Extended Hardy-Littlewood Inequalities and Some Applications

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Dedicated to my Mother: To you Omi.

Abstract

We establish conditions under which the extended Hardy-Littlewood inequality

$$\int_{\mathbb{R}^N} H\big(|x|,\, u_1(x), \ldots, u_m(x)\big)\, dx \leq \int_{\mathbb{R}^N} H\big(|x|,\, u_1^*(x), \ldots, u_m^*(x)\big)\, dx, \qquad (1.1)$$

where each u_i is non-negative and u_i^* denotes its Schwarz symmetrization, holds. We also determine appropriate monotonicity assumptions on H such that equality occurs in (1.1) if and only if each u_i is Schwarz symmetric. We end this paper with some applications of our results in the calculus of variations and partial differential equations.

1 Introduction

The most novel part of this paper is the establishement of cases of equality in the extended Hardy-Littlewood inequalities. Our approach also enables us to prove (1.1) in a simple and original way recovering the main result of [5]; let us point out at this early stage that there is no optimal result, including the one of this paper, concerning (1.1).

For dealing with (1.1) (and equality cases in (1.1)) in the calculus of variations and in some other domains, it is fundamental to establish it for integrands H which are not necessarily continous with respect to the distance $|x|$ (see the introduction of [3] for more details). In Theorem 4.2 of [2], we proved (1.1) under minimal regularity assumptions when $m = 2$, our approach set out in [2] still applies to $m > 2$, thus we can easily extend Proposition 4.1 of [2] (and consequently Theorem 4.2). However, this method uses approximation arguments in such a way that it seems impossible to read off cases of equality in (1.1). In case $H \equiv G$ is absolutely continous with respect to the second variable, we developed in [1] a self-contained method thanks to which we determined optimal monotonicity assumptions under which

$$\int_{\mathbb{R}^N} G\big(|x|,\, u(x)\big)\, dx = \int_{\mathbb{R}^N} G\big(|x|,\, u^*(x)\big)\, dx \qquad (1.2)$$

if and only if $u = u^*$ a.e.

Moreover, under minor modifications, this method still applies to establish cases of equality in (1.1) when, except for one u_i, the other functions are already radially decreasing but it cannot apply directly to determine cases of equality in (1.1) when $m > 1$ and each u_i is arbitrary. This case is the most interesting.

In this paper, we present two approaches to solve this problem:

• The first one consists of reducing the study of cases of equality in (1.1) to the one of integrands which are products of "derivatives" of H and step functions depending only on one u_i (see (3.9) and (3.10) in the proof of Theorem 3.1). We then obtain m equations having the form studied in Lemma 3.6 of [1], we can conclude using this result.

• It also turns out that the introduction of an appropriate intermediate step, enabling us to reduce the treatment of cases of equality in (1.1) to m equations such that each equation has the form (1.2) for each u_i (see (3.17) and (3.18) in the proof of Theorem 3.3), is fruitful. This work is completed in Theorem 3.3 in case $m = 2$ and can be easily extended to $m > 2$.

Let us remark that classes of functions given in Theorem 3.1 and Theorem 3.3 are different (see example 3.2). Together, they include a large class of functions arising in mathematical physics and economy. Note also that a "subtle" combination of our first approach and the second one gives us other classes of functions H for which (1.1) holds with equality if and only if each u_i is Schwarz symmetric (see Remark 3.6).

Proofs of results in Section 3 are the most "tricky", if not the most novel part of this paper. Despite the numerous applications of such results, we are not aware of any previous papers dealing with the establishement of equality cases in (1.1) (apart from [1]).

We end this paper with some applications of our results in the calculus of variations and partial differential equations (see Theorem 4.1, Theorem 4.2).

2 Notation, Definitions and Preliminaries

All statements about measurability refer to the Lebesgue measure, mes, on $\mathbb{R}^N$ or $[0, \infty)$. For $r \geq 0$, $B(0, r) = \{x \in \mathbb{R}^N \mid |x| < r\}$. There is a constant $V_N > 0$ such that $B(0, r) = V_N r^N$ for all $r > 0$.

For a measurable subset A of $\mathbb{R}^N$ with finite measure, $A^* = B(0, r)$ where $V_N r^N = \text{mes}A$. The characteristic function of a subset A of $\mathbb{R}^N$ is denoted by 1_A. Let M_N denote the set of all extended real-valued functions which are measurable on $\mathbb{R}^N$. For $u \in M_N$ and $t \in \mathbb{R}$, let $d_u(t) = \text{mes}\{x \in \mathbb{R}^N \mid u(x) > t\}$, be its distribution function and set

$$F_N = \{u \in M_N \mid 0 \leq u < \infty \ a.e. \text{ on } \mathbb{R}^N \text{ and } d_u(t) < \infty \text{ for all } t > 0\},$$

the set of Schwarz symmetrizable functions. For $u \in F_N$, its Schwarz symmetrization, denoted by u^*, is the unique function such that $d_{u^*}(t) = d_u(t)$ for every $t > 0$ and $u^*(x) = h(|x|)$ where $h : (0, \infty) \longrightarrow [0, \infty)$ is a non-increasing right-continous function. We say that an element $u \in F_N$ is Schwarz symmetric if $u = u^*$ a.e.

