Sharp Embedding of Sobolev Spaces Involving General Kernels and its Application

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Abstract. The purpose of this paper is to extend the embedding theorem of Sobolev spaces involving general kernels and we provide a sharp critical exponent in the embedding. As an application, solutions for equations driven by a general integro-differential operator, with homogeneous Dirichlet boundary conditions, is established by using the Mountain Pass Theorem.

1. Introduction

In the study of weak solutions for semilinear elliptic equations, the embedding from corresponding Sobolev space to L^q space plays a fundamental role, especially the compact embedding. In a recent work, Di Nezza, Palatucci and Valdinoci in [11] made a clear description for the fractional Sobolev space $W^{s,p}(\Omega)$ and gave an elegant proof for the embedding theorem from $W^{s,p}(\Omega)$ to $L^q(\Omega)$, which is continuous when $q \in [1, \frac{Np}{N-sp}]$ and compact when $q \in [1, \frac{Np}{N-sp})$, where $s \in (0, 1)$, $sp < N$ and Ω is a bounded extension domain in $\mathbb{R}^N$ with $N \geq 2$.

Motivated by the above work, our purpose of this paper is to build a sharp embedding theorem of Sobolev space involving general kernel K and by using this embedding theorem to search for weak solutions to problem

$$\begin{aligned} \mathcal{L}_K u + f(x,u) &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \Omega^c, \end{aligned} \tag{1.1}$$

where Ω is an open bounded C^2 domain in $\mathbb{R}^N$ with $N \geq 2$ and the nonlocal operator $\mathcal{L}_K$ is defined by

$$\mathcal{L}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^N} [u(x+y) + u(x-y) - 2u(x)] K(y) dy$$

with the kernel $K : \mathbb{R}^N \setminus \{0\} \to [0, +\infty)$ satisfying

$$\int_{\mathbb{R}^N} \min\{|x|^2, 1\} K(x)dx < +\infty \tag{1.2}$$

and

$$K(x) = K(-x), \qquad x \in \mathbb{R}^N \setminus \{0\}. \tag{1.3}$$

Moreover, we assume that K is decreasing monotone in the sense

$$K(x) \ge c_0 K(y) \qquad \text{if} \quad |x| \le |y|, \tag{1.4}$$

for some $c_0 \in (0, 1]$. A typical example for K is given by $K(x) = |x|^{-(N+2s)}$ with $s \in (0, 1)$ and then $\mathcal{L}_K$ is the fractional Laplacian operator $-(-\Delta)^s$.

During the last years, non-linear equations involving general integro-differential operators, especially, fractional Laplacian, have been studied by many authors. Caffarelli and Silvestre [4] studied the fractional Laplacian through extension theory. Caffarelli and Silvestre [5, 6], Ros-Oton and Serra [19] investigated regularity results for fractional elliptic equations. Sire and Valdinoci in [23], Felmer and Wang in [13], Hajaiej [16, 17] and Felmer, Quaas and Tan [12] obtained symmetry property of solutions for semilinear equation involving the fractional Laplacian. More interests on fractional elliptic equations see [7, 8, 9, 10, 15] and the references therein.

Recently, Servadei and Valdinoci in [22] obtained a solution of (1.1) via Mountain Pass Theorem under the hypothesis that there exist $\lambda > 0$ and $s \in (0, 1)$ such that

$$K(x) \ge \lambda |x|^{-(N+2s)}, \quad x \in \mathbb{R}^N \setminus \{0\}$$

and nonlinear term f is subcritical, that is,

$$|f(x, t)| \le a_1 + a_2|t|^{q-1} \quad \text{a.e. } x \in \Omega, \ t \in \mathbb{R}$$

with $q \in (2, \frac{2N}{N-2s})$ and constants $a_1, a_2 > 0$. We say that $\frac{2N}{N-2s}$ is the critical exponent, denoted by $2^*(s)$.

In this paper, we are also interested in studying problem (1.1) with more general kernels and our purpose is to find new criterion for critical exponent, where we could deal with the following case

$$\liminf_{|x|\to 0^+} K(x)|x|^N \in (0, \infty). \tag{1.5}$$

To this end, we define

$$s_0 = \sup\{s \ge 0 \mid \lim_{r\to 0^+} r^{2s} \int_{B_r^c(0)} K(y)dy = +\infty\}. \tag{1.6}$$

We remark that if K satisfies (1.2) and is nonnegative, then the definition in (1.6) is equivalent to

$$s_0 = \sup\{s \ge 0 \mid \lim_{r\to 0^+} r^{2s} \int_{B_1(0)\setminus B_r(0)} K(x)dx = +\infty\}$$

by the fact that $\int_{B_1^c(0)} K(x)dx$ is bounded from (1.2).

