

SCHRÖDINGER SYSTEMS ARISING IN NONLINEAR OPTICS AND QUANTUM MECHANICS

PART I

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ABSTRACT. In this first part, we study existence and uniqueness of solutions of a general nonlinear Schrödinger system in the presence of diamagnetic field, local and nonlocal nonlinearities. This kind of systems model many important phenomena in nonlinear optics; multimodal optical fibers, optical pulse propagation, ferromagnetic film and optical pulse propagation in the birefringent fibers. They also govern the interaction of electron and nuclei through Coulombic potential and under the action of external magnetic field in quantum mechanics.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. In this paper, we aim to study the following Cauchy problem of an m -coupled nonlinear Schrödinger equations with electromagnetic potentials, local and nonlocal nonlinearities

$$(1.1) \quad \begin{cases} i\partial_t \Phi_j = L_A \Phi_j + V(x)\Phi_j - g_j(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)\Phi_j - \sum_{i=1}^m W_{ij} * h(|\Phi_i|) \frac{h'(|\Phi_j|)}{|\Phi_j|} \Phi_j, \\ \Phi_j(0, x) = \Phi_j^0(x), \\ 1 \leq j \leq m, \end{cases}$$

where, for all $1 \leq j \leq m$, $\Phi_j^0 : \mathbb{R}^N \rightarrow \mathbb{C}$, $\Phi_j : \mathbb{R}^+ \times \mathbb{R}^N \rightarrow \mathbb{C}$, $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continuous and non-decreasing, $V : \mathbb{R}^N \rightarrow \mathbb{R}$ and $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ represents electric and magnetic potentials satisfying suitable assumptions that will be stated in the following. When the nonlocal nonlinearity and the magnetic field are null, (1.1) has many applications in physical problems. It appears in the study of special

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solitons in nonlinear waveguides, [16], the theory of Bose-Einstein condensates, [2], [18], [19], [23], optical pulse propagation in the birefringent fibers, [12], [14], wavelength division multiplexed optical systems and ferromagnetic film model. For a more detailed account, the reader can refer to [8]. When the local nonlinearity and the magnetic field are null, $m = 1$ and $h(s) = s^2$, (1.1) is called the Choquard equation. It has important applications in physics and has been intensively studied in the last decades by many mathematicians, [9] and the references therein. Recently, Zhao, Ma, [13] were able to obtain a very clever classification of the positive solitary waves of (1.1) when $m = 1$. In the presence of magnetic field, very recent powerful results have been addressed to the existence and uniqueness of solutions of (1.1), [11], and references therein. However results were established only for $h = 0$.

Surprisingly, the literature is totally silent about existence and uniqueness result of solutions of (1.1) in the presence of magnetic field, local and nonlocal nonlinearities with $m > 1$. Let us point out that it is natural to consider such situations since the local nonlinearities are related to the index of refraction of the medium in which the waves propagate. The nonlocal terms describe interaction between different waves (particles) emitted by lasers placed at the entrance of the waveguide. [24]. System (1.1) also models interactions of m -waves with different polarizations and it is crucial in the description of many phenomena involving monochromatic waves, [14], [20] and references therein. By I. P. Kaminow, [21], we know that a single mode optical fiber ($m = 1$) is not a realistic model for many important problems. Indeed, it is multimodal due to the presence of birefringence, which deeply influences the way in which waves evolve during the propagation along a fiber. Moreover, the linear birefringence makes a pulse split in m , while nonlinear birefringence traps them together against splitting. In [14], C. R. Menyuk has showed that the evolution of m -orthogonal pulse envelopes in birefringent optical fibers is governed by (1.1) (with A and h null but $m > 1$). When $G = 0$, $m = 1$, (1.1) is the so-called generalized Hartree-Fock equations with magnetic field, which forms in itself an important branch in mathematical physics. It has been intensively studied by many brilliant mathematicians; E. H. Lieb, A. Bahri, P. L. Lions, see [15] and [21] for more recent results. It is very important to extend these results to the case $m > 1$. Indeed, Hartree-Fock equations model the interaction of electron and nuclei through Coulombic potential and under the action of external magnetic field. In the context

of quantum mechanics, the first term in the energy functional (see below), corresponds to the kinetic energy of electrons, the second one is the 1-body attractive interaction between the electrons and the nuclei. The third term is related to the influence of the medium and the last term is the 2-body repulsive interaction between electrons. Part 2 will be entirely devoted to the existence, symmetry and monotonicity properties of the constrained minimizers of the energy functional; ground state solutions, which are the most important solutions of system (1.1).

The aim of this paper is to discuss (1.1) in its more generality.

Physically, the solution Φ is the i -th component of the beam kerr-like photorefractive. The magnetic operator L_A is defined as

$$L_A \phi_i := \left(\frac{\nabla}{i} - A(x) \right)^2 \phi_i = -\Delta \phi_i - \frac{2}{i} A(x) \cdot \nabla \phi_i + |A(x)|^2 \phi_i - \frac{1}{i} \operatorname{div} A(x) \phi_i.$$

The magnetic field B is $B = \nabla \times A$ in \mathbb{R}^3 and can be thought (and identified) in general dimension as a 2-form \mathcal{H}^B of coefficients $(\partial_i A_j - \partial_j A_i)$. We will keep using the notation $B = \nabla \times A$ in any dimension N .

In various relevant cases, it is possible to write (1.1) in the following vectorial form

$$\begin{cases} i \frac{\partial \Phi}{\partial t} = F'_A(\Phi) \\ \Phi(0, x) = \Phi^0(x) \end{cases}$$

where $\Phi^0 = (\Phi_1^0, \dots, \Phi_m^0)$ and we have set

$$\begin{aligned} F_A(\Phi) &= \frac{1}{2} \sum_{j=1}^m \int \left| \left(\frac{\nabla}{i} - A(x) \right) \Phi_j \right|^2 dx + \frac{1}{2} \int V(x) |\Phi|^2 dx - \int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2) dx \\ &\quad - \frac{1}{2} \sum_{i,j=1}^m \iint W_{ij}(|x-y|) h(|\Phi_i(x)|) h(|\Phi_j(y)|) dx dy, \end{aligned}$$

where $W_{ij} = W_{ji}$ and such as h satisfy suitable assumptions that will be stated in the following. Moreover, we observe that $G : (0, \infty) \times \mathbb{R}_+^m \rightarrow \mathbb{R}$ satisfy the following conditions

$$\frac{\partial G}{\partial s_j} = \frac{1}{2} g_j(|x|, s_1^2, \dots, s_m^2),$$

for every $j = 1, \dots, m$. We look for a soliton or standing wave of (1.1), namely a solution of the form $\Phi(t, x) = (\Phi_1(t, x), \dots, \Phi_m(t, x))$, where for $1 \leq j \leq m$, $\Phi_j(t, x) = e^{i\lambda_j t} u_j(x)$, λ_j real numbers and $u_j : \mathbb{R}^N \rightarrow \mathbb{C}$.

Therefore, $\mathcal{U} = (u_1, \dots, u_m)$ is a solution of the following $m \times m$ elliptic problem:

$$(1.2) \quad \begin{cases} L_A u_j + (V(x) - \lambda_j)u_j - g_j(|x|, |u_1|^2, \dots, |u_m|^2)u_j - \sum_{i=1}^m W_{ij} * h(|u_i|) \frac{h'(|u_j|)}{|u_j|} u_j = 0 \\ \text{for all } 1 \leq j \leq m. \end{cases}$$

We point out that the general Schrödinger system (1.1) we aim to study contains, as particular cases, physically meaningful situations as in Section 1.1 in [?].

1.2. Preliminaries and Notations. Since we want that the composite functions

$$x \mapsto G(|x|, u_1(x), \dots, u_m(x))$$

are measurable on \mathbb{R}^N for every $u_1, \dots, u_m \in M(\mathbb{R}^N)$, where $M(\mathbb{R}^N)$ is the set of measurable functions on \mathbb{R}^N , we deal with the following G of Carathéodory type:

Definition 1.1. *A function $G : (0, \infty) \times \mathbb{R}^m \rightarrow \mathbb{R}$ is an m -Carathéodory function if*

- (1) $G(\cdot, s_1, \dots, s_m) : (0, \infty) \rightarrow \mathbb{R}$ is measurable on $(0, \infty) \setminus \Gamma$, where Γ is a subset of $(0, \infty)$ having one dimension measure zero, for all $s_1, \dots, s_m \geq 0$,
- (2) For all $1 \leq n \leq m$, every $(m-1)$ tuple $s_i \geq 0$ and $r \in (0, \infty) \setminus \Gamma$, the function

$$\begin{aligned} \mathbb{R} &\rightarrow \mathbb{R} \\ s_n &\mapsto G(r, \dots, s_n, \dots) \end{aligned}$$

is continuous on \mathbb{R} .

Throughout this paper we denote by $\mathcal{H}_A^1 = \mathcal{H}_A^1(\mathbb{R}^N) = (H_A^1(\mathbb{R}^N))^m$ where $H_A^1 = H_A^1(\mathbb{R}^N)$ is the Hilbert space defined as the closure of $C_c^\infty(\mathbb{R}^N; \mathbb{C})$ under the scalar product

$$(u, v)_{H_A^1} = \Re \int (Du \cdot \overline{Dv} + u\overline{v}) dx$$

where $Du = (D_1u, \dots, D_Nu)$ and $D_j = i^{-1}\partial_j - A_j(x)$, with induced norm

$$\|u\|_{H_A^1}^2 = \int \left| \frac{1}{i} \nabla u - A(x)u \right|^2 dx + \int |u|^2 dx < \infty.$$

Recall that the diamagnetic inequality

$$(1.3) \quad |\nabla|u|| \leq \left| \left(\frac{\nabla}{i} - A(x) \right) u \right|$$

holds for every $u \in H_A^1(\mathbb{R}^N)$. The space \mathcal{H}_A^1 is equipped with the standard norm $\|\Phi\|_{\mathcal{H}_A^1}^2 = \|(\frac{\nabla}{i} - A(x))\Phi\|_{\mathcal{L}^2}^2 + \|\Phi\|_{\mathcal{L}^2}^2$ where $\mathcal{L}^p = \mathcal{L}^p(\mathbb{R}^N) = (L^p(\mathbb{R}^N))^m$ and the norm $\|\Phi\|_{\mathcal{L}^p}^2 = \sum_{i=1}^m \|\Phi_i\|_{L^p}^2$ for every $\Phi = (\Phi_1, \dots, \Phi_m) \in \mathcal{L}^p$. We denote by $L_w^q(\mathbb{R}^N)$ with $q > 1$ the weak L^q -space (see [10]) defined as the set of measurable functions f equipped with the norm

$$\|f\|_{q,w} = \sup_{D \subset \mathbb{R}^N, \mathcal{M}(D) < \infty} (\mathcal{M}(D))^{-1/q'} \int_D |f(x)| dx < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

where \mathcal{M} denotes the Lebesgue measure on \mathbb{R}^N . The dual space of \mathcal{H}_A^1 is denoted by \mathcal{H}'_A . We denote $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^N) = (H^1(\mathbb{R}^N))^m$ equipped with the standard norm $\|\Phi\|_{\mathcal{H}^1}^2 = \|\nabla\Phi\|_{\mathcal{L}^2}^2 + \|\Phi\|_{\mathcal{L}^2}^2$ and $\mathcal{H}^{-1}(\mathbb{R}^N) = (H^{-1}(\mathbb{R}^N))^m$. Clearly, by (1.3) the following Lemma holds:

Lemma 1.2. *The space \mathcal{H}_A^1 is continuously embedded in \mathcal{L}^p for all $p \in [2, 2^*]$ where $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and there exists $C > 0$ independent on A such that*

$$\|u\|_{\mathcal{L}^p} \leq C \|u\|_{\mathcal{H}_A^1}$$

Furthermore, $(\mathcal{L}^p)' = \mathcal{L}^{p'} \subset \mathcal{H}'_A$ where p' denotes the conjugate of p .