Simple functions can be symmetrized in a very simple way. Let

$$E_N = \{u \in F_N \mid u \text{ is a simple function}\}.$$

That is to say, E_N is the set of all functions which can be written as

$$u = \sum_{i=0}^{k} a_i 1_{A_i} \quad \text{for some } k \in \mathbb{N}, \tag{2.1}$$

where $a_i \in (0, \infty)$ with $a_i > a_{i+1}$, A_i are measurable subsets of $\mathbb{R}^N$ with $\text{mes}A_i < \infty$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

In [2], we proved that any element $u \in E_N$ can be rewritten as

$$u = \sum_{i=0}^{k} h_i u_i, \tag{2.2}$$

where

$$h_i = a_i - a_{i+1}, \qquad \text{for} \quad 0 \leq i \leq k-1,$$
$$h_k = a_k$$

and

$$u_i = \sum_{j=0}^{i} 1_{A_j}, \qquad \text{for} \quad 0 \leq i \leq k.$$

In this case

$$u^* = \sum_{i=0}^{k} h_i u_i^*. \tag{2.3}$$

Let us also recall that if $f \in F_N$ then there exists

$$\{f_n\} \subset E_N \text{ such that } f_n \nearrow f \text{ and } f_n^* \nearrow f^*. \tag{2.4}$$

Definition 2.1
A function $H : (0, \infty) \times \mathbb{R}_+^m \longrightarrow \mathbb{R}$ is called a m-Carathéodory function when

1) $H(\cdot, s_1, \ldots, s_m) : (0, \infty) \longrightarrow \mathbb{R}$ is measurable on $(0, \infty)\backslash\Gamma$, where Γ is a subset of $(0, \infty)$ having one dimensional measure zero, for all $s_1, \ldots, s_m \geq 0$,

2) For all $1 \leq n \leq m$, every $(m-1)$ tuple $s_i \geq 0$ and $r \in (0, \infty)\backslash\Gamma$

$$\begin{array}{rcl} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ s_n & \longmapsto & H(\ldots, s_n, \ldots) \end{array}$$

is continous on $\mathbb{R}_+$.

This definition establishes the standard context for handling the measurability of the composite functions involved in (1.1). An important property of a m-Carathéodory function is that the compositon $x \mapsto H\big(|x|, u_1(x), \ldots, u_m(x)\big)$ is measurable on $\mathbb{R}^N$ for every $u_1, \ldots, u_m \in F_N$.

Definition 2.2
• A function $F : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ has the property (CZR) when:

$$F(b, d) - F(b, c) - F(a, d) + F(a, c) \geq 0$$

for all $b \geq a \geq 0$ and $d \geq c \geq 0$.

• A function $G : (0, \infty) \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ has the property (CZR-2) when:

$$G(b, d) - G(b, c) - G(a, d) + G(a, c) \leq 0$$

for all $b \geq a > 0$ and $d \geq c \geq 0$.

• A function $H : (0, \infty) \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ has the property (CZR-3) when the function $H(\cdot, \cdot, y) - H(\cdot, \cdot, z)$ has the property (CZR-2) for all $y \geq z \geq 0$.

From now on, in an integral where no domain of integration is indicated, it is to be understood that integration extends over all $\mathbb{R}^N$.

Lemma 2.1 Let f, g and h be three elements of F_N, then

$$\int f(x)\, g(x)\, h(x)\, dx \leq \int f^*(x)\, g^*(x)\, h^*(x)\, dx.$$

Proof:
We first prove the result for functions in E_N. Let $f = a\,1_A$, $g = b\,1_B$ and $h = c\,1_C$.

$$\begin{aligned}\int 1_A\,1_B\,1_C &= \int 1_{A\cap B\cap C} \le \min\{\text{mes}A,\ \text{mes}B,\ \text{mes}C\} = \\ &= \int 1_{A^*\cap B^*\cap C^*} = \int 1_{A^*}\,1_{B^*}\,1_{C^*} = \\ &= \int (1_A)^*\,(1_B)^*\,(1_C)^*. \qquad (2.5)\end{aligned}$$

Now, let f, g and h be three functions in E_N rewritten with respect to (2.2).

$$f(x) = \sum_{i=0}^{n} a_i\,1_{A_i}(x), \quad g(x) = \sum_{j=0}^{m} b_j\,1_{B_j}(x) \quad \text{and} \quad h(x) = \sum_{k=0}^{\ell} c_k\,1_{C_k}(x).$$