Our basic setting is that $s_0 > 0$. In section 2, we will prove that $s_0 \le 1$ and exhibit an example in which the kernel K satisfying (1.5) makes $s_0 \in$

$(0,1)$. We note that the limit of $r^{2s_0}\int_{B_r^c(0)} K(y)dy$, as $r\to 0$, could be in $[0,\infty]$ or even no exists. Denote

$$l_\infty = \liminf_{r\to 0^+} r^{2s_0}\int_{B_r^c(0)} K(y)dy, \tag{1.7}$$

then it occurs one of the cases: Case 1: $l_\infty = 0$ and Case 2: $l_\infty \in (0,\infty]$.

Our first aim is to study the Sobolev space involving general kernel K. To be convenient for our analysis, we denote by X the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\int_{\mathbb{R}^{2N}\setminus\mathcal{O}} (g(x)-g(y))^2K(x-y)dxdy < +\infty,$$

where $\mathcal{O} := \Omega^c\times\Omega^c$. The space X is endowed with the norm as

$$\|g\|_X = (\|g\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^{2N}\setminus\mathcal{O}} (g(x)-g(y))^2K(x-y)dxdy)^{1/2}. \tag{1.8}$$

We define the following Sobolev space

$$X_0 = \{g\in X \mid g = 0 \ \text{ a.e. in } \Omega^c\}$$

equipped the norm (1.8). By the fact of (1.2), we stress that $C_0^2(\Omega)\subseteq X_0$, see [22], and so X and X_0 are nonempty.

Now we are ready for an embedding theorem.

Theorem 1.1. *Assume that $N\geq 2$, K satisfies (1.2) and (1.4), $s_0\in(0,1]$ defined by (1.6) and l_∞ is defined by (1.7). Let*

$$2^*(s_0) = \begin{cases} \frac{2N}{N-2s_0}, & \text{if} \quad \mathrm{N-2s_0>0}, \\ \infty, & \text{if} \quad \mathrm{N-2s_0=0}. \end{cases} \tag{1.9}$$

Then $(X_0, \|\cdot\|_X)$ is a Hilbert space and
(i) if $l_\infty = 0$, the embedding

$$X_0 \hookrightarrow L^q(\Omega) \tag{1.10}$$

is continuous and compact for $q\in[1,2^*(s_0))$, moreover, for $q\in[1,2^*(s_0))$ there exists $c_1>0$ such that

$$\|g\|_{L^q(\Omega)} \leq c_1\|g\|_X, \quad \forall g\in X_0; \tag{1.11}$$

(ii) if $l_\infty\in(0,\infty]$, the embedding (1.10) is continuous for $q\in[1,2^*(s_0)]$ if $2^*(s_0)<\infty$ or $q\in[1,\infty)$ if $2^*(s_0)=\infty$ and compact for $q\in[1,2^*(s_0))$. Furthermore, (1.11) holds for $q\in[1,2^*(s_0)]$ if $2^*(s_0)<\infty$ or for $q\in[1,\infty)$ if $2^*(s_0)=\infty$.

In what follows, we give a typical example for the kernel K, which is illuminating for Theorem 1.1.

Example 1.1. *Let*

$$K(x)=\frac{1}{|x|^{N+2s_0}}\left[(-\log|x|)_+ +1\right]^{\sigma},\quad x\in\mathbb{R}^N\setminus\{0\},\tag{1.12}$$

where $s_0\in(0,1]$, $\sigma\in\mathbb{R}$ *and* $(-\log|x|)_+=\max\{-\log|x|,0\}$.

When $s_0\in(0,1)$, $\sigma\in\mathbb{R}$ *or* $s_0=1$, $\sigma<-1$, *the kernel* K *defined by* (1.12) *satisfies* (1.2) *and* (1.4).

We note that $l_\infty=0$ *if* $\sigma<0$, $l_\infty\in(0,\infty)$ *if* $\sigma=0$ *and* $l_\infty=\infty$ *if* $\sigma>0$. *In particular,* $s_0\in(0,1)$ *and* $\sigma=0$, *the embedding* (1.10) *coincides the results in* [11]. *When* $s_0=1$, $2^*(s_0)=2^*$ *the critical exponent for* $H^1_0(\Omega)\hookrightarrow L^{2^*}(\Omega)$.

We note that Felsinger, Kassmann and Voigt in [14] studied Poincaré-Friedrichs inequality:

$$\|g\|^2_{L^2(\mathbb{R}^N)}\le c_2\int_{\mathbb{R}^{2N}}[g(x)-g(y)]^2K(x-y)dxdy,\qquad g\in L^2(\mathbb{R}^N),$$

where $c_2>0$, the kernel K satisfies (1.2) and

$$\int_{\mathbb{R}^{2N}}[h(x)-h(y)]^2K(x-y)dxdy\ge\int_{\mathbb{R}^{2N}}\frac{[h(x)-h(y)]^2}{|x-y|^{N+2\alpha}}dxdy,\quad \forall h\in L^2(\mathbb{R}^N)$$

for some $\alpha\in(0,1)$.