Recall that by $C(\bar{I}, X)$ is the space of continuous functions $I \rightarrow X$ equipped with the uniform norm when I is bounded. By $\mathcal{D}(I, X)$, we denote the space $C_c^\infty(I, X)$ of the C^∞ functions $I \rightarrow X$ with compact support in I , equipped with the uniform norm of all derivatives on I . By $L^p(I, X)$ the Banach space of measurable function $I \rightarrow X$ such that the norm

$$\|u\|_{L^p(I, X)} = \begin{cases} \left(\int_I \|u(t)\|_X^p dt \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \text{Esssup } \|u(t)\|_X, & \text{if } p = \infty \end{cases}$$

is finite. We denote by $W^{m,p}(I, X)$ the Banach space of measurable functions $I \rightarrow X$ such that $\frac{d^j u}{dt^j} \in L^p(I, X)$ for every $j = 1, \dots, m$ equipped with the norm

$$\|u\|_{W^{m,p}(I, X)} = \sum_{j=1}^m \left\| \frac{d^j u}{dt^j} \right\|_{L^p(I, X)}.$$

$C^{m,\alpha}(\bar{I}, X)$ for $0 \leq \alpha \leq 1$ is the Banach space of uniformly continuous and bounded functions with all their derivatives respect to t such that

$$\|u\|_{C^{m,\alpha}(\bar{I}, X)} = \|u\|_{W^{m,\infty}(I, X)} + \sup \left\{ \frac{\left| \frac{d^m u(t)}{dt^m} - \frac{d^m u(s)}{dt^m} \right|}{|t - s|^\alpha}; \quad s, t \in I \right\} < \infty.$$

Furthermore, we denote by $\mathcal{L}(X, Y)$ the Banach space of linear, continuous operators from the Banach spaces X and Y equipped with the norm topology.

2. LOCAL WELL-POSEDNESS

2.1. Assumptions on the magnetic potential. We suppose that A is a smooth function, namely $A \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and there exist some constant $C_\alpha > 0$, $\alpha \in \mathbb{N}^n$ such that:

- (A) $\forall \alpha \in \mathbb{N}^n$, $|\alpha| \geq 1$, $\sup_{x \in \mathbb{R}^N} |\partial_x^\alpha A| \leq C_\alpha$
- (B) $\exists \varepsilon > 0$, $\forall |\alpha| \geq 1$, $\sup_{x \in \mathbb{R}^N} |\partial_x^\alpha B| \leq C_\alpha \langle x \rangle^{-1-\varepsilon}$.

2.2. Assumptions on the external potentials. We suppose that the external potentials satisfy:

- (V) $V \in L^p(\mathbb{R}^N, \mathbb{R})$ or $V : \mathbb{R}^N \rightarrow \mathbb{R}$, $V \in L^p + L^\infty$ for some $p \geq 1$, $p \geq N/2$;
- (W) $W_{ij} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an even, non-increasing function, $W_{ij}(|x|) \in L_w^q(\mathbb{R}^N)$ with $q > \max\{1, N/4\}$ and $W_{ij} = W_{ji}$ for all $i, j = 1, \dots, m$.

2.3. Assumptions on the local nonlinearities. On the local nonlinearities, we assume that

(g) For every $j = 1, \dots, m$, the complex valued functions $f_j(x, \Phi) = g_j(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2) \Phi_j$ are measurable in $x \in \mathbb{R}^N$ and continuous in $\Phi \in \mathbb{C}^m$ almost everywhere on \mathbb{R}^N . Assume that there exist constant C and $\alpha \in [0, \frac{4}{N-2})$ ($\alpha \in [0, \infty)$ if $N = 2$) such that

$$|f_j(x, \Psi) - f_j(x, \Phi)| \leq C (|\Phi|^\alpha + |\Psi|^\alpha) |\Psi - \Phi|, \quad \text{for almost all } x \in \mathbb{R}^N \text{ and all } \Phi, \Psi \in \mathbb{C}^m.$$

Observe that $f_j(x, \mathbf{0}) = 0$.

(G) There exists $K > 0$ such that, for all $r > 0$ and $s_1, \dots, s_m \geq 0$, we have

$$0 \leq G(r, s_1, \dots, s_m) \leq K \left(\sum_{j=1}^m s_j + \sum_{j=1}^m s_j^{\frac{l_j+2}{2}} \right), \quad 0 < l_j < \frac{4}{N-2}.$$

Remark 2.1. The local term in the energy functional F_A is finite thanks to (G). Indeed, we have

$$\int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2) \leq K \int |\Phi|^2 + K \sum_{j=1}^m \int |\Phi_j|^{l_j+2}.$$

For every $j = 1, \dots, m$, by the Gagliardo-Nirenberg and Diamagnetic inequalities, we have that

$$\|\Phi_j\|_{l_j+2} \leq c \|\Phi_j\|_{L^2}^{1-\sigma_j} \|\nabla|\Phi_j|\|_{L^2}^{\sigma_j} \leq c \|\Phi_j\|_{L^2}^{1-\sigma_j} \|\Phi_j\|_{\mathcal{H}_A^1}^{\sigma_j}$$

where $\sigma_j = \frac{Nl_j}{2(l_j+2)}$. Since σ_j must belong to $[0, 1)$, we find that $0 < l_j < \frac{4}{N-2}$.

2.4. Assumptions on the nonlocal nonlinearities. On the nonlocal nonlinearity, we assume:

(h) $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is C^1 and non-decreasing, $h(0) = 0$ and there exist $C, D, E > 0$ such that

$$h(s) \leq Cs^\mu, \quad |h'(s)| \leq Ds^{\mu-1}, \quad |h''(s)| \leq Es^{\mu-2},$$

for all $s \in \mathbb{R}^+$, where

$$(2.1) \quad 2 \leq \mu \leq \frac{6q-1}{2q} - \frac{N-2}{N}.$$

Notice that this inequality is nonempty due to the condition $q > N/4$.

Remark 2.2. If we set $H(s) = \frac{h'(s)}{s}$ for all $s \in \mathbb{R}^+$, there is a positive constant $C > 0$ such that

$$(2.2) \quad |H(|z|)z - H(|w|)w| \leq C(|z|^{\mu-2} + |w|^{\mu-2})|z - w|, \quad \text{for all } z, w \in \mathbb{C},$$

$$(2.3) \quad |h(|z|) - h(|w|)| \leq C(|z|^{\mu-1} + |w|^{\mu-1})|z - w|, \quad \text{for all } z, w \in \mathbb{C}.$$

This easily follows in light of the growth conditions of the maps $\{s \mapsto h'(s), h''(s)\}$ contained in assumption (h). For the proof of inequality (2.2), see for instance [6, inequality (2-1) of Lemma 2.1] applied with $A(\eta) := H(|\eta|)\eta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. The only required condition is [6, condition (1-3)], which is indeed fulfilled in view of the growths for h' and h'' assumed in (h).

Remark 2.3. Since assumptions (h) and (W) hold true, by the Hardy-Littlewood-Sobolev inequality for weak L^q kernels (cf. [10, formula (7), p.107]), by the Gagliardo-Nirenberg and the diamagnetic inequality,

one can prove that the nonlocal term involved in the energy functional F_A is always finite. Indeed, we have

$$\begin{aligned}
& \iint W_{ij}(|x-y|)h(|\Phi_i(x)|)h(|\Phi_j(y)|) dx dy \leq C \|W_{ij}\|_{L_w^q} \|\Phi_i\|_{L^{\frac{2q\mu}{2q-1}}}^\mu \|\Phi_j\|_{L^{\frac{2q\mu}{2q-1}}}^\mu \\
& \leq C \|W_{ij}\|_{L_w^q} \|\nabla|\Phi_i|\|_{L^2}^{N\mu(\frac{1}{2}-\frac{(2q-1)}{2q\mu})} \|\Phi_i\|_{L^2}^{\mu[1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} \|\nabla|\Phi_j|\|_{L^2}^{N\mu(\frac{1}{2}-\frac{(2q-1)}{2q\mu})} \|\Phi_j\|_{L^2}^{\mu[1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} \\
& \leq C \|W_{ij}\|_{L_w^q} \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi_i \right\|_{L^2}^{N\mu(\frac{1}{2}-\frac{(2q-1)}{2q\mu})} \|\Phi_i\|_{L^2}^{\mu[1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} \\
& \times \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi_j \right\|_{L^2}^{N\mu(\frac{1}{2}-\frac{(2q-1)}{2q\mu})} \|\Phi_j\|_{L^2}^{\mu[1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} \\
& \leq C \|W_{ij}\|_{L_w^q} \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi \right\|_{\mathcal{L}^2}^{N\mu(\frac{1}{2}-\frac{(2q-1)}{2q\mu})} \|\Phi_i\|_{L^2}^{\mu[1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} \\
& \times \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi \right\|_{\mathcal{L}^2}^{N\mu(\frac{1}{2}-\frac{(2q-1)}{2q\mu})} \|\Phi_j\|_{L^2}^{\mu[1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} \\
& \leq C \|W_{ij}\|_{L_w^q} m_i^{\frac{\mu}{2}[1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} m_j^{\frac{\mu}{2}[1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} \\
& \times \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi \right\|_{\mathcal{L}^2}^{2N\mu(\frac{1}{2}-\frac{(2q-1)}{2q\mu})},
\end{aligned}$$

where $m_i = \|\Phi_i(t)\|_{L^2}^2$ and $m_j = \|\Phi_j(t)\|_{L^2}^2$ for all $i, j = 1, \dots, m$. Observe that, in order to have $2 \leq 2q\mu/(2q-1) \leq 2^*$, μ must belong to the range $2 \leq \mu \leq 2^*(2q-1)/2q$, which is compatible with the one in (h), which is smaller.

2.5. Local well-posedness. In this section, we want to establish the local well posedness of the Cauchy problem (1.1)

$$\begin{cases}
i\partial_t \Phi_j - L_A \Phi_j - V(x)\Phi_j + g_j(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)\Phi_j + \sum_{i=1}^m W_{ij} * h(|\Phi_i|) \frac{h'(|\Phi_j|)}{|\Phi_j|} \Phi_j = 0, \\
\Phi_j(0, x) = \Phi_j^0(x), \\
\text{for all } 1 \leq j \leq m,
\end{cases}$$

or equivalently,

$$(2.4) \quad \begin{cases} i\partial_t \Phi - L_A \Phi + \tilde{g}(\Phi) = 0, \\ \Phi(0, x) = \Phi^0(x), \end{cases}$$

where, for every $j = 1, \dots, m$, $\tilde{g}_j(\Phi) = -\tilde{g}_{1,j}(\Phi) + \tilde{g}_{2,j}(\Phi) + \tilde{g}_{3,j}(\Phi)$ with $\tilde{g}_{1,j}(\Phi) = V(x)\Phi_j$, $\tilde{g}_{2,j}(\Phi) = g_j(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)\Phi_j$ and $\tilde{g}_{3,j}(\Phi) = \sum_{i=1}^m W_{ij} * h(|\Phi_i|) \frac{h'(|\Phi_j|)}{|\Phi_j|} \Phi_j$. Observe that L_A is a self-adjoint, ≥ 0 operator on \mathcal{L}^2 , iL_A is skew-adjoint and generates a group of isometries $\{T(t)\}_{t \in \mathbb{R}}$ where $T(t) = e^{-itL_A}$ in \mathcal{L}^2 . Furthermore, the following lemma holds

