1}\varepsilon_1\wedge\ldots\wedge e.$$

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