As an application of Theorem 1.1, we are able to obtain the existence of weak solutions of (1.1). Before stating the existence result we make precise the definition of weak solution that we use in the article. We say that a function $u\in X_0$ is a weak solution of (1.1) if

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}[u(x)-u(y)][\varphi(x)-\varphi(y)]K(x-y)dxdy=\int_\Omega f(x,u(x))\varphi(x)dx,\tag{1.13}$$

for any $\varphi\in X_0$.

The existence result is stated as follows.

Theorem 1.2. *Assume that* $N\ge2$, $f(x,u)=|u|^{p-2}u$ *and* K *satisfies* (1.2), (1.3) *and* (1.4). *Let* $s_0\in(0,1]$ *defined in* (1.6).

Then problem (1.1) *admits a nontrivial weak solution for* $p\in(2,2^*(s_0))$, *where* $2^*(s_0)$ *is given by* (1.9).

Remark 1.1. (*i*) K *is given as Example* 1.1 *with* $s_0\in(0,1)$ *and* $\sigma\in\mathbb{R}$ *or* $s_0=1$ *and* $\sigma<-1$, *problem* (1.1) *admits a weak solution for* $f(x,u)=|u|^{p-2}u$ *with* $p\in(2,2^*(s_0))$.

(*ii*) K *is taken as Example* 2.1, *problem* (1.1) *admits a weak solution for* $f(x,u)=|u|^{p-2}u$ *with* $p\in(2,2^*(s_0))$.

We remark that Ros-Oton and Serra in [20] studied the nonexistence of weak solution to (1.1). They obtained that when $K(y)|y|^{N+2\sigma}$ with $\sigma\in(0,1)$ is nondecreasing alone rays form the origin and K satisfies some extra condition, problem (1.1) only admits trivial solution in the case of $f(u)=u^p$ with $p\ge2^*(\sigma)$. In particular, the assumption on K in [20] implies that $s_0\le\sigma$ by our definition of s_0.

The paper is organized as follows. In Section 2, we analyze some basic properties of the kernel K and give an example showing that s_0 makes sense. In Section 3, we study the Sobolev embedding theorem in our setting. Finally, we prove the existence of weak solution to (1.1) in Section 4.

2. Discussion to the kernel K

This section is devoted to the properties of the kernel K.

Proposition 2.1. *Assume that s_0 is defined by (1.6) and K satisfies (1.2). Then (i) for any $s < s_0$, we have*

$$\lim_{r\to 0^+} r^{2s}\int_{B_r^c(0)} K(y)dy = +\infty;$$

(ii)

$$s_0 \le \inf\{s\ge 0 \mid \lim_{r\to 0^+} r^{2s}\int_{B_r^c(0)} K(y)dy = 0\} \le 1; \tag{2.1}$$

(iii) if there exists $s_1 \le s_2$ such that

$$\liminf_{|x|\to 0^+} K(x)|x|^{N+2s_1} > 0 \quad \text{and} \quad \limsup_{|x|\to 0^+} K(x)|x|^{N+2s_2} < +\infty, \tag{2.2}$$

then $s_0 \in [s_1, s_2]$.

Proof. (i) By the definition of s_0, there at least are a sequence of positive numbers $\{s_n\}$ such that

$$s_n < s_0, \quad \lim_{n\to\infty} s_n = s_0, \quad \lim_{r\to 0^+} r^{2s_n}\int_{B_r(0)} K(y)dy = +\infty.$$

Then for any $s < s_0$, there exists $n \in \mathbb{N}$ such that $s < s_n$ and then

$$\lim_{r\to 0^+} r^{2s}\int_{B_r(0)} K(y)dy \ge \lim_{r\to 0^+} r^{2s_n}\int_{B_r(0)} K(y)dy = +\infty.$$

(ii) By (1.2) and K being nonnegative, we have that for any $r \in (0,1)$,

$$\begin{aligned}\infty &> \int_{\mathbb{R}^N} \min\{|x|^2,1\}K(x)dx\\ &> \int_{B_1(0)\setminus B_r(0)} |x|^2K(x)dx + \int_{\mathbb{R}^N\setminus B_1(0)} K(x)dx\\ &\ge r^2\int_{\mathbb{R}^N\setminus B_r(0)} K(x)dx.\end{aligned}$$

Then for any $s > 1$, we have that

$$r^{2s}\int_{B_r^c(0)} K(x)dx = r^{2(s-1)}\left[r^2\int_{B_r^c(0)} K(x)dx\right] \to 0 \quad \text{as } r\to 0.$$

Thus, $\inf\{s\ge 0 \mid \lim_{r\to 0^+} r^{2s}\int_{B_r^c(0)} K(y)dy = 0\} \le 1$.

