QUANTITATIVE STABILITY ESTIMATE OF THE HARDY-LITTLEWOOD FUNCTIONAL UNDER CONSTRAINTS

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Abstract. We prove the uniqueness of the maximizers of a Hardy-Littlewood type functional under constraints. We also establish a quantitative stability estimate.

Introduction

Strenghtened versions of: Hardy-Littlewood inequality [3], Hardy inequality [4,10], Sobolev inequality [5], Pölya-Szegö inequality [6], isoperimetric inequality [7,9] have received quite recently the attention of many mathematicians. Most of them have taken advantage of the breaking through papers of Brenier [1] and McCann [14,15] dealing with mass transportation theory. In fact, if turned out that the approches developped in the latter papers are very fruitful to quantify some functional inequalities connected to rearrangement. In this paper, we follow the traces of the mathematicians cited above to treat an optimization problem under constraints. More precisely, thanks to techniques developped in [1,14,15], we prove the uniqueness of the maximizer of the Hardy-Littlewood functional

$$\mathcal{F}(u) = \int_{\mathbb{R}^n} F(|x|, u(x))dx \,, \quad \forall u \in X \,,$$

where

$$X := \left\{ u \in L^p(\mathbb{R}^n) : 0 \le u \le a \,, \int_{\mathbb{R}^n} u^p \le 1 \right\} \,,$$

a is a positive number and $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ (here $\mathbb{R}^+ := [0, \infty)$) is Lebesgue measurable with respect to the first variable and continuous with respect to the second one. We are interested in the maximization problem:

$$\max \mathcal{F}(u), u \in X \tag{1}$$

(1) is connected to two relevant problems arising in physics and non-linear optics. The first one is related to the study of standing waves for the non-linear Schrödinger equation which leads to the minimization of the energy functional:

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx - \int_{\mathbb{R}^n} F(|x|, u(x)) \, dx \,,$$

over all $u \in H^1(\mathbb{R}^n)$, $u \ge 0$ having a prescribed L^2 norm[11]. The function F describes the index of refraction of the media in which the wave propagates. The model function is:

$$F(r, s) = p(r)s^2 + q(r)s^d \,, \quad 2 < d < 2 + \frac{4}{n} \,,$$

where p and q are positive dacreasing functions. The constraint on d has to be assumed to avoid non-existence issues due to unbalanced scalings. The two terms of the energy functional are in competition. Indeed if we try to maximize $\int_{\mathbb{R}^n} F(|x|, u(x)) \, dx$

under the additional natural constraint $u \leq a$, then the unique maximizer is given by the function $a\,1_{rB}$, having infinite dirichlet integral (here B is the Euclidian unit ball and $r > 0$ is such that $\int_{\mathbb{R}^n} (a\,1_{rB})^P = 1$). In this note, we identify assumptions on the integrand F ensuring that $\int_{\mathbb{R}^n} F(|x|, u(x))dx$ presents this behavior. (1) is also connected to variational problems for steady axisymmetric vortex-rings in which kinetic energy is maximized subject to prescribed impulse. This involves the maximization of Riesz-type functionals with functions in X,[2], where G.R.Burton has proved the existence of maximizers in an extended constraint set. The generalization of his method hinges on the resolution of an optimization problem which reduces to (1), [12].

Our main result is the following:

Theorem 1. *Let $a > 0$, $p \geq 1$, $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ (here $\mathbb{R}^+ := [0, \infty)$) be Carathéodory function satisfying:*

(H1) *for every $s \in \mathbb{R}^+$, $F(\cdot, s)$ is non-increasing,*

(H2) *for a.e $r > 0$ and every $0 < t < 1$,*

$$F(r, ta) \leq t^p F(r, a),$$

then:

(C1) *The function $w = a1_E$ is a maximum of $\mathcal{F}$ on X, E is the ball centred in the origin having Lebesgue measure $|E| = 1/a^p$.*

If additionally (H1) holds with a strict sign, then

(C2) *$w = a1_E$ is the unique maximizer of $\mathcal{F}$ on X.*

Moreover if F satisfies:

(H3) *There exists $\lambda > 0$ such that: for every $0 < r_1 < r_2$:*

$$F(r_1, a) \geq F(r_2, a) + \lambda(r_2 - r_1)\,, \textit{then}$$

(C3) *for every $u \in X$, we have*

$$\int_{\mathbb{R}^n} |u - w|^p \leq K(n, p, a)\sqrt{\frac{\mathcal{F}(w) - \mathcal{F}(u)}{\lambda}}\,,$$

where $K(n, p, a)$ is a constant depending only on n, p and a.