Then, by (2.5)

$$\begin{aligned}\int f\,g\,h &= \sum_{i,\,j,\,k} a_i b_j c_k \int 1_{A_i}\,1_{B_j}\,1_{C_k} \le \sum_{i,\,j,\,k} a_i b_j c_k \int (1_{A_i})^*\,(1_{B_j})^*\,(1_{C_k})^* = \\ &= \int f^*\,g^*\,h^* \quad \text{by} \quad (2.3).\end{aligned}$$

Using (2.4), we can extend the result to elements of F_N thanks to the monotone convergence theorem. $\lozenge$

Lemma 2.2 (Generalized Hardy-Littlewood inequality).
Let $\{f_i\}_{0\le i\le n}$, where $n \in \mathbb{N}$, be n elements of F_N, then

$$\int \prod_{i=0}^{n} f_i(x)\,dx \le \int \prod_{i=0}^{n} f_i^*(x)\,dx.$$

3 Main Results

Theorem 3.1
Let $u,\, v \in F_N$ and $H : (0,\,\infty) \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ be a 2-Carathéodory function verifying:

(H1) $H(r,\,s,\,t) = \int_0^t \int_0^s \partial_2\partial_3 H(r,\,a,\,b)\,da\,db$ for almost every $r > 0$ and every $s,\,t \ge 0$,

(H2) $\partial_2\partial_3 H(r,\,s,\,t) \ge 0$ for every $r > 0$, $s,\,t \ge 0$,

(H3) $\partial_2\partial_3 H(\cdot,\,s,\,t)$ is decreasing on $(0,\infty)$ for every $s,\,t \ge 0$ then

$$\int H\big(|x|,\,u(x),\,v(x)\big)\,dx \le \int H\big(|x|,\,u^*(x),\,v^*(x)\big)\,dx.$$

If in addition, we assume:

(H4) $\int\limits_0^\infty \int\limits_0^\infty \int \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u^*(x)\geq s\}}\, 1_{\{v^*(x)\geq t\}}\, dx\, dt\, ds < \infty,$

(H5) (a) $\partial_2\partial_3 H(r,\, \cdot,\, t)$ is continous on zero for almost every $r > 0$ and $t \geq 0$,
(b) $\partial_2\partial_3 H(r,\, s,\, \cdot)$ is continous on zero for almost every $r > 0$ and $s \geq 0$,

(H6) (a) For any $s \leq 1$, there exist f_t^1 and $f_t^2 \in L^1(\mathbb{R}^N)$ such that

$$\partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}} \leq f_t^1(x)$$

and

$$\partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u^*(x)\geq s\}}\, 1_{\{v^*(x)\geq t\}} \leq f_t^2(x)$$

for almost every $x \in \mathbb{R}^N$ and $t \geq 0$,

(b) For any $t \leq 1$, there exist g_s^1 and $g_s^2 \in L^1(\mathbb{R}^N)$ such that

$$\partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}} \leq g_s^1(x)$$

and

$$\partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u^*(x)\geq s\}}\, 1_{\{v^*(x)\geq t\}} \leq g_s^2(x)$$

for almost every $x \in \mathbb{R}^N$ and $s \geq 0$,

(H7) (a) $\partial_2\partial_3 H(\cdot,\, 0,\, t)$ is strictly decreasing for almost every $t \geq 0$,
(b) $\partial_2\partial_3 H(\cdot,\, s,\, 0)$ is strictly decreasing for almost every $s \geq 0$.

Then

$$\int H\big(|x|,\, u(x),\, v(x)\big)\, dx = \int H\big(|x|,\, u^*(x),\, v^*(x)\big)\, dx$$

if and only if $(u,\, v) = (u^*,\, v^*)$ almost everywhere.

Proof:
Using (H1), we have

$$\int H\big(|x|,\, u(x),\, v(x)\big)\, dx = \int \int\limits_0^\infty \int\limits_0^\infty \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{s\leq u(x)\}}\, 1_{\{t\leq v(x)\}}\, ds\, dt\, dx. \quad (3.1)$$

(H2) enables us to use Tonelli's theorem, we then can intervert the ds, dt and dx integrations in (3.1), thus

$$\int H\big(|x|,\, u(x),\, v(x)\big)\, dx = \int\limits_0^\infty \int\limits_0^\infty \int \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}}\, dx\, ds\, dt. \quad (3.2)$$

(H3) ensures that $\partial_2\partial_3 H\big(|x|,\, s,\, t\big)$ is radially decreasing for any $s,\, t \geq 0$. For $s,\, t \geq 0$, set $\ell_{s,\,t} = \lim\limits_{|x|\to\infty} \partial_2\partial_3 H(|x|,\, s,\, t)$, then

$$\int \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}}\, dx = \int \{\partial_2\partial_3 H(|x|,\, s,\, t) - \ell_{s,\,t}\}\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}}\, dx + \\ + \int \ell_{s,\,t}\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}}\, dx. \qquad (3.3)$$