Lemma 2.4. *If $\varepsilon > 0$ and $1 \leq p < \infty$, then $(I + \varepsilon L_A)^{-1}$ is continuous $\mathcal{L}^p \rightarrow \mathcal{L}^p$ and $\|(I + \varepsilon L_A)^{-1}\|_{\mathcal{L}(\mathcal{L}^p, \mathcal{L}^p)} \leq 1$.*

Proof. We adapt the proof of Lemma 9.1.3 in [3]. In our case, we deal with \mathcal{L}^p and the operator A is L_A . \square

Proposition 2.5. *By assumptions (V), (W), (g) and (h) for suitable choice of r_k and ρ_k (see Examples 1 to 4 in Section 4.2 in [3]), we have that $\tilde{g}_k, k = 1, 2, 3$ satisfy the following conditions*

- (1) $\tilde{g}_k \in C(\mathcal{H}_A^1, \mathcal{H}'_A)$ and there exists $\tilde{G}_k \in C^1(\mathcal{H}_A^1, \mathbb{R})$ such that $\tilde{g}_k = \tilde{G}'_k$;
- (2) there exist $r_k, \rho_k \in [2, \frac{2N}{N-2})$ ($r_k, \rho_k \in [2, \infty)$ if $n = 1$, $r_k, \rho_k \in [2, \infty)$ if $N = 2$) such that $\tilde{g}_k : \mathcal{H}_A^1 \rightarrow \mathcal{L}^{\rho'_k} \hookrightarrow \mathcal{H}'_A$;
- (3) for every $M > 0$, there exists $C(M) < \infty$ such that

$$(2.5) \quad \|\tilde{g}_k(\Psi) - \tilde{g}_k(\Phi)\|_{\mathcal{L}^{\rho'_k}} \leq C(M) \|\Psi - \Phi\|_{\mathcal{L}^{r_k}}, \quad \text{for } k = 1, 2, 3,$$

for every $\Psi, \Phi \in \mathcal{H}_A^1$ such that $\|\Psi\|_{\mathcal{H}_A^1} + \|\Phi\|_{\mathcal{H}_A^1} \leq M$;

- (4) for every $\Phi \in \mathcal{H}_A^1$ and $j = 1, \dots, m$,

$$(2.6) \quad \Im(\tilde{g}_{k,j}(\Phi)\overline{\Phi_j}) = 0 \quad \text{a.e. on } \mathbb{R}^N.$$

Proof. We start with \tilde{g}_1 . Condition (1) is satisfied since for every $j = 1, \dots, m$, $\tilde{g}_{1,j}$ satisfies it on H_A^1 and H_A^{-1} and $\tilde{G}_1(\Phi) = \frac{1}{2} \int V(x)|\Phi|^2$ for all $\Phi \in \mathcal{H}_A^1$. (2) and (2.5) follow since they are satisfied by each component $\tilde{g}_{1,j}$ for every $j = 1, \dots, m$ in the spaces L^s with $s = \rho'_1, r_1$ where $r_1 = \rho_1 = \frac{2p}{p-1}$ and by the definition of the norm in the spaces

\mathcal{L}^s , the diamagnetic and Sobolev inequalities. Indeed,

$$\begin{aligned} \|\tilde{g}_{1,j}(\Psi) - \tilde{g}_{1,j}(\Phi)\|_{L^{\rho'_1}}^{\rho'_1} &\leq C \int |V|^{\rho'_1} |\Psi_j - \Phi_j|^{\rho'_1} \leq C \left(\int |V|^p \right)^{\frac{\rho'_1}{p}} \left(\int |\Psi_j - \Phi_j|^{\frac{p\rho'_1}{p-\rho'_1}} \right)^{\frac{p-\rho'_1}{p}} \\ &\leq C \|V\|_{L^p}^{\rho'_1} \|\Psi_j - \Phi_j\|_{L^{\frac{p\rho'_1}{p-\rho'_1}}}^{\rho'_1} \end{aligned}$$

for $\rho_1 \in [2, 2^*)$ and r_1 which satisfies

$$2 \leq r_1 = \frac{p\rho'_1}{p-\rho'_1} = \frac{p\rho_1}{p(\rho_1) - \rho_1} \leq 2^*.$$

The choice of $\rho_1 = \frac{2p}{p-1}$ leads to $r_1 = \rho_1 \in [2, 2^*)$ since $p \geq \frac{N}{2}$. Finally, (2.6) follows since V is real valued. Condition (1) is satisfied by \tilde{g}_2 since each component $\tilde{g}_{2,j}$, for every $j = 1, \dots, m$, satisfies it on H_A^1 and H_A^{-1} and $\tilde{G}_2(\Phi) = \int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)$. (2) and (2.5) follow easily from the local Lipschitz assumption in (g), diamagnetic inequality, Sobolev embedding $H^1 \hookrightarrow L^{\alpha+2}$ and for $r_2 = \rho_2 = \alpha + 2$. Indeed, by assumption (g),

$$\begin{aligned} \|\tilde{g}_{2,j}(\Psi) - \tilde{g}_{2,j}(\Phi)\|_{L^{\rho'_2}}^{\rho'_2} &= \int |g_j(|x|, |\Psi_1|^2, \dots, |\Psi_m|^2) \Psi_j - g_j(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2) \Phi_j|^{\rho'_2} \\ &\leq C \left(\|\Psi\|_{\mathcal{L}^\theta}^{\alpha\rho'_2} + \|\Phi\|_{\mathcal{L}^\theta}^{\alpha\rho'_2} \right) \|\Psi - \Phi\|_{\mathcal{L}^{r_2}}^{\rho'_2} \leq C(M) \|\Psi - \Phi\|_{\mathcal{L}^{r_2}}^{\rho'_2}. \end{aligned}$$

where $\rho_2, r_2 \in [2, 2^*)$ satisfy

$$2 \leq \theta = \frac{\alpha\rho'_2 r_2}{r_2 - \rho'_2} \leq 2^*.$$

The choice of $r_2 = \rho_2 = \alpha + 2$ leads to $2 \leq \alpha + 2 < 2^*$ which is compatible with the range of α . Also (2.6) is obvious since each $g_{2,j}$ is real valued. Finally, we deal with \tilde{g}_3 . (1) holds since for every $j = 1, \dots, m$, $\tilde{g}_{3,j}$ satisfies it on H_A^1 and H_A^{-1} and $\tilde{G}_3(\Phi) = \frac{1}{2} \sum_{i,j=1}^m \iint W_{ij}(|x-y|) h(|\Phi_i(x)|) h(|\Phi_j(y)|) dx dy$. (2) holds since, for each component, by

Hölder inequality, assumption (h) and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
 \|\tilde{g}_{3,j}(\Phi)\|_{L^{\rho'_3}}^{\rho'_3} &\leq C \sum_{i=1}^m \int |W_{ij} * h(|\Phi_i|)|^{\rho'_3} \left| \frac{h'(|\Phi_j|)}{|\Phi_j|} \Phi_j \right|^{\rho'_3} \leq C \sum_{i=1}^m \int |W_{ij} * h(|\Phi_i|)|^{\rho'_3} |h'(|\Phi_j|)|^{\rho'_3} \\
 &\leq C \sum_{i=1}^m \left(\int |W_{ij} * h(|\Phi_i|)|^{\frac{\rho'_3 \rho_3}{\rho_3 - \rho'_3}} \right)^{\frac{\rho_3 - \rho'_3}{\rho_3}} \left(\int |h'(|\Phi_j|)|^{\rho_3} \right)^{\frac{\rho'_3}{\rho_3}} \\
 &\leq C \sum_{i=1}^m \left(\int |W_{ij} * h(|\Phi_i|)|^{\frac{\rho_3}{\rho_3 - 2}} \right)^{\frac{\rho_3 - \rho'_3}{\rho_3}} \left(\int |\Phi_j|^{\rho_3(\mu-1)} \right)^{\frac{\rho'_3}{\rho_3}} \\
 &\leq C \sum_{i=1}^m \|W_{ij} * h(|\Phi_i|)\|_{L^{\frac{\rho_3}{\rho_3 - 2}}}^{\frac{\rho_3}{\rho_3 - 1}} \|\Phi_j\|_{L^{\rho_3(\mu-1)}}^{\rho'_3(\mu-1)} \leq C \sum_{i=1}^m \|W_{i,j}\|_{L_w^q}^{\rho'_3} \|\Phi_i\|_{L^{\mu p}}^{\rho'_3 \mu} \|\Phi_j\|_{L^{\rho_3(\mu-1)}}^{\rho'_3(\mu-1)}
 \end{aligned}$$

where $p = \frac{\rho_3 q}{2\rho_3 q - 2q - \rho_3}$ for $\rho_3 \in [2, 2^*)$ which satisfies

$$\begin{cases} 2 \leq \rho_3(\mu - 1) \leq 2^* \\ \frac{2}{\mu} \leq p = \frac{\rho_3 q}{2\rho_3 q - 2q - \rho_3} \leq \frac{2^*}{\mu}. \end{cases}$$

Observe that the choice of $\rho_3 = \frac{4q}{2q-1}$ in the above inequalities leads to the restriction

$$2 \leq \mu < 1 + \frac{2^*(2q-1)}{4q},$$

which is compatible with the range of μ in condition (h). Condition (2.5) is also satisfied since, for each component, we have

$$\begin{aligned}
 \|\tilde{g}_{3,j}(\Psi) - \tilde{g}_{3,j}(\Phi)\|_{L^{\rho'_3}}^{\rho'_3} &\leq C \sum_{i=1}^m \int \left| W_{ij} * h(|\Psi_i|) \frac{h'(|\Psi_j|)}{|\Psi_j|} \Psi_j - W_{ij} * h(|\Phi_i|) \frac{h'(|\Phi_j|)}{|\Phi_j|} \Phi_j \right|^{\rho'_3} \\
 &\leq C \sum_{i=1}^m (I_i + J_i),
 \end{aligned}$$

where, for $i = 1, \dots, m$ we have set

$$\begin{aligned}
 I_i &= \int \left| W_{ij} * h(|\Psi_i|) \right|^{\rho'_3} \left| \frac{h'(|\Psi_j|)}{|\Psi_j|} \Psi_j - \frac{h'(|\Phi_j|)}{|\Phi_j|} \Phi_j \right|^{\rho'_3} \\
 J_i &= \int \left| W_{ij} * (h(|\Psi_i|) - h(|\Phi_i|)) \right|^{\rho'_3} \left| h'(|\Phi_j|) \right|^{\rho'_3}.
 \end{aligned}$$

By virtue of Hölder inequality, condition (2.2) related to (h), Hardy-Littlewood-Sobolev and Sobolev inequalities, we have that

$$\begin{aligned}
I_i &= \int \left| W_{ij} * h(|\Psi_i|) \right|^{\rho'_3} \left| H(|\Psi_j|)\Psi_j - H(|\Phi_j|)\Phi_j \right|^{\rho'_3} \\
&\leq \left(\int \left| W_{ij} * h(|\Psi_i|) \right|^{\frac{\rho_3}{\rho_3-2}} \right)^{\frac{\rho_3-2}{\rho_3-1}} \left(\int \left| H(|\Psi_j|)\Psi_j - H(|\Phi_j|)\Phi_j \right|^{\rho_3} \right)^{\frac{\rho'_3}{\rho_3}} \\
&\leq C \|W_{i,j}\|_{L_w^q}^{\rho'_3} \|\Psi_i\|_{L^{\mu p}}^{\rho'_3 \mu} \left(\int (|\Psi_j|^{\rho_3(\mu-2)} + |\Phi_j|^{\rho_3(\mu-2)}) |\Psi_j - \Phi_j|^{\rho_3} \right)^{\frac{\rho'_3}{\rho_3}} \\
&\leq C \|W_{i,j}\|_{L_w^q}^{\rho'_3} \|\Psi_i\|_{L^{\mu p}}^{\rho'_3 \mu} \left\{ \|\Psi_j\|_{L^{\frac{(\mu-2)r_3\rho_3}{r_3-\rho_3}}}^{\rho'_3(\mu-2)} + \|\Phi_j\|_{L^{\frac{(\mu-2)r_3\rho_3}{r_3-\rho_3}}}^{\rho'_3(\mu-2)} \right\} \|\Psi_j - \Phi_j\|_{L^{r_3}}^{\rho'_3} \\
&\leq C(M) \|\Psi_j - \Phi_j\|_{L^{r_3}}^{\rho'_3},
\end{aligned}$$

where $\rho_3 \in [2, 2^*)$ and $r_3 \in [2, 2^*)$ satisfy the following conditions

$$(2.7) \quad \begin{cases} 2 \leq \mu p \leq 2^* \\ 2 \leq \frac{(\mu-2)r_3\rho_3}{r_3-\rho_3} \leq 2^*. \end{cases}$$