(C2) improves [Theorem3.3, 8] where the supermodularity of F is needed.

(C3) contains quantitative information of the gap between any u in X and w in L^p proportionally to the gap of their images by $\mathcal{F}$. It tells us, among other things, that if u is such that $\mathcal{F}(u)$ is close to $\mathcal{F}(w)$ then u, w inherit this property. This quantitative estimates are crucial in proving the stability arguments developped in [2]. More details about this issue are given in [13].

In the following, if not precised, all statements about measurability refer to Lebesgue measure.

Proof of our result

The proof of Theorem 1 is based on a basic result in mass transportation theory, namely the Brenier Theorem [1] (see also [14]): given two Radon measures μ_1, μ_2 on $\mathbb{R}^n$, both absolutely continuous with respect to the Lebesgue measure and such that $\mu_1(\mathbb{R}^n) = \mu_2(\mathbb{R}^n)$, there exists a convex function $\varphi : \mathbb{R}^n \to [0, \infty]$ and a Borel measurable map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(x) = \nabla\varphi(x)$ at a.e. $x \in \mathbb{R}^n$ and T pushes forward μ_1 into μ_2, i.e.

$$\int_{\mathbb{R}^n} H(y) d\mu_2(y) = \int_{\mathbb{R}^n} H(T(x)) d\mu_1(x) , \tag{2}$$

for every Borel function $H : \mathbb{R}^n \to [0, \infty]$.

We pass now to prove Theorem 1.

Proof of Theorem 1. By (H2), as $F(r, \cdot)$ is continuous for a.e. $r \in \mathbb{R}^+$, we deduce that $F(r, 0) = 0$ for a.e. $r \in \mathbb{R}^+$. We let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and denote by σ the $(n-1)$-dimensional Hausdorff measure restricted to S^{n-1}.

Step one: Let us fix $u \in X$ and construct an auxiliary function $v = a\,1_G$ by letting

$$G := \left\{ x \in \mathbb{R}^n : |x| < \kappa\left(\frac{x}{|x|}\right) \right\} ,$$

where we have introduced $\kappa : S^{n-1} \to \mathbb{R}^+$,

$$\kappa(\nu) := \left(\frac{n}{a^p} \int_0^\infty u(r\nu)^p r^{n-1} dr \right)^{1/n} , \quad \nu \in S^{n-1} . \tag{3}$$

Note that $v(r\nu) = a\,1_{[0,\kappa(\nu)]}(r)$, and that the value of $\kappa(\nu)$ has been chosen so that the measures

$$1_{\mathbb{R}^+}(r) u(r\nu)^p\, r^{n-1}\, dr \quad \text{and} \quad 1_{\mathbb{R}^+}(r) v(r\nu)^p\, r^{n-1}\, dr ,$$

have the same total mass on $\mathbb{R}$. For every $\nu \in S^{n-1}$, let T_ν denote the map given by Brenier theorem. By construction T_ν is increasing on $\mathbb{R}$, moreover, thanks to (2) we have

$$\int_{\mathbb{R}^+} H(r) v(\nu r)^p r^{n-1} dr = \int_{\mathbb{R}^+} H(T_\nu(r)) u(\nu r)^p r^{n-1} dr , \tag{4}$$

for every Borel function $H : \mathbb{R} \to [0, \infty]$: in particular $T_\nu(r) \in [0, k(\nu)]$ for a.e. $r \in \mathbb{R}$. Note also that, as $0 \le u \le a$, we clearly have