(H3) implies that, for all $s,\, t \geq 0$,

$$\big(\partial_2\partial_3 H(|x|,\, s,\, t) - \ell_{s,\,t}\big)^* = \partial_2\partial_3 H(|x|,\, s,\, t) - \ell_{s,\,t} \quad \text{for almost every } x \in \mathbb{R}^N.$$

Thus, by Lemma 2.1,

$$\int \{\partial_2\partial_3 H(|x|,\, s,\, t) - \ell_{s,\,t}\}\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}}\, dx \;\leq \\ \leq \int \{\partial_2\partial_3 H(|x|,\, s,\, t) - \ell_{s,\,t}\}\, 1_{\{u^*(x)\geq s\}}\, 1_{\{v^*(x)\geq t\}}\, dx \qquad (3.4)$$

and since $\ell_{s,\,t} \geq 0$, we certainly have

$$\int \ell_{s,\,t}\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}}\, dx \leq \int \ell_{s,\,t}\, 1_{\{u^*(x)\geq s\}}\, 1_{\{v^*(x)\geq t\}}\, dx. \quad (3.5)$$

Combining (3.3) to (3.5), we obtain for all $s,\, t \geq 0$:

$$\int \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}}\, dx \leq \\ \leq \int \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u^*(x)\geq s\}}\, 1_{\{v^*(x)\geq t\}}\, dx. \qquad (3.6)$$

Clearly (3.2) holds when u is replaced by u^* and v by v^*. Thus (3.2) together with (3.6) imply that:

$$\int H\big(|x|,\, u(x),\, v(x)\big)\, dx \leq \int H\big(|x|,\, u^*(x),\, v^*(x)\big)\, dx$$

proving the first part of our result.

Now suppose that

$$\int H\big(|x|,\, u(x),\, v(x)\big)\, dx = \int H\big(|x|,\, u^*(x),\, v^*(x)\big)\, dx. \qquad (3.7)$$

that is to say:

$$\int_0^\infty \int_0^\infty \int \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}}\, dx\, ds\, dt =$$

$$= \int_0^\infty \int_0^\infty \int \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u^*(x)\geq s\}}\, 1_{\{v^*(x)\geq t\}}\, dx\, ds\, dt.$$

Then (H4) and (3.6) imply that equality occurs in the previous line if and only if: For almost every $(s,\, t) \in \mathbb{R}_+^2$,

$$\int \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u(x)\geq s\}}\, 1_{\{v(x)\geq t\}}\, dx =$$

$$= \int \partial_2\partial_3 H(|x|,\, s,\, t)\, 1_{\{u^*(x)\geq s\}}\, 1_{\{v^*(x)\geq t\}}\, dx \ (< \infty). \qquad (3.8)$$

(3.8) may not hold for $D = \{(s,\, 0) \mid s \geq 0\}$, however we can certainly construct a sequence $\{t_n\} \subset \mathbb{R}_+$ such that $t_n \searrow 0$ and (3.8) occurs for $D_n = \{(s,\, t_n) \mid s \geq 0\}$ for all $n \in \mathbb{N}$.

Thanks to (H5) (b) and (H6) (b), we can use the dominated convergence theorem obtaining: for almost every $s \geq 0$,

$$\int \partial_2\partial_3 H(|x|,\, s,\, 0)\, 1_{\{u(x)\geq s\}}\, dx = \int \partial_2\partial_3 H(|x|,\, s,\, 0)\, 1_{\{u^*(x)\geq s\}}\, dx < \infty. \qquad (3.9)$$

Similarly: for almost every $t \geq 0$,

$$\int \partial_2\partial_3 H(|x|,\, 0,\, t)\, 1_{\{v(x)\geq t\}}\, dx = \int \partial_2\partial_3 H(|x|,\, 0,\, t)\, 1_{\{v^*(x)\geq t\}}\, dx < \infty. \qquad (3.10)$$

Finally (H7) enables us to use Lemma 3.6 of [1], concluding that (3.9) occurs if and only if $u = u^*$ a.e. and (3.10) holds if and only if $v = v^*$ a.e. $\Diamond$

Example 3.1 $H(r,\, s,\, t) = \frac{1}{1+r^2} st$, $u,\, v \in F_1$ such that

$$\int \frac{1}{1+|x|^2} u^*(x) v^*(x)\, dx < \infty$$

satisfy all the hypotheses of Theorem 3.1.