We now prove the first inequality (2.1). To this end, we denote

$$s_{00} = \inf\{s \geq 0 \mid \lim_{r\to 0^+} r^{2s} \int_{B_r^c(0)} K(y)dy = 0\}.$$

Since for any $s > s_{00}$, we have that

$$\lim_{r\to 0^+} r^{2s} \int_{B_r^c(0)} K(x)dx = 0.$$

By the definition of s_0, we have that $s_0 \leq s$ and then by arbitrary of $s > s_{00}$, we obtain that $s_0 \leq s_{00}$.

(iii) For any $s < s_1$ by (2.2), we have that

$$\begin{aligned} r^{2s} \int_{B_r^c(0)} K(x)dx &= r^{2(s-s_1)}[r^{2s_1} \int_{B_r^c(0)} K(x)dx] \\ &\geq r^{2(s-s_1)} \inf_{|x|\in(0,1)} (K(x)|x|^{N+2s_1})[r^{2s_1} \int_r^1 \tau^{-2s_1-1} d\tau] \\ &\geq r^{2(s-s_1)} \int_r^1 \tau^{-1} d\tau \inf_{|x|\in(0,1)} (K(x)|x|^{N+2s_1}) \\ &\to \infty \quad \text{as } r \to 0. \end{aligned}$$

By the definition of s_0, we have $s_0 \geq s$ and then by arbitrary of $s < s_1$, we obtain that $s_0 \geq s_1$. Similarly to prove $s_0 \leq s_2$. □

Lemma 2.1. *(i) Assume that the kernel K satisfies (1.4) with $c_0 = 1$ and is continuous in $\mathbb{R}^N \setminus \{0\}$, then K is radially symmetric about the origin.*

(ii) Assume that the kernel K satisfies (1.2), (1.4) and (1.6) with $s_0 > 0$. Then for any $s \in (0, s_0)$, there exists a sequence $\{r_n\}$ of positive numbers which converges to 0 and

$$\lim_{r_n\to 0^+} r_n^{N+2s} \inf_{|x|=r_n} K(x) = +\infty. \tag{2.3}$$

Proof. (i) By contradiction, we may assume that there exist $x_1, y_1 \in \mathbb{R}^N \setminus \{0\}$ such that $|x_1| = |y_1|$ and $K(x_1) > K(y_1)$. Since K is continuous in $\mathbb{R}^N \setminus \{0\}$, then there exists $x_2 \in \mathbb{R}^N \setminus \{0\}$ such that $|x_2| > |x_1|$ and

$$K(x_2) \geq K(x_1) - \frac{K(x_1) - K(y_1)}{2} > K(y_1),$$

which is impossible with the assumption (1.4).

(ii) By Proposition 2.1 (i), we have that for $s \in (0, s_0)$ and $\epsilon \in (0, s_0 - s)$,

$$\lim_{r\to 0^+} r^{2(s+\epsilon)} \int_{B_r^c(0)} K(x)dx = +\infty. \tag{2.4}$$

Let $\tilde{K}(r) = \inf_{|x|=r} K(x)$, then by (1.4), we have $\tilde{K}(r_2) \geq c_0 \tilde{K}(r_1)$ for $r_1 \geq r_2$ and $\tilde{K}(r) \geq c_0 K(x)$ for any $|x| > r$.

If (2.3) doesn't hold, then there no exist any sequence $\{r_n\}$ converging to zero such that (2.3) holds, that is

$$\limsup_{r\to 0^+} r^{N+2s} \tilde{K}(r) < +\infty.$$

Then there exists $c_3 > 0$ such that

$$\tilde{K}(r) \leq c_3 r^{-N-2s}, \quad r \in (0,1).$$

For any $x \in B_1(0) \setminus \{0\}$, we have $\tilde{K}(\frac{|x|}{2}) \geq c_0 K(x)$,

$$\begin{aligned} r^{2(s+\epsilon)} \int_{B_1(0)\setminus B_r(0)} K(x)dx &\leq \frac{1}{c_0} r^{2(s+\epsilon)} \int_{B_1(0)\setminus B_r(0)} \tilde{K}(\frac{|x|}{2})dx \\ &\leq c_4 r^{2(s+\epsilon)} \int_r^1 \tau^{-1-2s} d\tau \\ &\leq c_5 r^{2\epsilon}, \end{aligned}$$

where $c_4, c_5 > 0$. Together with (1.2), we have

$$\lim_{r\to 0^+} r^{2(s+\epsilon)} \int_{B_r^c(0)} K(x)dx = 0.$$

which contradicts with (2.4). The proof is complete. □

To end this section, we construct an example of K satisfying (1.5) for which $s_0 \in (0,1)$.