$$T_\nu(r) \le r , \quad \text{for a.e. } r \in \mathbb{R}^+. \tag{5}$$

We are going to prove that $\mathcal{F}(u) \le \mathcal{F}(v)$. By (H2) we have that

$$\mathcal{F}(u) = \int d\sigma(\nu) \int_{\mathbb{R}^+} F(r, u(r\nu)) r^{n-1} dr \le \int d\sigma(\nu) \int_{\mathbb{R}^+} \frac{F(r, a)}{a^p} u(r\nu)^p r^{n-1} dr , \tag{6}$$

while at the same time, thanks to (4)

$$\begin{aligned} \mathcal{F}(v) &= \int d\sigma(\nu) \int_0^{\kappa(\nu)} F(r, a) r^{n-1} dr = \int d\sigma(\nu) \int_{\mathbb{R}^+} \frac{F(r, a)}{a^p} v(r\nu)^p r^{n-1} dr \\ &= \int d\sigma(\nu) \int_{\mathbb{R}^+} \frac{F(T_\nu(r), a)}{a^p} u(r\nu)^p r^{n-1} dr , \end{aligned}$$

By (H1) and (5) it follows immediately that $\mathcal{F}(u) \le \mathcal{F}(v)$.

Step two: We are going to prove that $\mathcal{F}(v) \leq \mathcal{F}(w)$. We start by noticing that $|E| = |G|$. Indeed by (3)

$$|G| = \int \frac{\kappa(\nu)^n}{n}\, d\sigma(\nu) = \frac{1}{a^p} \int_{\mathbb{R}^n} u^p = |E|\,.$$

In particular $|E \setminus G| = |G \setminus E|$, and, without loss of generality, $|E \setminus G| > 0$. Consider the Brenier map $T : \mathbb{R}^n \to \mathbb{R}^n$ between $1_{E\setminus G}(x)dx$ and $1_{G\setminus E}(y)dy$. By (2),

$$\int_{E\setminus G} H(y)dy = \int_{G\setminus E} H(T(x))dx\,, \tag{7}$$

for every Borel function $H : \mathbb{R}^n \to [0, \infty]$. On choosing $H(y) = F(y, a)$ we find

$$\int_{E\setminus G} F(|x|, a)dx = \int_{G\setminus E} F(|T(x)|, a)dx\,, \tag{8}$$

while, on taking $H(y) = 1_{E\setminus G}(y)$, we prove that $T(x) \in E \setminus G$ for a.e. $x \in G \setminus E$. As E is a ball, this last remark implies that

$$|T(x)| \leq |x|\,, \quad \text{for a.e. } x \in G \setminus E\,. \tag{9}$$

On combining (9) with (8) we get

$$\begin{aligned}
\mathcal{F}(w) &= \int_{G\cap E} F(|x|, a)dx + \int_{E\setminus G} F(|x|, a)dx \\
&= \int_{G\cap E} F(|x|, a)dx + \int_{G\setminus E} F(|T(x)|, a)dx \\
&\geq \int_{G\cap E} F(|x|, a)dx + \int_{G\setminus E} F(|x|, a)dx = \mathcal{F}(v)\,,
\end{aligned} \tag{10}$$

and the conclusion follows.

Proof of (C2): Let us now assume that for every $s \in \mathbb{R}^+$ the function $F(\cdot, a)$ is strictly decreasing, and consider a function $u \in X$ that maximizes $\mathcal{F}$ on X, i.e. such that $\mathcal{F}(u) = \mathcal{F}(w)$. We want to show that $u = w$ a.e. on $\mathbb{R}^n$. Let us prove that $G = E$ up to null sets. Indeed, let R denote the radius of the ball E. If $|G \setminus E| > 0$, then we can consider T and repeat the above argument. Since $F(\cdot, a)$ is strictly decreasing and equality holds in (10), we find that $|T(x)| = |x|$ for a.e. $x \in G \setminus E$. Thus $|T(x)| \geq R$ for a.e. $x \in \mathbb{R}$; but $T(x) \in E \setminus G$ for a.e. $x \in G \setminus E$, therefore it must be $|G \setminus E| = 0$, a contradiction. As $G = E$ up to null sets, we have $\kappa(\nu) = R$ for every $\nu \in S^{n-1}$. The equality sign in (6) implies that, for σ-a.e. $\nu \in S^{n-1}$, $T_\nu(r) = r$ for a.e. $r \in \{t : u(\nu t) > 0\}$. As $0 \leq T_\nu \leq \kappa(\nu) = R$, by (4) and (5) we deduce that $\{t : u(\nu t) > 0\} \subset [0, R]$ for σ-a.e. $\nu \in S^{n-1}$. On applying (4) to $H = 1_{\{t:u(\nu t)>0\}}$ we deduce $u(\nu r) = a$ on $\{t : u(\nu t) > 0\}$, therefore that $u(\nu r) = a\, 1_{[0,R]}(r)$. In particular $u = w$ a.e. on $\mathbb{R}^n$.