Remarks 3.1
i) In (H1), it is not required that $\partial_2\partial_3 H$ exists everywhere. Class of functions satisfying (H1) are discussed in details in [4].

ii) a) If (H1) only holds for $s \in \mathrm{Im}u \cup \mathrm{Im}u^*$ and $t \in \mathrm{Im}v \cup \mathrm{Im}v^*$, then Theorem 3.1 remains true.
b) If (H2) and (H3) hold for almost every $s \in \mathrm{Im}u \cup \mathrm{Im}u^*$ and almost every $t \in \mathrm{Im}v \cup \mathrm{Im}v^*$, then the conclusion of Theorem 3.1 remains valid.

iii) In (H5), we can replace the continuity of the functions by the existence of

$$\lim_{s\to 0^+} \partial_2\partial_3 H(r,\, s,\, t) \quad \text{and} \quad \lim_{t\to 0^+} \partial_2\partial_3 H(r,\, s,\, t),$$

we then modify (H7).

iv) If $H \in C^3\big((0,\,\infty) \times \mathbb{R}_+^2\big)$:

$$\begin{array}{lll}
(\mathrm{H3}) & \Longleftrightarrow\ \partial_1\partial_2\partial_3 H(r,\,s,\,t) \le 0, & \text{for all } r>0,\ s,\,t \ge 0,\\
(\mathrm{H7})(\mathrm{a}) & \Longleftrightarrow\ \partial_1\partial_2\partial_3 H(r,\,0,\,t) < 0, & \text{for all } r>0, \text{ almost every } t \ge 0,\\
(\mathrm{H7})(\mathrm{b}) & \Longleftrightarrow\ \partial_1\partial_2\partial_3 H(r,\,s,\,0) < 0, & \text{for all } r>0, \text{ almost every } s \ge 0.
\end{array}$$

Remark 3.2
If (H1), (H2), (H3), (H4) and (a) of (H5), (H6) and (H7) hold then

$$\int H\big(|x|,\,u(x),\,v(x)\big)\,dx = \int H\big(|x|,\,u^*(x),\,v^*(x)\big)\,dx$$

implies that $v = v^*$ a.e.

Remark 3.3
A priori Theorem 3.1 does not apply to functions as

$$H(r,\,s,\,t) = \frac{1}{1+r^2}s^n t^m, \qquad n,\,m > 1.$$

However, by a suitable change of integrands, we can determine equality cases for such functions using this result. More precisely, suppose that

$*)$ $\quad \int \frac{1}{1+|x|^2} f^n(x) g^m(x)\,dx = \int \frac{1}{1+|x|^2}(f^*(x))^n (g^*(x))^m\,dx < \infty$, where f and $g \in F_1$.

Set $F = f^n$, then $F \in F_1$ and $F^* = (f^n)^* = (f^*)^n$ a.e. and $G = g^m \in F_1$ with $G^* = (g^m)^* = (g^*)^m$ a.e.

$*)$ $\quad$ is equivalent to $\int \frac{1}{1+|x|^2} F(x)G(x)\,dx = \int \frac{1}{1+|x|^2} F^*(x)G^*(x)\,dx < \infty$.

Now setting $\tilde{H}(r,\,s,\,t) = \frac{1}{1+r^2}st$ and applying Example 3.1, we conclude that $F = F^*$ a.e. and $G = G^*$ a.e., thus $(f,\,g) = (f^*, g^*)$ a.e.

Theorem 3.1 can be easily generalized to m functions.

Theorem 3.2
Let $u_1, \ldots, u_m \in F_N$, $H : (0,\,\infty) \times \mathbb{R}_+^m \longrightarrow \mathbb{R}$ be a m-Carathéodory function such that

i) $\quad H(r,\,b_1, \ldots, b_m) = \int_0^{b_m} \ldots \int_0^{b_1} \partial_2 \ldots \partial_{m+1} H(r,\,a_1,\,\ldots, a_m)\,da_1 \ldots da_m$
for all $r > 0$, $b_1, \ldots, b_m \ge 0$,

ii) $\quad \partial_2 \ldots \partial_{m+1} H(r,\,a_1,\,\ldots, a_m) \ge 0$ for all $r > 0$, $a_1,\,\ldots, a_m \ge 0$,

iii) $r \mapsto \partial_2 \ldots \partial_{m+1} H(r,\, a_1,\, \ldots, a_m)$ is non-increasing for all $a_1,\, \ldots, a_m \geq 0$, then

$$\int H\big(|x|,\, u_1(x), \ldots, u_m(x)\big)\, dx \leq \int H\big(|x|,\, u_1^*(x), \ldots, u_m^*(x)\big)\, dx$$

If we suppose in addition that

iv) $\int\limits_{\mathbb{R}_+^m} \int \partial_2 \ldots \partial_{m+1} H(|x|,\, a_1,\, \ldots, a_m)\, 1_{\{u_1^*(x) \geq a_1\}} \ldots 1_{\{u_m^*(x) \geq a_m\}}\, dx\, da_1 \ldots da_m < \infty$,

v) For any fixed n, $1 \leq n \leq m$, $\partial_2 \ldots \partial_{m+1} H(r,\, a_n,\, \cdot)$ is continous on $0_{\mathbb{R}^{m-1}}$ for all $r > 0$ and $a_n \geq 0$,

vi) There exists an integrable function $K : \mathbb{R}^N \longrightarrow \mathbb{R}_+$ such that

$$\partial_2 \ldots \partial_{m+1} H(|x|,\, a_1,\, \ldots, a_m) \leq K(x) \quad \text{for all} \quad x \in \mathbb{R}^N,\ a_1,\, \ldots, a_m \geq 0,$$

vii) For all fixed n, $1 \leq n \leq m$, the function $r \mapsto \partial_2 \ldots \partial_{m+1} H(r,\, a_n,\, 0_{\mathbb{R}^{m-1}})$ is strictly decreasing on $(0,\, \infty)$ for all $a_n \geq 0$. Then

$$\int H\big(|x|,\, u_1(x), \ldots, u_m(x)\big)\, dx = \int H\big(|x|,\, u_1^*(x), \ldots, u_m^*(x)\big)\, dx$$

if only if each u_i is Schwarz symmetric.