Example 2.1. *Let*

$$K(x) = \begin{cases} a_n^{-N-2s}, & |x| \in [a_{n+1}, a_n), \\ |x|^{-N}, & |x| \in [a_1, 1), \\ |x|^{-N-2s}, & |x| \in [1, +\infty). \end{cases} \tag{2.5}$$

where $s \in (0,1)$, $a_0 \in (0,1)$, $a_n = a_0^{b^n}$ with $n \in \mathbb{N}$ and $b = \frac{N+2s}{N}$.

Then

$$\liminf_{r\to 0^+} K(r) r^N = 1 \quad \text{and} \quad s_0 \in (0,s).$$

Proof. We observe that $\lim_{n\to+\infty} a_n = 0$ and

$$K(a_n)a_n^N = a_{n-1}^{-N-2s} a_n^N = a_0^{-b^{n-1}(N+2s)} a_n^N = (a_0^{b^n})^{-N} a_n^N = 1,$$

then we have

$$\liminf_{r\to 0^+} K(r) r^N = 1.$$

Combining Proposition 2.1 (iii) and the fact of $\limsup_{r\to 0^+} K(r) r^{N+2s} \leq 1$, we have that

$$s_0 \in [0,s).$$

Now we prove that $s_0 > 0$. For $r \in (0, a_1)$, there exists $n \in \mathbb{N}$ such that $a_{n+1} \leq r < a_n$. If n big enough, we have $a_n \leq \frac{1}{2} a_{n-1}$. Combining with $b > 1$, then

$$\begin{aligned} \int_{B_{a_1(0)}\setminus B_r(0)} K(y)dy &= |\omega_N| [(a_n - r)^N a_n^{-N-2s} + \sum_{k=2}^{n} (a_{k-1} - a_k)^N a_{k-1}^{-N-2s}] \\ &\geq |\omega_N| \sum_{k=2}^{n} (a_{k-1} - a_k)^N a_{k-1}^{-N-2s} \\ &\geq |\omega_N| 2^{-N} a_{n-1}^{-2s}, \end{aligned}$$

where ω_N is the unit sphere of $\mathbb{R}^N$. Choose $\beta = b^{-2}s > 0$, then we obtain that

$$a_{n-1}^{-2s} \geq a_{n+1}^{-2\beta}.$$

Therefore,

$$\liminf_{r\to 0^+} r^{2\beta} \int_{B_{a_1(0)}\setminus B_r(0)} K(y)dy \geq 2^{-N}|\omega_N|.$$

By Proposition 2.1 (iii), we obtain that $s_0 \geq \beta > 0$. □

3. Sobolev spaces

In this section, we will consider some embedding results inspired from [11]. First we introduce some basic spaces and some useful tools to prove embedding theorems.

Lemma 3.1. *Assume that $N \geq 2$, K satisfies (1.2) and (1.4), $s_0 \in (0,1]$ defined by (1.6) and l_∞ is defined by (1.7). Let $x \in \mathbb{R}^N$ and $E \subset \mathbb{R}^N$ be a measurable set with $|E| \in (0,+\infty)$, then*
(i) if $l_\infty = 0$, for any $s \in (0,s_0)$, there exists $c_6 > 0$ such that

$$\int_{E^c} K(x-y)dy \geq c_6|E|^{-\frac{2s}{N}}; \tag{3.1}$$

(ii) if $l_\infty \in (0,\infty]$, (3.1) holds with $s \in (0,s_0]$.

Proof. We just need to prove that the conclusion of Lemma 3.1 holds for a sequence E_n with $|E_n| > 0$ and $\lim_{n\to\infty} |E_n| = 0$. Let $\rho_n = (\frac{|E_n|}{\omega_N})^{1/N}$, then it follows that $|E_n^c \cap B_{\rho_n}(x)| = |E_n \cap B_{\rho_n}^c(x)|$ for $x \in \mathbb{R}^N$. Therefore, by (1.4), we have that

$$K(x-y) \geq c_0 \inf_{|z|=\rho_n} K(z), \quad y \in E_n^c \cap B_{\rho_n}(x),$$

$$\inf_{|z|=\rho_n} K(z) \geq c_0 K(x-y), \quad y \in E_n \cap \bar{B}_{\rho_n}^c(x).$$