Observe that, by the choice of $\rho_3 = \frac{4q}{2q-1}$, the above inequalities are satisfied by

$$(2.8) \quad \frac{4q2^*}{2^*(2q-1) - 4q(\mu-2)} \leq r_3 \leq \frac{4q}{6q-1-2q\mu}.$$

The range of μ in assumption (h) ensures that $r_3 \in [2, 2^*)$. Dealing with the second term in the above sum, by means of condition (2.3) related to (h), we have

$$\begin{aligned}
J_i &= \int \left| W_{ij} * (h(|\Psi_i|) - h(|\Phi_i|)) \right|^{\rho'_3} \left| h'(|\Phi_j|) \right|^{\rho'_3} \\
&\leq \left(\int \left| W_{ij} * (h(|\Psi_i|) - h(|\Phi_i|)) \right|^{\frac{\rho_3}{\rho_3-2}} \right)^{\frac{\rho_3-2}{\rho_3-1}} \left(\int \left| h'(|\Phi_j|) \right|^{\rho_3} \right)^{\frac{\rho'_3}{\rho_3}} \\
&\leq \|W_{ij} * (h(|\Psi_i|) - h(|\Phi_i|))\|_{L^{\frac{\rho_3}{\rho_3-2}}}^{\rho'_3} \|h'(|\Phi_j|)\|_{L^{\rho_3}}^{\rho'_3} \\
&\leq C \|W_{ij}\|_{L_w^q}^{\rho'_3} \|h(|\Psi_i|) - h(|\Phi_i|)\|_{L^p}^{\rho'_3} \|\Phi_j\|_{L^{\rho_3(\mu-1)}}^{\rho'_3(\mu-1)} \\
&\leq C \|W_{ij}\|_{L_w^q}^{\rho'_3} \left(\|\Phi_i\|_{L^{\frac{(\mu-1)r_3\rho_3}{r_3-p}}}^{\rho'_3(\mu-1)} + \|\Psi_i\|_{L^{\frac{(\mu-1)r_3\rho_3}{r_3-p}}}^{\rho'_3(\mu-1)} \right) \times \\
&\times \|\Psi_i - \Phi_i\|_{L^{r_3}}^{\rho'_3} \|\Phi_j\|_{L^{\rho_3(\mu-1)}}^{\rho'_3(\mu-1)} \leq C(M) \|\Psi_i - \Phi_i\|_{L^{r_3}}^{\rho'_3}
\end{aligned}$$

where $p = \frac{\rho_3 q}{2\rho_3 q - 2q - \rho_3}$ and $\rho_3 \in [2, 2^*)$ and $r_3 \in [2, 2^*)$ satisfy the following conditions

$$\begin{cases} 2 \leq \rho_3(\mu - 1) \leq 2^* \\ 2 \leq \frac{(\mu-1)pr_3}{(r_3-p)} = \frac{(\mu-1)\rho_3 q r_3}{2\rho_3 q r_3 - 2q r_3 - \rho_3 r_3 - \rho_3 q} \leq 2^*. \end{cases}$$

Observe again that, choosing $\rho_3 = \frac{4q}{2q-1}$, the second inequalities are satisfied by the same value of r_3 found for the second equations in (2.7). So

$$\|\tilde{g}_3(\Psi) - \tilde{g}_3(\Phi)\|_{\mathcal{L}^{\rho'_3}} \leq C(M)\|\Psi - \Phi\|_{\mathcal{L}^{r_k}}$$

(2.6) is obvious since $W_{i,j}$, h and h' are real-valued. \square

So \tilde{g} satisfy (1) to (2.6) and, in particular, assumptions like (4.2.1) – (4.2.4) in Section 4.2 or like (4.6.3) – (4.6.6) in Section 4.6 in Cazenave [3]. Consequently, by Remark 4.2.9 and Remark 4.2.13 or Remark 4.6.3 in [3], we are able to establish the local well-posedness of the above Cauchy problem in \mathcal{H}_A^1 . Observe that we assume the “a priori” information that solutions are unique since uniqueness is proved by methods which are strictly related to the type of nonlinearity and, in the following, we establish a result which ensures it. Now, we give the details of the proof. Recall that the energy functional is defined as

$$\begin{aligned} F_A(\Phi) &= \frac{1}{2} \sum_{j=1}^m \int \left| \left(\frac{\nabla}{i} - A(x) \right) \Phi_j \right|^2 dx + \frac{1}{2} \int V(x) |\Phi|^2 dx - \int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2) dx \\ &\quad - \frac{1}{2} \sum_{i,j=1}^m \iint W_{ij}(|x-y|) h(|\Phi_i(x)|) h(|\Phi_j(y)|) dx dy \\ &= \frac{1}{2} \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi \right\|_{\mathcal{L}^2}^2 + \tilde{G}_1(\Phi) - \tilde{G}_2(\Phi) - \tilde{G}_3(\Phi) \end{aligned}$$

and we denote by $\tilde{G}(\Phi) = \tilde{G}_1(\Phi) - \tilde{G}_2(\Phi) - \tilde{G}_3(\Phi)$ for every $\Phi \in \mathcal{H}_A^1$. It follows that $F_A \in C^1(\mathcal{H}_A^1, \mathbb{R})$ and that

$$F'_A(\Phi) = L_A \Phi - \tilde{g}(\Phi).$$

Remark. In assumption (1), we require that $\tilde{g} : \mathcal{H}_A^1 \rightarrow \mathcal{H}'_A$ as L_A does and that \tilde{g} is the gradient of some functional \tilde{G} since we can define the energy. Indeed, the conservation of energy is essential in our proof of local existence. (2) requires that \tilde{g} is slightly better than a mapping $\mathcal{H}_A^1 \rightarrow \mathcal{H}'_A$ and assumption (2.5) is a form of local Lipschitz condition. Finally, (2.6) implies the conservation of charge that is essential for our proof.

Proposition 2.6. *Let A satisfy (A) and (B) with some constants $(C_\alpha)_{\alpha \in \mathbb{N}^n}$ and assume (V), (W), (g) and (h) so that, in particular, \tilde{g}_k , $k = 1, 2, 3$ satisfy assumptions (1)-(2.6). For every $M > 0$, there exists $T(M)$ (depending only on M) and the (C_α) 's with the following property. For every $\Phi^0 \in \mathcal{H}_A^1$ such that $\|\Phi^0\|_{\mathcal{H}_A^1} \leq M$, there exists a solution $\Phi \in L^\infty(I, \mathcal{H}_A^1) \cap W^{1,\infty}(I, \mathcal{H}'_A)$ of the problem*

$$(2.9) \quad \begin{cases} i\partial_t \Phi - L_A \Phi + \tilde{g}(\Phi) = 0, \\ \Phi(0, x) = \Phi^0(x) \end{cases}$$

with $I_M = (-T(M), T(M))$. In addition,

$$(2.10) \quad \|\Phi(t)\|_{L^\infty((-T(M), T(M)); \mathcal{H}_A^1)} \leq 2M.$$

Furthermore,

$$(2.11) \quad \|\Phi(t)\|_{\mathcal{L}^2} = \|\Phi^0\|_{\mathcal{L}^2}$$

$$(2.12) \quad F_A(\Phi(t)) \leq F_A(\Phi^0)$$

for all $t \in I_M = (-T(M), T(M))$.

Remark. Note that both the equations in (2.9) make sense respectively in \mathcal{H}'_A and in \mathcal{H}_A^1 . Indeed, since $\Phi \in W^{1,\infty}((-T(M), T(M)), \mathcal{H}'_A)$, by Remark 2.3.10 in [3] easily adapted to our case, Φ is continuous $[-T(M), T(M)] \rightarrow \mathcal{H}'_A$; and so, since $\Phi \in L^\infty((-T(M), T(M)), \mathcal{H}_A^1)$, it follows that $\Phi : [-T(M), T(M)] \rightarrow \mathcal{H}_A^1$ is weakly continuous. Furthermore, it follows from the duality inequality $\|\Phi\|_{\mathcal{L}^2}^2 \leq \|\Phi\|_{\mathcal{H}'_A} \|\Phi\|_{\mathcal{H}_A^1}$ that $\Phi \in C([-T(M), T(M)], \mathcal{L}^2)$. Furthermore, also (2.11) and (2.12) make sense.

In order to prove the above proposition, we establish the following two elementary lemmas.

Lemma 2.7. *Let $I \subset \mathbb{R}$ be an interval. Then, for every $\Phi \in L^\infty(I, \mathcal{H}_A^1) \cap W^{1,\infty}(I, \mathcal{H}'_A)$, we have*

$$\|\Phi(t) - \Phi(s)\|_{\mathcal{L}^2} \leq C|t - s|^{1/2}, \quad \text{for all } s, t \in I$$

where $C = \max\{\|\Phi\|_{L^\infty(I, \mathcal{H}_A^1)}, \|\Phi'\|_{L^\infty(I, \mathcal{H}'_A)}\}$

Proof. The result follows from Remark 2.3.10 in [3] applied to the case of $X = \mathcal{H}_A^1$ and $p = \infty$ and the duality inequality $\|\Phi\|_{\mathcal{L}^2}^2 \leq \|\Phi\|_{\mathcal{H}'_A} \|\Phi\|_{\mathcal{H}_A^1}$ (see Remark 2.3.8 (iii) in [3] adapted to our case).

Lemma 2.8. *Let \tilde{g}_k for $k = 1, 2, 3$ satisfy assumptions (1)-(2.6). Then, after possibly modifying the function $C(M)$ independent of A (see Lemma 4.2.5, pag.55 in [3] for $A = 0$ and one equation)*

(i) *for all $k = 1, 2, 3$, it holds*

$$\|\tilde{g}_k(\Psi) - \tilde{g}_k(\Phi)\|_{\mathcal{L}^{\rho'_k}} \leq C(M)\|\Psi - \Phi\|_{\mathcal{L}^2}^{\alpha_k}$$

for every $\Phi, \Psi \in \mathcal{H}_A^1$ such that $\|\Psi\|_{\mathcal{H}_A^1} + \|\Phi\|_{\mathcal{H}_A^1} \leq M$;

(ii) *for all $k = 1, 2, 3$, it holds*

$$|\tilde{G}_k(\Psi) - \tilde{G}_k(\Phi)| \leq C(M)\|\Psi - \Phi\|_{\mathcal{L}^2}^{\beta_k}$$

for every $\Phi, \Psi \in \mathcal{H}_A^1$ such that $\|\Psi\|_{\mathcal{H}_A^1} + \|\Phi\|_{\mathcal{H}_A^1} \leq M$,

with $\alpha_k = 1 - N(\frac{1}{2} - \frac{1}{r_k})$ and $\beta_k = 1 - N(\frac{1}{2} - \frac{1}{\rho_k})$.