Remark 3.4
vi) can be expressed in a same way as (H6) which is important for applications of such result in the calculus of variations.

All the hypotheses can be weakened in a same manner as in Remark 3.1.

Theorem 3.3
Let $u,\, v \in F_N$, $H : (0,\, \infty) \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ be a 2-Carathéodory function satisfying:

1) $H(r,\, \cdot,\, \cdot)$ has the property (CZR) for almost every $r > 0$,

2) H has the property (CZR-3),

3) (a) For almost every $r > 0$ and all $t \geq 0$:

$$H(r,\, 0,\, t) - H(r,\, 0,\, 0) = \int_0^t \partial_3 H(r,\, 0,\, b)\, db,$$

(b) For almost every $r > 0$ and all $s \geq 0$:

$$H(r,\, s,\, 0) - H(r,\, 0,\, 0) = \int_0^s \partial_2 H(r,\, a,\, 0)\, da,$$

4) (a) $\int\limits_0^\infty \int \left|\partial_3 H\big(|x|,\, 0,\, t\big)\right| 1_{\{v(x)\geq t\}}\, dx\, dt$ and
$\int\limits_0^\infty \int \left|\partial_3 H\big(|x|,\, 0,\, t\big)\right| 1_{\{v^*(x)\geq t\}}\, dx\, dt$ are finite,

(b) $\int\limits_0^\infty \int \left|\partial_2 H\big(|x|,\, s,\, 0\big)\right| 1_{\{u(x)\geq s\}}\, dx\, ds$ and
$\int\limits_0^\infty \int \left|\partial_2 H\big(|x|,\, s,\, 0\big)\right| 1_{\{u^*(x)\geq s\}}\, dx\, ds$ are finite,

5) (a) $r \mapsto \partial_3 H(r,\, 0,\, t)$ is strictly decreasing on $(0,\, \infty)$ for almost every $t \geq 0$,
(b) $r \mapsto \partial_2 H(r,\, s,\, 0)$ is strictly decreasing on $(0,\, \infty)$ for almost every $s \geq 0$,

6) $\int H\big(|x|,\, 0,\, 0\big)\, dx < \infty$.

Then if $\int H\big(|x|,\, u^*(x),\, v^*(x)\big)\, dx < \infty$

$$\int H\big(|x|,\, u(x),\, v(x)\big)\, dx = \int H\big(|x|,\, u^*(x),\, v^*(x)\big)\, dx$$

if and only if $(u,\ v) = (u^*,\, v^*)$ a.e.

Proof:
For $r > 0,\ s,\, t \geq 0$, set

$$\tilde{H}(r,\, s,\, t) = H(r,\, s,\, t) - H(r,\, s,\, 0) - H(r,\, 0,\, t) + H(r,\, 0,\, 0). \qquad (3.11)$$

1) and 2) imply that $\tilde{H}$ satisfies all the hypotheses of Proposition 4.1 of [2], then for any $u,\, v \in F_N$

$$\int \tilde{H}\big(|x|,\, u(x),\, v(x)\big)\, dx \leq \int \tilde{H}\big(|x|,\, u^*(x),\, v^*(x)\big)\, dx. \qquad (3.12)$$

We can easily check using 3) $\rightarrow$ 6) that

$$-\infty < \int H\big(|x|,\, u(x),\, 0\big)\, dx \leq \int H\big(|x|,\, u^*(x),\, 0)\big)\, dx < \infty \qquad (3.13)$$

$$-\infty < \int H\big(|x|,\, 0,\, v(x)\big)\, dx \leq \int H\big(|x|,\, 0,\, v^*(x)\big)\, dx < \infty. \qquad (3.14)$$

Now suppose that

$$\int H\big(|x|,\, u(x),\, v(x)\big)\, dx = \int H\big(|x|,\, u^*(x),\, v^*(x)\big)\, dx. \qquad (3.15)$$