We come now to a quantitative stability estimate:

Proof of (C3): Let $\delta := \mathcal{F}(w) - \mathcal{F}(u)$. Thanks to (H3), from (6) and (10) we find that

$$\delta \;\geq\; \lambda \int_{G\setminus E} (|x| - |T(x)|)dx\,, \tag{11}$$

$$\delta \;\geq\; \lambda \int d\sigma(\nu) \int_0^\infty (r - T_\nu(r)) \frac{u(r\nu)^p}{a^p} r^{n-1} dr\,. \tag{12}$$

We now consider (11) and (12) separately:

Step one: Let $\varepsilon \in (0,R)$, then $(R+\varepsilon)^n \le R^n + C(n)R^{n-1}\varepsilon$. Thus

$$\begin{aligned}\frac{1}{a^p}\int_{\mathbb{R}^n}|w-v|^p &= |E\Delta G| = 2|G\setminus E| \le 2\{|G\setminus B_{R+\varepsilon}| + |B_{R+\varepsilon}\setminus B_R|\} \\ &\le C(n)\left\{|G\setminus B_{R+\varepsilon}| + \varepsilon R^{n-1}\right\}.\end{aligned}$$

If $x \in G\setminus B_{R+\varepsilon}$ then $|x| \ge R+\varepsilon \ge |T(x)|+\varepsilon$. By (11) we have $|G\setminus B_{R+\varepsilon}| \le (\delta/\varepsilon\lambda)$, therefore we come to

$$\frac{1}{a^p}\int_{\mathbb{R}^n}|w-v|^p \le C(n)\left\{\frac{\delta}{\lambda\varepsilon} + \varepsilon R^{n-1}\right\}.$$

We minimize over $\varepsilon \in [0,R]$ and find

$$\frac{1}{a^p}\int_{\mathbb{R}^n}|w-v|^p \le C(n)\max\left\{\sqrt{\frac{\delta R^{n-1}}{\lambda}}, \frac{\delta}{\lambda R}\right\}. \tag{13}$$

Step two: We start by noticing that

$$\begin{aligned}\int_{\mathbb{R}^n}|u-v|^p &= \int_G (a-u)^p + \int_{\mathbb{R}^n\setminus G} u^p \le \int_G (a^p-u^p) + \int_{\mathbb{R}^n\setminus G} u^p = 2\int_{\mathbb{R}^n\setminus G} u^p \\ &= 2\int \tau_2(\nu)d\sigma(\nu),\end{aligned}$$

where, for every $\nu \in S^{n-1}$, we have set

$$\tau_1(\nu) := \int_0^{\kappa(\nu)} u(r\nu)^p r^{n-1}dr, \quad \tau_2(\nu) := \int_{\kappa(\nu)}^{\infty} u(r\nu)^p r^{n-1}dr.$$

Since $a^p\kappa(\nu)^n/n = \tau_1(\nu)+\tau_2(\nu)$, we have that

$$a^p\frac{T_\nu(r)^n}{n} \le \tau_1(\nu) + \int_{\kappa(\nu)}^r u(t\nu)t^{n-1}dt \le \tau_1(\nu) + a^p\frac{r^n}{n} - a^p\frac{\kappa(\nu)^n}{n},$$

for every $r \ge \kappa(\nu)$, i.e.,

$$T_\nu(r) \le \left(r^n - \frac{n\tau_2(\nu)}{a^p}\right)^{1/n}, \quad \forall r \ge \kappa(\nu).$$