Proof. (i) follows from (2.5), the definition of the norm in \mathcal{L}^{r_k} , Lemma 1.2 and Gagliardo-Nirenberg inequality adapted in such case, namely $\|\Psi - \Phi\|_{\mathcal{L}^{r_k}} \leq C\|\Psi - \Phi\|_{\mathcal{L}^2}^{1-\tilde{\alpha}_k}\|\Psi - \Phi\|_{\mathcal{H}_A^1}^{\tilde{\alpha}_k}$ where $\tilde{\alpha}_k = N(1/2 - 1/r_k)$ (see [3, Theorem 2.3.7]). (ii) follows from the identity

$$\tilde{G}_k(\Psi) - \tilde{G}_k(\Phi) = \int_0^1 \frac{d}{ds} \tilde{G}_k(s\Psi + (1-s)\Phi) ds = \int_0^1 \langle \tilde{g}_k(s\Psi + (1-s)\Phi), \Psi - \Phi \rangle_{\mathcal{H}'_A, \mathcal{H}_A^1} ds$$

hypothesis (2) and $\|\Psi - \Phi\|_{\mathcal{L}^{\rho_k}} \leq C\|\Psi - \Phi\|_{\mathcal{L}^2}^{1-\tilde{\beta}_k}\|\Psi - \Phi\|_{\mathcal{H}_A^1}^{\tilde{\beta}_k}$ where $\tilde{\beta}_k = N(1/2 - 1/\rho_k)$. The fact that the constant $C(M)$ is independent of the magnetic field A follows from the uniformity of the constant in Lemma 1.2. \square

Proof of Proposition 2.6. Following Cazenave [3], the proof proceeds in three steps. In the first step, we approximate \tilde{g} by a family of “regular” nonlinearities. The importance of such regularization procedure is due to the necessity of obtaining the energy inequality (2.12). Dealing with the choice of the type of regularization which can be different for each nonlinearity we deal with, we apply $(I + \varepsilon L_A)^{-1}$ so that the proof applies to our different nonlinearities \tilde{g}_k , $k = 1, 2, 3$ and works as well when $\Omega = \mathbb{R}^N$. So we are able to construct approximate solutions. In the second step, we obtain uniform estimates on such solutions, by using the conservation laws and, in particular, the conservation of energy, in order to pass, in the third step, to the limit in the approximate equation. Observe that, even if there is a little bitter difficulty in the passage to the limit in the nonlinearity, we can recover

the conservation of charge by the limiting problem thanks to the global regularization.

Step 1: Construction of approximate solutions. From now on, we consider $\Phi^0 \in \mathcal{H}_A^1$ and we set $M = \|\Phi^0\|_{\mathcal{H}_A^1}$. Given a positive integer $n \in \mathbb{N}$, we define

$$J_n^A = \left(I + \frac{1}{n} L_A \right)^{-1}$$

so that, for every $f \in \mathcal{H}'_A$, $J_n^A f \in \mathcal{H}_A^1$ is the unique solution of the following system

$$\Phi - \frac{1}{n} L_A \Phi = f \quad \text{in } \mathcal{H}'_A.$$

From the self-adjointness of L_A , it is possible to deduce the following main properties of the self-adjoint operator J_n^A (see Section 2.4 and Theorem 4.6.1 in Section 4.6 in [3]), namely

$$(2.13) \quad \|J_n^A\|_{\mathcal{L}(\mathcal{H}'_A, \mathcal{H}_A^1)} \leq \frac{1}{n};$$

$$(2.14) \quad \|J_n^A\|_{\mathcal{L}(X, X)} \leq 1, \quad \text{whenever } X = \mathcal{H}_A^1, \mathcal{L}^2, \mathcal{H}'_A;$$

$$(2.15)$$

$$J_n^A \Phi \xrightarrow{n \rightarrow \infty} \Phi \quad \text{in } X, \text{ for every } \Phi \in X, \text{ whenever } X = \mathcal{H}_A^1, \mathcal{L}^2, \mathcal{H}'_A;$$

if Φ^n is bounded in X , whenever $X = \mathcal{H}_A^1, \mathcal{L}^2, \mathcal{H}'_A$, then $J_n^A \Phi^n - \Phi^n \rightarrow 0$ in X , as $n \rightarrow \infty$.

From Lemma 2.4, we have that

$$(2.16) \quad \|J_n^A\|_{\mathcal{L}(\mathcal{L}^p, \mathcal{L}^p)} \leq 1, \quad \text{for } 1 \leq p < \infty$$

We define, for every $\Phi \in \mathcal{L}^2$, the approximation of \tilde{g} as

$$\begin{aligned} \tilde{g}_n(\Phi) &= J_n^A(\tilde{g}(J_n^A \Phi)) = J_n^A((\tilde{g}_1 + \tilde{g}_2 + \tilde{g}_3)(J_n^A \Phi)) \\ &= J_n^A(\tilde{g}_1(J_n^A \Phi) + \tilde{g}_2(J_n^A \Phi) + \tilde{g}_3(J_n^A \Phi)) \\ &= J_n^A(\tilde{g}_1(J_n^A \Phi)) + J_n^A(\tilde{g}_2(J_n^A \Phi)) + J_n^A(\tilde{g}_3(J_n^A \Phi)) \\ &= \tilde{g}_{1,n}(J_n^A \Phi) + \tilde{g}_{2,n}(J_n^A \Phi) + \tilde{g}_{3,n}(J_n^A \Phi) \end{aligned}$$

and, for every $\Phi \in \mathcal{H}_A^1$, the approximation of \tilde{G}

$$(2.17)$$

$$\tilde{G}_n(\Phi) = \tilde{G}(J_n^A \Phi) = \tilde{G}_1(J_n^A \Phi) + \tilde{G}_2(J_n^A \Phi) + \tilde{G}_3(J_n^A \Phi) = \tilde{G}_{1,n}(\Phi) + \tilde{G}_{2,n}(\Phi) + \tilde{G}_{3,n}(\Phi).$$

We observe that from (2.13) the above definition make sense. Furthermore, by (2.13) and (2.5), we have that \tilde{g}_n is Lipschitz continuous on

bounded sets of \mathcal{L}^2 , and by (2.13) and (1) that $\tilde{G}_n \in C^1(\mathcal{H}_A^1, \mathbb{R})$ and $\tilde{G}'_n = \tilde{g}_n$. From (2.6), it follows that

$$(\tilde{g}_n(\Phi), i\Phi)_{\mathcal{L}^2} = (\tilde{g}(J_n^A \Phi), J_n^A \Phi)_{\mathcal{L}^2} = 0, \quad \text{for every } \Phi \in \mathcal{L}^2.$$

Therefore, there exists a sequence $(\Phi^n)_{n \in \mathbb{N}}$ of functions of $C(\mathbb{R}, \mathcal{H}_A^1) \cap C^1(\mathbb{R}, \mathcal{H}'_A)$ such that

$$(2.18) \quad \begin{cases} i\Phi_t^n - L_A \Phi^n + \tilde{g}_n(\Phi^n) = 0, \\ \Phi^n(0) = \Phi^0. \end{cases}$$

Furthermore,

$$(2.19) \quad \|\Phi^n(t)\|_{\mathcal{L}^2} = \|\Phi^0\|_{\mathcal{L}^2}$$

and

$$(2.20) \quad \begin{aligned} F_A^n(\Phi^n(t)) &= \frac{1}{2} \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi^n(t) \right\|_{\mathcal{L}^2}^2 + \tilde{G}_n(\Phi^n(t)) \\ &= \frac{1}{2} \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi^0 \right\|_{\mathcal{L}^2}^2 + \tilde{G}_n(\Phi^0) = F_A^n(\Phi^0) \end{aligned}$$

for all $t \in \mathbb{R}$.

Step 2. Estimates on the sequence Φ^n . We denote by $C(M)$ various constants depending only on M . Remark again that the independence on A follows from the uniformity of the constants involved in Lemma 1.2 and consequently Lemmas 2.7 and 2.8. Let

$$(2.21) \quad \theta_n = \sup\{\tau > 0 : \|\Phi^n(t)\|_{\mathcal{H}_A^1} \leq 2M \quad \text{on } (\tau, \tau)\}.$$

Note that by (2.14) and (2.16),

$$(2.22) \quad \tilde{g}_n \quad \text{verifies (2) and (2.5) uniformly in } n \in \mathbb{N} \text{ and } A.$$

Therefore, by (2.18)

$$(2.23) \quad \sup_{n \in \mathbb{N}} \|\Phi_t^n\|_{L^\infty(-T(M), T(M); \mathcal{H}'_A)} \leq C(M).$$

From (2.21) and (2.23), we can apply Lemma 2.7 so

$$(2.24) \quad \|\Phi^n(t) - \Phi^n(s)\|_{\mathcal{L}^2} \leq C(M)|t - s|^{1/2}, \quad \text{for all } s, t \in (-\theta_n, \theta_n).$$

Applying (2.19), (2.20), Lemma 2.8 (ii), (2.21) and (2.24) for $s = 0$, we obtain

(2.25)

$$\begin{aligned}
\|\Phi^n(t)\|_{\mathcal{H}_A^1}^2 &\leq \|\Phi^0\|_{\mathcal{L}^2} + \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi^0 \right\|_{\mathcal{L}^2} + 2 \left| \tilde{G}_n(\Phi^n(t)) - \tilde{G}_n(\Phi^0) \right| \\
&\leq \|\Phi^0\|_{\mathcal{H}_A^1}^2 + 2 \sum_{k=1}^3 \left| \tilde{G}_{k,n}(\Phi^n(t)) - \tilde{G}_{k,n}(\Phi^n(0)) \right| \\
&\leq \|\Phi^0\|_{\mathcal{H}_A^1}^2 + C(M) \sum_{k=1}^3 |t|^{\beta_k/2} \\
(2.26) \quad &\leq \|\Phi^0\|_{\mathcal{H}_A^1}^2 + C(M) |t|^{\beta/2}
\end{aligned}$$

where $\beta = \max\{\beta_k : k = 1, 2, 3\}$ for all $t \in (-\theta_n, \theta_n)$. If we define $T(M)$ by

$$C(M)T(M)^{\beta/2} = 3M^2,$$

recalling that $\|\Phi^0\|_{\mathcal{H}_A^1}$, it follows from (2.25) that

$$\|\Phi^n\|_{L^\infty(-T, T; \mathcal{H}_A^1)} \leq 2M,$$

for $T = \min\{T(M), \theta_n\}$. This implies that $T(M) \leq \theta_n$; and so

$$(2.27) \quad \|\Phi^n\|_{L^\infty(-T(M), T(M); \mathcal{H}_A^1)} \leq 2M,$$

and by (2.23)

$$(2.28) \quad \|\Phi_t^n\|_{L^\infty(-T(M), T(M); \mathcal{H}'_A)} \leq C(M).$$

Step 3. Passage to the limit. It follows from (2.27) and (2.28) and Proposition 2.3.13 (i) in [3] adapted to our case that there exists $\Phi \in L^\infty(-T(M), T(M); \mathcal{H}_A^1) \cap W^{1,\infty}(-T(M), T(M); \mathcal{H}'_A)$ and a subsequence, which we still denote by (Φ^n) such that for all $t \in [-T(M), T(M)]$,

$$(2.29) \quad \Phi^n(t) \rightharpoonup \Phi(t) \quad \text{in } \mathcal{H}_A^1, \text{ as } n \rightarrow \infty.$$