Thus

$$\begin{aligned}
\int_{E_n^c} K(x-y)dy &= \int_{E_n^c\cap B_{\rho_n}^c(x)} K(x-y)dy + \int_{E_n^c\cap B_{\rho_n}(x)} K(x-y)dy \\
&\geq \int_{E_n^c\cap B_{\rho_n}^c(x)} K(x-y)dy + c_0 \inf_{|z|=\rho_n} K(z)|E_n^c \cap B_{\rho_n}(x)| \\
&\geq \int_{E_n^c\cap B_{\rho_n}^c(x)} K(x-y)dy + c_0 \inf_{|z|=\rho_n} K(z)|E_n \cap \bar{B}_{\rho_n}^c(x)| \\
&\geq \int_{E_n^c\cap B_{\rho_n}^c(x)} K(x-y)dy + c_0^2 \int_{E_n\cap B_{\rho_n}^c(x)} K(x-y)dy \\
&\geq c_0^2 \int_{B_{\rho_n}^c} K(x-y)dy, \qquad (3.2)
\end{aligned}$$

where the last inequality holds since $c_0 \in (0,1]$.

(i) By Proposition 2.1 (i) and $s_0 > 0$, we observe that for any $s \in (0, s_0)$,

$$\lim_{r\to 0^+} r^{2s} \int_{B_r^c(0)} K(y)dy = \infty. \tag{3.3}$$

Then by (3.2), there exists $c_7 > 0$ such that

$$\int_{E_n^c} K(x-y)dy \geq c_7 |E_n|^{-\frac{2s}{N}}.$$

(ii) For $l_\infty \in (0, \infty)$, there exists $c_8 \in (0,1)$ such that for $r \in (0,1)$,

$$r^{2s_0} \int_{B_r^c(0)} K(y)dy \geq c_8 l_\infty,$$

which, together with (3.2), implies that

$$\int_{E_n^c} K(x-y)dy \geq c_8 c_0^2 l_\infty |E_n|^{-\frac{2s_0}{N}}.$$

For $l_\infty = \infty$, there exists $c_9 > 0$ such that for $r \in (0,1)$,

$$r^{2s_0} \int_{B_r^c(0)} K(y)dy \geq c_9,$$

which, together with (3.2), implies that

$$\int_{E_n^c} K(x-y)dy \geq c_9 c_0^2 |E_n|^{-\frac{2s_0}{N}}.$$

For $s \in (0, s_0)$, it is the same as the proof of (i). □

Lemma 3.2. [11, Lemma 6.2] *Assume that $s \in (0,1)$, $2s < N$ and $T > 1$. Let $n \in \mathbb{Z}$ and $\{a_k\}$ be a bounded, nonnegative, decreasing sequence with $a_k = 0$ for any $k \geq n$. Then,*

$$\sum_{k\in\mathbb{Z}} a_k^{1-\frac{2s}{N}} T^k \leq c_{10} \sum_{k\in\mathbb{Z}, a_k\neq 0} a_{k+1} a_k^{-\frac{2s}{N}} T^k,$$

where $c_{10} > 0$ depends on s, T, N but is independent of n.

Lemma 3.3. *Assume that $N \geq 2$, K satisfies (1.2) and (1.4), $s_0 \in (0,1]$ defined by (1.6) and l_∞ is defined by (1.7). Let $f \in L^\infty(\mathbb{R}^N)$ be compactly supported, then*

$$\int_{\mathbb{R}^{2N}} |f(x) - f(y)|^2 K(x-y)dxdy \geq c_{11} \sum_{k\in\mathbb{Z}, a_k\neq 0} a_{k+1} a_k^{-\frac{2s}{N}} 2^{2k},$$

where $a_k = |\{|f| > 2^k\}|$, $k \in \mathbb{Z}$, $c_{11} > 0$ dependent of N, K and the choice of s is the same as in Lemma 3.1.

Proof. The proof is similar to Lemma 6.3 in [11] just replaced the kernel by K. For reader's convenience, we give the detail below. Firstly, we assume that f is nonnegative. If not, we replace f by $|f|$. Let $A_k := \{f > 2^k\}$, $D_k := A_k \setminus A_{k+1}$, $d_k := |D_k|$ and $S := \sum_{j\in\mathbb{Z}, a_{j-1}\neq 0} 2^{2j} a_{j-1}^{-\frac{2s}{N}} d_j$. Then

$$\{(i,j) \in \mathbb{Z}^2 \ s.t. \ a_{i-1} \neq 0 \text{ and } a_{j-1}^{-\frac{2s}{N}} d_j \neq 0\} \subset \{(i,j) \in \mathbb{Z}^2 \ s.t. \ a_{j-1} \neq 0\}. \tag{3.4}$$