Then by (12) we deduce that

$$\begin{aligned}\frac{\delta}{\lambda} &\ge \int d\sigma(\nu)\int_{\kappa(\nu)}^{\infty}\left[r - \left(r^n - \frac{n\tau_2(\nu)}{a^p}\right)^{1/n}\right]u(r\nu)^p r^{n-1}dr \\ &\ge \int d\sigma(\nu)\int_{\kappa(\nu)}^{\infty}\frac{\tau_2(\nu)}{a^p}\left(r^n - \frac{n\tau_2(\nu)}{a^p}\right)^{(1/n)-1} u(r\nu)^p r^{n-1}dr \\ &\ge \int \frac{\tau_2(\nu)^2}{a^p}\left(\kappa(\nu)^n - \frac{n\tau_2(\nu)}{a^p}\right)^{(1/n)-1} d\sigma(\nu) \\ &= \frac{c(n)}{a^{p/n}}\int \tau_2(\nu)^2\tau_1(\nu)^{(1/n)-1}d\sigma(\nu).\end{aligned}$$

By Hölder inequality

$$\int_{\mathbb{R}^n}|u-v|^p \le 2\int \tau_2(\nu)d\sigma(\nu) \le C(n)\sqrt{a^{p/n}\frac{\delta}{\lambda}}\sqrt{\int \tau_1(\nu)^{1-(1/n)}d\sigma(\nu)}.$$

By Jensen inequality for concave functions,

$$\begin{aligned}\int \tau_1(\nu)^{1-(1/n)} d\sigma(\nu) &\leq C(n)\left(\int \tau_1(\nu) d\sigma(\nu)\right)^{1-(1/n)} \\ &\leq C(n)\left(\int a^p \frac{\kappa(\nu)^n}{n} d\sigma(\nu)\right)^{1-(1/n)} = C(n)\,,\end{aligned}$$

and we come to conclude that

$$\int_{\mathbb{R}^n} |u-v|^p \leq C(n)\sqrt{a^{p/n}\frac{\delta}{\lambda}} \tag{14}$$

Step three: As $\int |w-u|^p \leq 2^p$, (C3) follows trivially whenever $\delta \geq \lambda$. Let us now assume that $\delta \leq \lambda$, then (C3) is easily deduced from (13) and (14). □

Remark:
(H3) can be weakened and replaced by
(H3') $\inf_{r\in[0,m]} |(dF/dr)(r,a)| > 0$ for every $m > 0$.
Let u be a function in the constraint, define u_0 as follows:
$u_0 = u$ on B_R, $u_0 = a$ on $B_{R+m} \setminus B_R$ and $u_0 = 0$ elswhere.
where m is chosen so that

$$a^p|B_{R+m} \setminus B_R| = \int_{\mathbb{R}^n\setminus B_R} u^p\,.$$

In this way we have

$$\int |w-u|^p = \int |w-u_0|^p and \int u_0^p = \int u^p$$

so we can reduce to prove the theorem on u_0 instead that on u. On the other hand u_0 is compactly supported on B_{R+m} and m is uniformly bounded from above if the deficit of u satisfies some uniform bound, more precisely

$$a^p((R+m)^n - R^n)|B| = \int_{\mathbb{R}^n\setminus B_R} u^p \leq \int_{\mathbb{R}^n} |u-w|^p\,,$$

and since $F(.,a)$ is strictly decreasing , for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mathcal{F}(w) - \mathcal{F}(u) \leq \delta$$

we have

$$\int_{\mathbb{R}^n} |u-w|^p \leq \varepsilon$$

So, from the above relation for $\varepsilon = 1$ we should find that if $\mathcal{F}(w) - \mathcal{F}(u)$ is small enough then

$$m \leq Constant(a,n)$$

Therefore we have reduced to work on a class of functions u_0 wich are supported in a set , where $F(.,a)$ satisfies the required lower bound, and there we can repeat the argument of our theorem.

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