In addition, by (2.27) and (2.29), Lemma 2.7, (2.22) and Lemma 2.8, we have that $\tilde{g}_{k,n}(\Phi^n)$ is bounded in the space $C^{0,\alpha_k/2}(-T(M), T(M); \mathcal{L}^{\rho'_k})$ for $k = 1, 2$ and $k = 3$ for $\mu \in [2, 3)$ or in the space $C^{0,\alpha_k(\mu-2)/2}(-T(M), T(M); \mathcal{L}^{\rho'_k})$ for $k = 3$ and $\mu \geq 3$. Therefore, it follows from Proposition 2.1.7 in [3] adapted to our case that there exists f_k which belongs to $C^{0,\alpha_k/2}(-T(M), T(M); \mathcal{L}^{\rho'_k})$ for $k = 1, 2$ and $k = 3$ for $\mu \in [2, 3)$ or to $C^{0,\alpha_k(\mu-2)/2}(-T(M), T(M); \mathcal{L}^{\rho'_k})$

for $k = 3$ and $\mu \geq 3$ and a subsequence, which we still denote by $(\tilde{g}_{k,n}(\Phi^n))$ such that for all $t \in [-T(M), T(M)]$,

$$(2.30) \quad \tilde{g}_{k,n}(\Phi^n(t)) \rightharpoonup f_k(t) \quad \text{in } \mathcal{L}^{\rho'_k}, \text{ as } n \rightarrow \infty.$$

On the other hand, it follows from (2.18) that for every $\Psi \in \mathcal{H}_A^1$ and for every $\phi \in \mathcal{D}(-T(M), T(M))$, we have

$$\int_{-T(M)}^{T(M)} \left\{ -\langle i\Phi^n, \Psi \rangle_{\mathcal{H}'_A, \mathcal{H}_A^1} \phi'(t) - \left\langle L_A \Phi^n - \sum_{k=1}^3 \tilde{g}_{k,n}(\Phi^n), \Psi \right\rangle_{\mathcal{H}'_A, \mathcal{H}_A^1} \phi(t) \right\} dt = 0$$

Applying (2.29), (2.30) and the Dominated Convergence Theorem, it follows that

$$\int_{-T(M)}^{T(M)} \left\{ i\langle \Phi, \Psi \rangle_{\mathcal{H}'_A, \mathcal{H}_A^1} \phi'(t) - \langle L_A \Phi - f, \Psi \rangle_{\mathcal{H}'_A, \mathcal{H}_A^1} \phi(t) \right\} dt = 0,$$

where $f = f_1 + f_2 + f_3$. This means that Φ satisfies

$$(2.31) \quad \begin{cases} i\Phi_t - L_A \Phi + f = 0, & \text{for almost } t \in (-T(M), T(M)), \\ \Phi(0) = \Phi^0. \end{cases}$$

Now we prove the following crucial result according to which the limit problem enjoys the conservation of charge.

Lemma 2.9. *For all $t \in (-T(M), T(M))$, we have $\Im(f(t)\overline{\Phi(t)}) = 0$ almost everywhere on \mathbb{R}^N .*

Proof. It's not so different respect to the one in Lemma 4.2.6 in [3]. Indeed, it's sufficient to show that for every bounded subsets B of \mathbb{R}^N , we have for every $k = 1, 2, 3$,

$$\langle f_k(t)|_B, i\Phi(t)|_B \rangle_{\mathcal{L}^{\rho'_k(B)}, \mathcal{L}^{\rho_k(B)}} = 0.$$

For simplicity, we omit the time dependence and we write

$$\begin{aligned} \langle f_k, i\Phi \rangle_{\mathcal{L}^{\rho'_k(B)}, \mathcal{L}^{\rho_k(B)}} &= \langle f_k - J_n^A(\tilde{g}_k(J_n^A \Phi^n)), i\Phi \rangle + \langle J_n^A(\tilde{g}_k(J_n^A \Phi^n)) - \tilde{g}_k(J_n^A \Phi^n), i\Phi \rangle \\ &\quad + \langle \tilde{g}_k(J_n^A \Phi^n), i(\Phi - \Phi^n) \rangle + \langle \tilde{g}_k(J_n^A \Phi^n), i(\Phi^n - J_n^A \Phi^n) \rangle \\ &\quad + \langle \tilde{g}_k(J_n^A \Phi^n), i(J_n^A \Phi^n) \rangle \rightarrow_{n \rightarrow \infty} A + B + C + D + E. \end{aligned}$$

Note first that, by (2.30), $J_n^A(\tilde{g}_k(J_n^A \Phi^n)) = \tilde{g}_{k,n}(\Phi^n) \rightharpoonup f_k$ in $\mathcal{L}^{\rho'_k}$, hence in $\mathcal{L}^{\rho'_k(B)}$. Therefore, $A = 0$. Next, we observe that $\tilde{g}_k(J_n^A \Phi^n)$ is bounded in $\mathcal{L}^{\rho'_k}$. It follows from (2.16) and (2.16) that $J_n^A(\tilde{g}_k(J_n^A \Phi^n)) - \tilde{g}_k(J_n^A \Phi^n) \rightharpoonup 0$ in \mathcal{H}'_A , hence in $\mathcal{L}^{\rho'_k(B)}$. Therefore, $B = 0$. Since by (2.29), $\Phi^n \rightharpoonup \Phi$ in \mathcal{H}_A^1 , we have $\Phi^n \rightarrow \Phi$ in $\mathcal{L}^{\rho_k(B)}$. Since $\tilde{g}_k(J_n^A \Phi^n)$ is bounded in $\mathcal{L}^{\rho'_k(B)}$, it follows that $C = 0$. By (2.14) and (2.16),

$\Phi^n - J_n^A \Phi^n$ is bounded in \mathcal{H}_A^1 and converges weakly to 0 in \mathcal{H}'_A . It follows that $\Phi^n - J_n^A \Phi^n \rightarrow 0$ in $\mathcal{L}^{\rho_k}(B)$. Since $\tilde{g}_k(J_n^A \Phi^n)$ is bounded in $\mathcal{L}^{\rho'_k}(B)$, it follows that $D = 0$. Finally, $E = 0$ by (2.6). Hence the result. \square

End of the proof of Proposition 2.6. Taking the $\mathcal{H}'_A - \mathcal{H}_A^1$ duality product of the first equation in (2.31) with $i\Phi$, it follows that

$$\frac{d}{dt} \|\Phi(t)\|_{\mathcal{L}^2} = 0, \quad \text{for all } t \in (-T(M), T(M));$$

and so

$$(2.32) \quad \|\Phi(t)\|_{\mathcal{L}^2} = \|\Phi^0\|_{\mathcal{L}^2}.$$

It follows from (2.19), (2.32) and Proposition 2.3.13 (ii) in [3] adapted to our case that

$$(2.33) \quad \Phi^n \rightarrow \Phi \quad \text{in } C(-T(M), T(M); \mathcal{L}^2).$$

Applying (2.27), (2.33) and Gagliardo-Nirenberg inequality (see Theorem 2.3.7 in [3]), it follows that

$$(2.34) \quad \Phi^n \rightarrow \Phi \quad \text{in } C(-T(M), T(M); \mathcal{L}^p), \text{ for every } 2 \leq p < \frac{2N}{N-2}.$$

It follows from (2.5), (2.15) and (2.34) that

$$J_n^A(\tilde{g}_k(J_n^A \Phi^n)) = \tilde{g}_{k,n}(\Phi^n(t)) \rightarrow \tilde{g}_k(\Phi(t)) \quad \text{in } \mathcal{L}^{\rho'_k}, \text{ for all } t \in (-T(M), T(M)).$$

Therefore, $f = \tilde{g}(\Phi)$ and so, Φ satisfies (2.9). (2.10) follows from (2.27) and (2.11) from (2.32). It remains to prove (2.12). This follows from (2.20), weak lower semicontinuity of the \mathcal{H}_A^1 -norm and the fact that $\tilde{G}_n(\Phi^n(t)) \rightarrow \tilde{G}(\Phi(t))$ as $n \rightarrow \infty$ by (2.34) and Lemma 2.8 (ii). This completes the proof.

Before proceeding further, we make the following definition.

Definition 2.10. *In all what follows, we say that we have uniqueness for problem (2.9) if the following hold. For every interval J containing 0 and for every $\Phi^0 \in \mathcal{H}_A^1$, any two solutions of (2.9) in $L^\infty(J, \mathcal{H}_A^1) \cap W^{1,\infty}(J, \mathcal{H}'_A)$ coincide.*

The main result of this section is the following.

Theorem 2.11. *Let A satisfies (A) and (B) and assume (V), (W), (g) and (h) so that, in particular, \tilde{g} satisfy assumptions (1)-(2.6) and assume that we have uniqueness for problem (2.9). Then the following properties hold.*

- (1) *For every $\Phi^0 \in \mathcal{H}_A^1(\mathbb{R}^N)$, there exists $T_*(\Phi^0), T^*(\Phi^0) > 0$ and there exists a unique, maximal solution $\Phi \in C((-T_*(\Phi^0), T^*(\Phi^0)), \mathcal{H}_A^1) \cap C^1((-T_*(\Phi^0), T^*(\Phi^0)), \mathcal{H}'_A)$ of problem (2.9). Φ is maximal in the sense that if $T^*(\Phi^0) < \infty$ (resp., $T_*(\Phi^0) < \infty$), then $\|\Phi(t)\|_{\mathcal{H}_A^1} \rightarrow \infty$, as $t \uparrow T^*(\Phi^0)$ (resp., as $t \downarrow -T_*(\Phi^0)$);*
- (2) *in addition, we have conservation of charge and energy, that is*

$$\|\Phi(t)\|_{\mathcal{L}^2} = \|\Phi^0\|_{\mathcal{L}^2} \quad F_A(\Phi(t)) = F_A(\Phi^0)$$

for all $t \in (-T_(\Phi^0), T^*(\Phi^0))$;*

Proof. Following Cazenave [3], the proof proceeds in two steps. We first show that the solution Φ given by Proposition 2.6 belongs to $\Phi \in C((-T_*(\Phi^0), T^*(\Phi^0)), \mathcal{H}_A^1) \cap C^1((-T_*(\Phi^0), T^*(\Phi^0)), \mathcal{H}'_A)$, and that we have conservation of energy. Next, we consider the maximality result.

Step 1. Regularity. Let I be an interval and let $\Phi \in L^\infty(I, \mathcal{H}_A^1) \cap W^{1,\infty}(I, \mathcal{H}'_A)$ satisfy

$$i\Phi_t - L_A\Phi + \tilde{g}(\Phi) = 0, \quad \text{for all } t \in I$$

We claim that Φ enjoys both conservation of charge and energy and that $\Phi \in C(I, \mathcal{H}_A^1) \cap C^1(I, \mathcal{H}'_A)$. To see this, consider

$$M = \sup \left\{ \|\Phi(t)\|_{\mathcal{H}_A^1}, t \in I \right\},$$

and let us first show that $\|\Phi(t)\|_{\mathcal{L}^2}$ and $F_A(\Phi(t))$ are constant on every interval $J \subset I$ of length at most $T(M)$, where $T(M)$ is given by Proposition 2.6. Indeed, let J be as above and let $\sigma, \tau \in J$. Let $\Phi^0 = \Phi(\sigma)$ and let Ψ be the solution of (2.9) given by Proposition 2.6. $\Psi(\cdot - \sigma)$ is defined on J and by uniqueness, $\Psi(\cdot - \sigma) = \Phi(\cdot)$ on J . By (2.11) and (2.12), it follows in particular that

$$(2.35) \quad \|\Phi(\tau)\|_{\mathcal{L}^2} = \|\Phi(\sigma)\|_{\mathcal{L}^2}, \quad F_A(\Phi(\tau)) \leq F_A(\Phi(\sigma)).$$

Now let $\Phi^0 = \Phi(\tau)$ and let Z be the solution of (2.9) given by Proposition 2.6. $Z(\cdot - \tau)$ is defined on J and by uniqueness, $Z(\cdot - \tau) = \Phi(\cdot)$ on J . By (2.12), it follows in particular that

$$F_A(\Phi(\sigma)) \leq F_A(\Phi(\tau)).$$

Comparing with (2.35), this implies that both $\|\Phi(t)\|_{\mathcal{L}^2}$ and $F_A(\Phi(t))$ are constant on J . Since J is arbitrary, it follows that

(2.36)

$$\|\Phi(t)\|_{\mathcal{L}^2} = \|\Phi(s)\|_{\mathcal{L}^2} \quad \text{and} \quad F_A(\Phi(t)) = F_A(\Phi(s)), \quad \text{for all } s, t \in I.$$

Furthermore, note that by Lemma 2.7, $\Phi \in C^{0,1/2}(\bar{I}, \mathcal{L}^2)$; and so, by Lemma 2.8 (ii), the function $t \rightarrow \tilde{G}(\Phi(t)) = \sum_{k=1}^3 \tilde{G}_k(\Phi(t))$ is continuous $\bar{I} \rightarrow \mathbb{R}$. In view of (2.36), it follows that $\|\Phi(t)\|_{\mathcal{H}_A^1}$ is continuous $\bar{I} \rightarrow \mathbb{R}$. Therefore, by Lemma 2.1.5 in [3] for $X = \mathcal{H}_A^1$, $\Phi \in C(\bar{I}, \mathcal{H}_A^1)$, and by the equation, $\Phi \in C^1(\bar{I}, \mathcal{H}'_A)$.