Remarking that under our hypotheses

$$\int H_-\big(|x|,\, u(x),\, v(x)\big)\, dx \quad \text{and} \quad \int H_-\big(|x|,\, u^*(x),\, v^*(x)\big)\, dx$$

are finite (see proof of Theorem 4.2 of [2]), we certainly have by Lemma 3.1 of [2] that

$$\int H\big(|x|,\,u(x),\,v(x)\big)\,dx = \int \tilde{H}\big(|x|,\,u(x),\,v(x)\big)\,dx + \int H\big(|x|,\,0,\,v(x)\big)\,dx + \\ + \int H\big(|x|,\,u(x),\,0\big)\,dx - \int H\big(|x|,\,0,\,0\big)\,dx$$

and

$$\int H\big(|x|,\,u^*(x),\,v^*(x)\big)\,dx = \int \tilde{H}\big(|x|,\,u^*(x),\,v^*(x)\big)\,dx + \int H\big(|x|,\,0,\,v^*(x)\big)\,dx + \\ + \int H\big(|x|,\,u^*(x),\,0\big)\,dx - \int H\big(|x|,\,0,\,0\big)\,dx.$$

Combining (3.12) to (3.15), it follows, since by our hypotheses integrals in (3.12) are finite,

$$\int \tilde{H}\big(|x|,\,u(x),\,v(x)\big)\,dx = \int \tilde{H}\big(|x|,\,u^*(x),\,v^*(x)\big)\,dx \qquad (3.16)$$

$$\int H\big(|x|,\,u(x),\,0\big)\,dx = \int H\big(|x|,\,u^*(x),\,0\big)\,dx \qquad (3.17)$$

$$\int H\big(|x|,\,0,\,v(x)\big)\,dx = \int H\big(|x|,\,0,\,v^*(x)\big)\,dx. \qquad (3.18)$$

3) $\to$ 6) imply, using Theorem 6.1 of [1], that (3.17) and (3.18) hold if and only if $(u,\,v) = (u^*, v^*)$ a.e. $\diamondsuit$.

Remark 3.5
Note that 2) and 5) (a) imply that $r \mapsto \partial_3 H(r,\,s,\,t)$ is strictly decreasing for any $s \geq 0$ and almost every $t \geq 0$.

Example 3.2 $H(r,\,s,\,t) = \frac{1}{1+r^2}s + \frac{1}{1+r^2}t$, u and $v \in F_1$ such that

$$\int \frac{1}{1+|x|^2} u^*(x)\,dx \quad \text{and} \quad \int \frac{1}{1+|x|^2} v^*(x)\,dx < \infty$$

satisfy all the hypotheses of Theorem 3.3.

Clearly $\frac{1}{1+r^2}s + \frac{1}{1+r^2}t$ does not satisfy conditions of Theorem 3.1. Conversly the function given in Example 3.1 does not satisfy assumptions of Theorem 3.3.

Remark 3.6
In Theorem 3.1, it is assumed that $H(r,\,\cdot,\,0) \equiv H(r,\,0,\,\cdot) = 0$, however the approach developed to prove this result remains fruitful eventhough $H(r,\,\cdot,\,0)$ or $H(r,\,0,\,\cdot)$ are not as required in Theorem 3.1 if

$$H(r,\,s,\,t) = \int_0^t \int_0^s \partial_2\partial_3 H(r,\,a,\,b)\,da\,db + H(r,\,s,\,0) + H(r,\,0,\,t) - H(r,\,0,\,0),$$

for almost every $r > 0$, every $s \in \mathrm{Im}u \cup \mathrm{Im}u^*$ and every $t \in \mathrm{Im}v \cup \mathrm{Im}v^*$.

Closely following the proof of Theorem 3.3, we can easily determine hypotheses on H such that if (1.1) holds with equality then $v = v^*$ a.e. The establishement of cases of equality in (1.1) is then reduced to: $\int H\big(|x|,\, u(x),\, v^*(x)\big)\, dx = \int H\big(|x|,\, u^*(x),\, v^*(x)\big)\, dx$

Considering
$$\begin{array}{rcl} G\colon (0,\,\infty) \times \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ (r,\, s) & \longmapsto & H(r,\, s,\, h(r)), \end{array}$$

where $h : (0,\, \infty) \to \mathbb{R}$ is defined by $h(|x|) = v^*(x)$ for every $x \in \mathbb{R}^N$ and applying Theorem 6.1 of [1] to $(G,\, u)$, we can deduce that $u = u^*$ a.e.

Finally, let us point out that a combination of Theorem 3.1 and Theorem 6.1 of [1] gives also appropriate conditions on H such that equality holds in (1.1) if and only if $(u,\, v) = (u^*,\, v^*)$ a.e.

4 Some Applications

For $u = (u_1, u_2) \in \big(H^1(\mathbb{R}^N)\big)^2$, we set

$$\begin{aligned} \tilde{J}(u) = \tilde{J}(u_1,\, u_2) = & \frac{1}{2}\int |\nabla u_1|^2 + \frac{1}{2}\int |\nabla u_2|^2 - \frac{1}{2}\int p\big(|x|\big)|u(x)|^2\, dx \\ & - \int H\big(|x|,\, u_1(x),\, u_2(x)\big)\, dx. \end{aligned}$$

$(\tilde{P}_c)$: $\tilde{I}_c = \inf\limits_{u \in \tilde{S}_c} \tilde{J}(u)$, where $\tilde{S}_c = \{u \in \big(H^1(\mathbb{R}^N)\big)^2 \mid \|u\|_2^2 = c^2\}$ for a prescribed $c > 0$.