Then we have that

$$\begin{aligned}\sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \sum_{j\in\mathbb{Z}, j\geq i+1} 2^{2i} a_{i-1}^{-\frac{2s}{N}} d_j &= \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \sum_{j\in\mathbb{Z}, j\geq i+1, a_{i-1}^s d_j\neq 0} 2^{2i} a_{i-1}^{-\frac{2s}{N}} d_j \\ &\leq \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}, j\geq i+1, a_{i-1}\neq 0} 2^{2i} a_{i-1}^{-\frac{2s}{N}} d_j \\ &= \sum_{j\in\mathbb{Z}, a_{j-1}\neq 0} \sum_{i\in\mathbb{Z}, i\leq j-1} 2^{2i} a_{i-1}^{-\frac{2s}{N}} d_j \\ &\leq \sum_{j\in\mathbb{Z}, a_{j-1}\neq 0} \sum_{i\in\mathbb{Z}, i\leq j-1} 2^{2i} a_{j-1}^{-\frac{2s}{N}} d_j \\ &= \sum_{j\in\mathbb{Z}, a_{j-1}\neq 0} \sum_{k=0}^{+\infty} 2^{2j-2} 2^{-2k} a_{j-1}^{-\frac{2s_0}{N}} d_j \\ &\leq S.\end{aligned}$$

Fixed $i \in \mathbb{Z}$ and $x \in D_i$, for any $l \in \mathbb{Z}$ with $l \leq i-2$ and any $y \in D_l$, we have that

$$|f(x) - f(y)| \geq 2^{i-1}$$

and therefore,

$$\begin{aligned}\sum_{l\in\mathbb{Z}, l\leq i-2} \int_{D_j} |f(x)-f(y)|^2 K(x-y)dy &\geq 2^{2i-2} \sum_{l\in\mathbb{Z}, l\leq i-2} \int_{D_j} K(x-y)dy \\ &= 2^{2i-2} \int_{A_{i-1}^c} K(x-y)dy.\end{aligned}$$

By Lemma 3.1, we have

$$\sum_{l\in\mathbb{Z}, l\leq i-2} \int_{D_l} |f(x)-f(y)|^2 K(x-y)dy \geq c_{12} 2^{2i} a_{i-1}^{-\frac{2s}{N}},$$

for some suitable $c_{12} > 0$. As a consequence, for any $i \in \mathbb{Z}$,

$$\sum_{l\in\mathbb{Z}, l\leq i-2} \int_{D_i\times D_l} |f(x)-f(y)|^2 K(x-y)dxdy \geq c_{12} 2^{2i} a_{i-1}^{-\frac{2s}{N}} d_i$$

and then,

$$\sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \sum_{l\in\mathbb{Z}, l\leq i-2} \int_{D_i\times D_l} |f(x)-f(y)|^2 K(x-y)dxdy \geq c_{12} S.$$

Thus, we obtain

$$\begin{aligned}&\sum_{i\in\mathbb{Z},a_{i-1}\neq 0}\sum_{l\in\mathbb{Z},l\leq i-2}\int_{D_i\times D_l}|f(x)-f(y)|^2K(x-y)dxdy\\ &\geq c_{12}[\sum_{i\in\mathbb{Z},a_{i-1}\neq 0}2^{2i}a_{i-1}^{-\frac{2s}{N}}a_i-\sum_{i\mathbb{Z},a_{i-1}\neq 0}\sum_{j\in\mathbb{Z},j\geq i+1}2^{2i}a_{i-1}^{-\frac{2s}{N}}d_j]\\ &\geq c_{12}(2^{2i}a_{i-1}^{-\frac{2s}{N}}a_i-S).\end{aligned}$$

So, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}^N}|f(x)-f(y)|^2K(x-y)dxdy\\ &\geq 2\sum_{i\in\mathbb{Z},a_{i-1}\neq 0}\sum_{l\in\mathbb{Z},l\leq i-2}\int_{D_i\times D_l}|f(x)-f(y)|^2K(x-y)dxdy\\ &\geq c_{12}(\sum_{i\in\mathbb{Z},a_{i-1}\neq 0}2^{2i}a_{i-1}^{-\frac{2s}{N}}a_i).\end{aligned}$$

The proof ends. □

Lemma 3.4. *Assume that $q\in[1,+\infty)$, $f:\mathbb{R}^N\to\mathbb{R}$ is a measurable function. For any $n\in\mathbb{N}$,*

$$f_n(x):=\max\{\min\{f(x),n\},-n\},\quad \forall x\in\mathbb{R}^N.$$

Then

$$\lim_{n\to+\infty}\|f_n\|_{L^q(\mathbb{R}^N)}=\|f\|_{L^q(\mathbb{R}^N)}.$$

Proof. The details of the proof refers to [11, Lemma 6.4], but also [2]. □

Now we can give the statement of embedding theorem as follows:

Theorem 3.1. *Assume that Ω is a C^2 bounded open domain in $\mathbb{R}^N$ with $N\geq 2$, K satisfies (1.2) and (1.4), $s_0\in(0,1]$ defined by (1.6) and l_∞ is defined by (1.7).*

Then

(i) if $l_\infty=0$, then for $s\in(0,s_0)$ there exists $c_{13}>0$ independent of Ω such that for any $f\in X_0$, we have

$$\|f\|_{L^{2^*(s)}(\Omega)}\leq c_{13}(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|f(x)-f(y)|^2K(x-y)dxdy)^{\frac{1}{2}};\tag{3.5}$$

(ii) if $l_\infty\in(0,\infty]$, then (3.5) holds with $s\in(0,s_0]$ if $2^*(s_0)<\infty$ or $s\in(0,s_0)$ if $2^*(s_0)=\infty$.