Step 2. Maximality. Consider $\Phi^0 \in \mathcal{H}_A^1$ and let

$$T^*(\Phi^0) = \sup \{T > 0 : \text{there exists a solution of (2.9) on } [0, T]\}$$

$$T_*(\Phi^0) = \sup \{T > 0 : \text{there exists a solution of (2.9) on } [-T, 0]\}.$$

By uniqueness and Step 1, there exists a solution

$$\Phi \in C((-T_*(\Phi^0), T^*(\Phi^0)), \mathcal{H}_A^1) \cap C^1((-T_*(\Phi^0), T^*(\Phi^0)), \mathcal{H}'_A)$$

of (2.9). Suppose now that $T^*(\Phi^0) < \infty$ and assume that there exists $M < \infty$ and a sequence $t_j \uparrow T^*(\Phi^0)$ such that $\|\Phi(t_j)\|_{\mathcal{H}_A^1} \leq M$. Let k be such that $t_k + T(M) > T^*(\Phi^0)$. By Proposition 2.6 and Step 1 and starting from $\Phi(t_k)$, one can extend Φ up to $t_k + T(M)$, which is a contradiction with the maximality. Therefore, $\|\Phi(t)\|_{\mathcal{H}_A^1} \rightarrow \infty$, as $t \uparrow T^*(\Phi^0)$. One shows by the same argument that if $T_*(\Phi^0) < \infty$, then $\|\Phi(t)\|_{\mathcal{H}_A^1} \rightarrow \infty$, as $t \downarrow T_*(\Phi^0)$. Therefore, we have established statements (i) and (ii) of Theorem 2.11.

Remark. By Theorem 2.11, under a priori uniqueness assumption, we have proved the well posedness of problem (2.9) in \mathcal{H}_A^1 , in particular, under assumptions (1) through (2.6) on \tilde{g}_k for $k = 1, 2, 3$. We recall below a general sufficient condition for uniqueness by adapting Corollary 4.2.12 in [3]). It follows that

$$\Psi(t) - \Phi(t) = i \int_0^t T(t-s)(\tilde{g}(\Psi(s)) - \tilde{g}(\Phi(s))) ds,$$

for all $t \in I$, where $T(t)$ is the propagator e^{-itL_A} . By assumptions (A) and (B) on the potential and magnetic potentials, adapting the result in Yajima [17] proved for such $T(t)$, we have the following \mathcal{L}^p - \mathcal{L}^q estimates

Proposition 2.12. *Let $I_T = [-T, T]$. Then, for any p such that $2 \leq p \leq \infty$ and q conjugate to p , there exists a constant C independent of t such that for any $v \in \mathcal{L}^q$*

$$(2.37) \quad \|T(t)v\|_{\mathcal{L}^p} \leq \frac{C}{t^{N(\frac{1}{2}-\frac{1}{q})}} \|v\|_{\mathcal{L}^q}.$$

Corollary 2.13. *The conclusions of Theorem 2.11 holds true.*

Proof. We have to prove that the uniqueness condition in $L^\infty(I, \mathcal{H}_A^1) \cap W^{1,\infty}(I, \mathcal{H}'_A)$ is fulfilled. The argument follows the line of [3, proof of Theorem 4.3.1]. Let I be an interval containing 0 to be chosen sufficiently small. Let $\Psi, \Phi \in L^\infty(I, \mathcal{H}_A^1) \cap W^{1,\infty}(I, \mathcal{H}'_A)$ be two solutions of equation (2.9). Let r_i and ρ_i the exponents for which the nonlinearity g_i verifies the assumptions of Theorem 2.11. Therefore, setting $\frac{2}{q_j} = N(\frac{1}{2} - \frac{1}{r_j})$ and $\frac{2}{\gamma_j} = N(\frac{1}{2} - \frac{1}{\rho_j})$, there exists $\delta > 0$ such that

$$\|\Psi - \Phi\|_{L^{q_i}(I, L^{r_i})} \leq C \sum_{j=1}^m \|\tilde{g}(\Psi) - \tilde{g}(\Phi)\|_{L^{\gamma'_j}(I, L^{\rho'_j})} \leq C(|I| + |I|^\delta) \sum_{j=1}^m \|\Psi - \Phi\|_{L^{q_j}(I, L^{r_j})},$$

where the first inequality can be obtained by arguing as in the proof of [3, Theorem 3.5.2(ii)], where the property in [3, Theorem 3.2.1] is substituted by (2.37), the estimate by Yajima. In turn, adding the above inequality over $i = 1, \dots, m$ and choosing the size of $|I|$ such that $C(|I| + |I|^\delta) < 1$ we get the inequality

$$(1 - C(|I| + |I|^\delta)) \sum_{j=1}^m \|\Psi - \Phi\|_{L^{q_j}(I, L^{r_j})} \leq 0,$$

yielding the desired conclusion. □

3. GLOBAL WELL-POSEDNESS

We have established the local solvability of the Cauchy problem (2.9) in \mathcal{H}_A^1 . In order to show that the solution Φ is global, namely that exists for all times, it is sufficient to establish a priori estimates on $\|\Phi(t)\|_{\mathcal{H}_A^1}$ by using the conservation laws (charge and energy) under some appropriate assumptions on the nonlinearities.

Theorem 3.1. *Assume (A), (B), (V), (W), (G) and (h) with*

$$0 < l_j < \frac{4}{N}, \quad \mu < 2 - \frac{1}{q} + \frac{2}{N}, \quad \inf_{x \in \mathbb{R}^N} V(x) > 0.$$

Let $\Phi^0 \in \mathcal{H}_A^1(\mathbb{R}^N)$ be such that $\|\Phi^0\|_{\mathcal{H}_A^1} \leq M$ and let

$$\Phi \in C((-T_*(\Phi^0), T^*(\Phi^0)), \mathcal{H}_A^1) \cap C^1((-T_*(\Phi^0), T^*(\Phi^0)), \mathcal{H}'_A)$$

be the maximal solution of problem (2.9) given by Theorem 2.11. Then, Φ is global, namely $T_*(\Phi^0) = T^*(\Phi^0) = \infty$ and $\sup\{\|\Phi(t)\|_{\mathcal{H}_A^1} : t \in \mathbb{R}\} < \infty$.

Proof. Let $I_0 = (-T_*(\Phi^0), T^*(\Phi^0))$. By Theorem 2.11 (ii), we have the conservation of energy and charge, that is

$$\|\Phi(t)\|_{\mathcal{L}^2} = \|\Phi^0\|_{\mathcal{L}^2} \quad F_A(\Phi(t)) = F_A(\Phi^0)$$

for all $t \in I_0$. From the first equality we have that

$$\|\Phi_j(t)\|_{\mathcal{L}^2} = \|\Phi_j^0\|_{\mathcal{L}^2} \quad \text{for all } j = 1, \dots, m$$

and from the second

$$\begin{aligned} F_A(\Phi(t)) &= \frac{1}{2} \sum_{j=1}^m \int \left| \left(\frac{\nabla}{i} - A(x) \right) \Phi_j(t) \right|^2 + \frac{1}{2} \int V(x) |\Phi(t)|^2 \\ &\quad - \int G(|x|, |\Phi_1(t)|^2, \dots, |\Phi_m(t)|^2) \\ &\quad - \frac{1}{2} \sum_{i,j=1}^m \iint W_{ij}(|x-y|) h(|\Phi_i(t)|) h(|\Phi_j(t)|) dx dy = F_A(\Phi^0) = C_0. \end{aligned}$$

Since V is bounded from below we have that

$$\begin{aligned} C \|\Phi(t)\|_{\mathcal{H}_A^1}^2 &\leq \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi(t) \right\|_{\mathcal{L}^2}^2 + \int V(x) |\Phi(t)|^2 \\ &\leq C_0 + 2 \int G(|x|, |\Phi_1(t)|^2, \dots, |\Phi_m(t)|^2) \\ &\quad + \sum_{i,j=1}^m \iint W_{ij}(|x-y|) h(|\Phi_i(t)|) h(|\Phi_j(t)|) dx dy \end{aligned}$$

By assumptions (G0)-(G1), we have that

$$\begin{aligned} \int G(|x|, |\Phi_1(t)|^2, \dots, |\Phi_m(t)|^2) &\leq K \int |\Phi(t)|^2 + K \int \sum_{j=1}^m |\Phi_j(t)|^{l_j+2} \\ &= K \|\Phi(t)\|_{\mathcal{L}^2}^2 + K \sum_{j=1}^m \int |\Phi_j(t)|^{l_j+2} \end{aligned}$$

For $j = 1, \dots, m$, by the Gagliardo-Nirenberg inequality we have that:

$$\|\Phi_j(t)\|_{l_j+2} \leq c \|\Phi_j(t)\|_{L^2}^{1-\sigma_j} \|\nabla|\Phi_j(t)|\|_{L^2}^{\sigma_j}, \quad \sigma_j = \frac{Nl_j}{2(l_j+2)}.$$

Now let $p_j = \frac{4}{Nl_j}$ and q_j is such that $\frac{1}{p_j} + \frac{1}{q_j} = 1$. Applying Young and Diamagnetic Inequalities, we obtain

$$\begin{aligned} \|\Phi_j(t)\|_{l_j+2}^{l_j+2} &\leq C^{l_j+2} \|\Phi_j(t)\|_{L^2}^{(1-\sigma_j)(l_j+2)} \|\nabla|\Phi_j(t)|\|_{L^2}^{\sigma_j(l_j+2)} \\ &\leq \frac{1}{q_j} \left\{ \frac{C^{l_j+2}}{\varepsilon} \|\Phi_j(t)\|_{L^2}^{(1-\sigma_j)(l_j+2)} \right\}^{q_j} + \frac{Nl_j}{4} \varepsilon^{\frac{4}{Nl_j}} \|\nabla|\Phi_j(t)|\|_{L^2}^2 \\ &\leq \frac{1}{q_j} \left\{ \frac{C^{l_j+2}}{\varepsilon} m_j^{\frac{(1-\sigma_j)(l_j+2)}{2}} \right\}^{q_j} + \frac{Nl_j}{4} \varepsilon^{\frac{4}{Nl_j}} \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi_j(t) \right\|_{L^2}^2 \\ &\leq \frac{1}{q_j} \left\{ \frac{C^{l_j+2}}{\varepsilon} m_j^{\frac{(1-\sigma_j)(l_j+2)}{2}} \right\}^{q_j} + \frac{Nl_j}{4} \varepsilon^{\frac{4}{Nl_j}} \left\| \left(\frac{\nabla}{i} - A(x) \right) \Phi_j(t) \right\|_{\mathcal{L}^2}^2 \\ &\leq \frac{1}{q_j} \left\{ \frac{C^{l_j+2}}{\varepsilon} m_j^{\frac{(1-\sigma_j)(l_j+2)}{2}} \right\}^{q_j} + \frac{Nl_j}{4} \varepsilon^{\frac{4}{Nl_j}} \|\Phi_j(t)\|_{\mathcal{H}_A^1}^2. \end{aligned}$$