$\|\cdot\|_2$ is the norm in $\big(L^2(\mathbb{R}^N)\big)^2$. Closely following the proof of Example 4 of [3] whose principle ingredient is (1.1) with $m = 1$, we obtain:

Theorem 4.1
Suppose that $p : (0,\, \infty) \longrightarrow \mathbb{R}_+$ satisfies:

(P1) p is non-increasing and $\lim\limits_{r \to \infty} p(r) = 0$,

(P2)
- If $N = 1,\, 2$, there exists $a > 0$ such that $p(a) > 0$,
- If $N > 2$, there exists $R > 0$ such that

$$p(R) > \frac{j^2_{N/2-1,1}}{R^2}$$

where $j_{N/2-1,1}$ is the first zero of the Bessel function $J_{N/2-1}$.

Assume that $H : (0,\, \infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a 2-Carathéodory function such that

(V1) $H(r,\, s_1,\, s_2) \le H(r,\, |s_1|,\, |s_2|)$ for almost every $r > 0$ and all $s_1,\, s_2 \in \mathbb{R}$.

For $r > 0$, s_1 and $s_2 \ge 0$, set $s = (s_1,\, s_2)$ and suppose:

(V2) $0 \le \tilde{H}(r,\, s_1,\, s_2) \le K\big(|s|^2 + \sum_{i=1}^{2} s_i^{\ell+2}\big)$, where $K > 0$, $0 < \ell < \frac{4}{N}$, and $\tilde{H}$ is the restriction of H to $(0,\, \infty) \times \mathbb{R}_+^2$.

(V3) (a) $(r,\, s_1) \mapsto \tilde{H}(r,\, s_1,\, s_2)$ and $(r,\, s_2) \mapsto \tilde{H}(r,\, s_1,\, s_2)$ have the property (CZR-2) for all $s_2 \ge 0$ (respectively $s_1 \ge 0$),

(b) $\tilde{H}$ has the property (CZR) for almost every $r > 0$,

(c) $\tilde{H}$ has the property (CZR-3),

(d) $s_1 \mapsto \lim\limits_{r\to\infty} \tilde{H}(r,\, s_1,\, 0)$ and $s_2 \mapsto \lim\limits_{r\to\infty} \tilde{H}(r,\, 0,\, s_2)$ exist and are continous on $[0,\, \infty)$.

(V4) $\forall \varepsilon > 0$, $\exists r_0 > 0$ and $s_0 > 0$ such that

$$\tilde{H}(r,\, s_1,\, s_2) \le \varepsilon |s|^2 \quad \text{for all } r \ge r_0 \text{ and } |s| \le s_0.$$

(V5) $\tilde{H}(r,\, \theta s_1,\, \theta s_2) \ge \theta^2 \tilde{H}(r,\, s_1,\, s_2)$ for almost every $r > 0$, every $\theta \ge 1$ and $s_1,\, s_2 \ge 0$.

Then for any $c > 0$, there exists $u^c = (u_1^c,\, u_2^c)$ such that each component is Schwarz symmetric, $\|u^c\|_2^2 = c^2$ and $\tilde{J}(u^c) = \tilde{I}_c$.

Theorem 4.2
Suppose that $(p,\, H)$ satisfy conditions of Theorem 4.1, if in addition p is strictly decreasing on $(0,\, \infty)$ or H verifies 3) $\to$ 5) of Theorem 3.3, then all minimizers of $(\tilde{P}_c)$ are Schwarz symmetric. (This means that all the minimizers of $(\tilde{P}_c)$ can be written $(u_1^c,\, u_2^c)$ where u_1^c and u_2^c are Schwarz symmetric.)

Remark 4.1
Under appropriate growth conditions on H (see Theorem 6.1 (CV) of [1]), 4) of Theorem 3.3 is satisfied for every $u,\, v \in H^1(\mathbb{R}^N)$.

Remark 4.2
Under some regularity assumptions on H, a solution of $(\tilde{P}_c)$ satisfies the 2×2 elliptic eigenvalue system

$$S_{2\times 2}: \qquad \begin{cases} \Delta u(x) + \partial_2 H\big(|x|,\, u(x),\, v(x)\big) + p\big(|x|\big)u(x) + \lambda u(x) = 0 \\ \Delta v(x) + \partial_3 H\big(|x|,\, u(x),\, v(x)\big) + p\big(|x|\big)v(x) + \lambda v(x) = 0 \end{cases}$$

where $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier.

Theorem 4.1 assures us that $S_{2\times 2}$ admits a Schwarz symmetric ground state (that is to say that each component of the ground state is Schwarz symmetric).

Under conditions of Therem 4.2, all ground states of $S_{2\times 2}$ are Schwarz symmetric.

5 References

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