Consequently,

$$\begin{aligned} \int G(|x|, |\Phi_1(t)|^2, \dots, |\Phi_m(t)|^2) &\leq K \|\Phi(t)\|_{\mathcal{L}^2}^2 + K \sum_{j=1}^m \frac{1}{q_j} \left\{ \frac{C^{l_j+2}}{\varepsilon} m_j^{\frac{(1-\sigma_j)(l_j+2)}{2}} \right\}^{q_j} \\ &\quad + K \left(\sum_{j=1}^m \frac{Nl_j}{4} \varepsilon^{\frac{4}{Nl_j}} \right) \|\Phi(t)\|_{\mathcal{H}_A^1}^2 \\ &= C_1 + C_2(\varepsilon) + K \left(\sum_{j=1}^m \frac{Nl_j}{4} \varepsilon^{\frac{4}{Nl_j}} \right) \|\Phi(t)\|_{\mathcal{H}_A^1}^2. \end{aligned}$$

Following the calculations done in Remark 2.3,

$$\begin{aligned} &\sum_{i,j=1}^m \iint W_{ij}(|x-y|) h(|\Phi_i(t)|) h(|\Phi_j(t)|) dx dy \\ &\leq \sum_{i,j=1}^m \|W_{ij}\|_{L_w^q} m_i^{\frac{\mu}{2} [1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} m_j^{\frac{\mu}{2} [1-N(\frac{1}{2}-\frac{(2q-1)}{2q\mu})]} \|\Phi(t)\|_{\mathcal{H}_A^1}^{2N\mu(\frac{1}{2}-\frac{(2q-1)}{2q\mu})} \\ &\leq C_3 \|\Phi(t)\|_{\mathcal{H}_A^1}^{2N\mu(\frac{1}{2}-\frac{(2q-1)}{2q\mu})}. \end{aligned}$$

Consequently,

$$\|\Phi(t)\|_{\mathcal{H}_A^1}^2 \leq C_0 + C_1 + C_2(\varepsilon) + K \left(\sum_{j=1}^m \frac{Nl_j}{4} \varepsilon^{\frac{4}{Nl_j}} \right) \|\Phi(t)\|_{\mathcal{H}_A^1}^2 + C_3 \|\Phi(t)\|_{\mathcal{H}_A^1}^{2N\mu \left(\frac{1}{2} - \frac{(2q-1)}{2q\mu} \right)}$$

and taking ε such that $1 - K \left(\sum_{j=1}^m \frac{Nl_j}{4} \varepsilon^{\frac{4}{Nl_j}} \right)$ is positive, we have

$$\left(1 - K \left(\sum_{j=1}^m \frac{Nl_j}{4} \varepsilon^{\frac{4}{Nl_j}} \right) \right) \|\Phi(t)\|_{\mathcal{H}_A^1}^2 \leq C_0 + C_1 + C_2(\varepsilon) + C_3 \|\Phi(t)\|_{\mathcal{H}_A^1}^{2N\mu \left(\frac{1}{2} - \frac{(2q-1)}{2q\mu} \right)}.$$

By the hypothesis on μ , if $\|\Phi(t)\|_{\mathcal{H}_A^1}^2$ was unbounded respect to t , by the above inequality we would have a contradiction. So we have proved that $\|\Phi(t)\|_{\mathcal{H}_A^1}^2$ is bounded respect to t thus proving the global existence result 3.1 by standard arguments. \square

Remark 3.2. We observe in particular that condition $l_j < \frac{4}{N}$ in (G) and $\mu < 2 - \frac{1}{q} + \frac{2}{N}$ are fundamental for the proof of the above global existence result.

4. APPENDIX

Step 1: Construction of approximate solutions (Cazenave-Weissler [5]). We present a supposed adaptation of the arguments in Cazenave and Weissler [5] (see Theorem 2.1) in the case of systems (see Remark 2.7 in [5]) and for $A \neq 0$ and the arguments in Cazenave-Esteban [4] in the case of systems and not necessarily for magnetic potentials A of polynomial type and constant magnetic fields B .

We apply Lemma 1.2, i.e., $\mathcal{H}_A^1 \subset \mathcal{L}^p$ and $\mathcal{L}^{p'} \subset \mathcal{H}'_A$ in the place of usual Sobolev's inequalities and we use suitable estimates of the propagator $T(t) = e^{-itL_A}$.

By assumptions (V), (W) and (g) and (h), we have that each \tilde{g}_i and so \tilde{g} belong to $C(\mathcal{H}_A^1, \mathcal{H}'_A)$. Consider the problem

$$(4.1) \quad \begin{cases} i\partial_t \Phi^n - L_A \Phi^n + \tilde{g}_n(\Phi^n) = 0, \\ \Phi^n(0) = \varphi \in \mathcal{H}_A^1 \end{cases}$$

where $\tilde{g}_n = \sum_{k=1}^3 \tilde{g}_{k,n}$ and each of the $\tilde{g}_{k,n}$ is the natural regularization for every given type of nonlinearity we deal with as Examples 1, 2 and 3 in [5]. So each $\tilde{g}_{k,n}$, $\tilde{G}_{k,n}$ and, by Example 4 in [5], \tilde{g}_n and \tilde{G}_n satisfy all the assumptions in Section 2 of [5]. So we are able to prove the following

Lemma 4.1. *Let $M \geq 0$ and $\varphi \in \mathcal{H}_A^1$ such that $\|\varphi\|_{\mathcal{H}_A^1} \leq M$. Then, there exists $\tau_{n,A} > 0$ such that there exist a sequence $(\Phi^n)_{n \in \mathbb{N}}$ of functions of $C([0, \tau_{n,A}[, \mathcal{H}_A^1) \cap C^1([0, \tau_{n,A}[, \mathcal{H}'_A)$ solutions of (4.1). Furthermore, for any $t \in [0, \tau_{n,A}[$, we have*

$$(4.2) \quad F_{A,n}(\Phi^n) = F_{A,n}(\varphi)$$

$$(4.3) \quad \|\Phi^n(t)\|_{\mathcal{L}^2} = \|\varphi\|_{\mathcal{L}^2}$$

Proof. We expect that the proof is the same as in Lemma 2.7 in [11] in the case of systems which is obtained by Lemma 3.5 in [5] replacing usual derivatives by magnetic ones. In particular, since \tilde{g}_n is a globally Lipschitz-continuous nonlinearity, we can apply the classical result on $T(t)$ that generates the solution Φ^n above, contained in the Appendix in [5] that we recall in the following.

Remark: Let X be a Banach space and A the generator of a C_0 semigroup $T(t)$. Let $F \in C(X, X)$ be Lipschitz continuous on bounded sets of X . It is well-known that for any $\varphi \in X$, there exists $T_{\max}(\varphi) > 0$ and a unique solution $u \in C([0, T_{\max}], X)$ of

$$(4.4) \quad u(t) = T(t)\varphi + \int_0^t T(t-s)F(u(s)) ds, \quad \text{for } t \in [0, T_{\max}).$$

Moreover, if $T < \infty$ then $\|u(t)\|_{\max} \rightarrow +\infty$ as $t \uparrow T_{\max}$. Furthermore, the mapping $\varphi \rightarrow T_{\max}(\varphi)$ is lower semicontinuous. If $T \in (0, T_{\max}(\varphi))$ and if $\varphi^n \rightarrow \varphi$ in X as $n \rightarrow \infty$, then $u_{\varphi^n} \rightarrow u_\varphi$ in $C([0, T], X)$. If X is reflexive and $\varphi \in D(A)$, then $u_\varphi \in C([0, T], D(A)) \cap C([0, T], X)$ and u_φ solves the problem

$$u_t = Au + F(u) \quad \text{for } t \in [0, T_{\max}) \text{ and } u(0) = \varphi.$$

□

Step 2: Boundedness of the existence time. From the conservation laws (4.2) and (4.3) of the approximate problem (4.1), we show that the existence time $\tau_{n,A}$ can be bounded from below uniformly with respect to $n \in \mathbb{N}$ and A satisfying the Assumptions (A) and (B).

Lemma 4.2. *Let $M > 0$ and let A satisfy assumptions as above with some constants $(C_\alpha)_{\alpha \in \mathbb{N}^N}$. Then, there exists $T(M) > 0$ depending only*

on M and the (C_α) 's such that for all $\varphi \in \mathcal{H}_A^1$ such that $\|\varphi\|_{\mathcal{H}_A^1} \leq M$ we have

$$(4.5) \quad \|\Phi^n\|_{L^\infty([0, T(M)]; \mathcal{H}_A^1)} \leq 2M = 2\|\varphi\|_{\mathcal{H}_A^1}$$

$$(4.6) \quad \|\Phi_t^n\|_{L^\infty([0, T(M)]; \mathcal{H}'_A)} \leq C(M)$$

Proof. We expect that the proof is exactly the same as in Lemma 2.8 in [11] in the case of systems by using Lemma 4.1 (in particular, we use strongly the conservation of energy (4.2)). Recall that, in Lemma 1.2, the constant C is independent on A and by a result like Lemma 3.3 in [5] we get uniformity with respect to A . \square

Step 3: Passage to the limit. The final step is to prove the convergence of the Φ^n to a solution of the initial problem. First, we prove convergence in \mathcal{L}^2 .

Lemma 4.3. *Let $M > 0$ and let A satisfy assumptions as above with some constants $(C_\alpha)_{\alpha \in \mathbb{N}^N}$. Then, there exists $T(M) > 0$ depending only on M and the (C_α) 's such that for all $\varphi \in \mathcal{H}_A^1$ such that $\|\varphi\|_{\mathcal{H}_A^1} \leq M$, $(\Phi^n)_{n \in \mathbb{N}^N}$ is a Cauchy sequence in $C((0, T(M)]; \mathcal{L}^2)$.*

Proof. We expect that the proof is the same as in Lemma 2.9 in [11] in the case of systems, using Theorem 2, Lemma 2.4, Lemma 2.3 in [11] or in general Lemma 3.3 in [5] and Lemma 4.1. \square

We complete the proof of Theorem 2.11. We denote by Φ the limit of Φ^n in $C((0, T(M)]; \mathcal{L}^2)$. From Lemma 4.2, it follows that $\Phi \in L^\infty((-T(M), T(M)); \mathcal{H}_A^1)$ and by Lemma 1.2, Φ^n converges to Φ in $C((0, T(M)]; \mathcal{L}^r)$ for all $r \in [2, 2^*)$. hence, it follows from a result like Lemma 3.3 in [5] that $\tilde{g}_n(\Phi^n)$ converges to $\tilde{g}(\Phi)$ in $C((0, T(M)]; \mathcal{H}'_A)$ and Φ solves (4.1) in $L^\infty((0, T(M)]; \mathcal{H}'_A)$. Uniqueness is an immediate consequence of Proposition 2.12. Conservation laws are obtained from the passage to the limit. Indeed, combining Lemma 3.3 in [5] and Lemma 4.1 we get

$$F_A(\Phi(t)) = F_A(\varphi).$$

This shows that $\Phi \in C((0, T(M)]; \mathcal{H}_A^1)$ and hence $\Phi \in C^1((0, T(M)]; \mathcal{H}'_A)$. Theorem 2.11 follows from considering the maximal solution corresponding to the initial datum and the reverse equation.

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