Proof. First we note that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|f(x)-f(y)|^2K(x-y)dxdy<+\infty.\tag{3.6}$$

Without loss of generality, we can assume that $f \in L^\infty(\mathbb{R}^N)$. Indeed, let f_n be defined as in Lemma 3.4, then combining with Lemma 3.4 and (3.6), we make use of the Dominated Convergence Theorem to imply

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2N}} |f_n(x) - f_n(y)|^2 K(x-y)dxdy = \int_{\mathbb{R}^{2N}} |f(x) - f(y)|^2 K(x-y)dxdy,$$

which allows us to obtain estimate for function $f \in X_0$.

Take s, a_k and A_k defined as in Lemma 3.3, then we have that

$$\|f\|^{2^*(s)}_{L^{2^*(s)}(\mathbb{R}^N)} = \sum_{k\in\mathbb{Z}} \int_{A_k\setminus A_{k+1}} |f(x)|^{2^*(s)} dx \le \sum_{k\in\mathbb{Z}} 2^{2^*(s)(k+1)} a_k,$$

that is,

$$\|f\|^2_{L^{2^*(s)}(\mathbb{R}^N)} \le 4(\sum_{k\in\mathbb{Z}} 2^{2^*(s)k} a_k)^{2/2^*(s)}.$$

Since $2 < 2^*(s)$, then

$$\|f\|^2_{L^{2^*(s)}(\mathbb{R}^N)} \le 4\sum_{k\in\mathbb{Z}} 2^{2k} a_k^{2/2^*(s)}.$$

By Lemma 3.2 with $T = 4$, it follows that

$$\|f\|^2_{L^{2^*(s)}(\mathbb{R}^N)} \le c_{14} \sum_{k\in\mathbb{Z}} 2^{2k} a_{k+1} a_k^{-\frac{2s}{N}},$$

where $c_{14} > 0$ depending on N, K.

Finally, it suffices to apply Lemma 3.4 to obtain that

$$\|f\|_{L^{2^*(s)}(\mathbb{R}^N)} \le c_{15}(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x) - f(y)|^2 K(x-y)dxdy)^{\frac{1}{2}},$$

where $c_{15} > 0$. Since $f \in X_0$, $f = 0$ in Ω^c, then (3.5) holds. □

Corollary 3.1. *The norm (1.8) in X_0 is equivalent to*

$$\|f\|_{X_0} := (\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x) - f(y)|^2 K(x-y)dxdy)^{\frac{1}{2}}. \tag{3.7}$$

Proof. We only need to prove that there exists $c_{16} > 0$ such that for any $f \in X_0$,

$$\|f\|_X \le c_{16}\|f\|_{X_0}.$$

It follows by Theorem 3.1 that

$$\begin{aligned}
\|f\|_X^2 &= \int_\Omega f^2(x)dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x) - f(y)|^2 K(x-y)dxdy \\
&\le |\Omega|^{1-\frac{2}{2^*(s)}} (\int_\Omega |f|^{2^*(s)}(x)dx)^{\frac{2}{2^*(s)}} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x) - f(y)|^2 K(x-y)dxdy \\
&\le c_{17} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x) - f(y)|^2 K(x-y)dxdy,
\end{aligned}$$

where $c_{17} > 0$. The proof is complete. □

In what follows, we prove compact embedding result. To this end, we denote by $\mathcal{T}$ a sequence of functions in X_0 such that

$$\|f\|_X \le 1, \qquad \forall f \in \mathcal{T}.$$

Theorem 3.2. *Assume that $N \ge 2$, K satisfies (1.2) and (1.4), $s_0 \in (0,1]$ defined by (1.6) and l_∞ is defined by (1.7). Let $2^*(s_0) = \frac{2N}{N-2s_0}$ if $N-2s_0 > 0$, $2^*(s_0) = \infty$ if $N - 2s_0 = 0$.*

Then $\mathcal{T}$ is pre-compact in $L^q(\Omega)$, $q \in [1, 2^*(s_0))$.

Proof. We first prove that $\mathcal{T}$ is pre-compact in $L^2(\Omega)$. To this end, we only show that $\mathcal{T}$ is totally bounded in $L^2(\Omega)$. By Lemma 2.1(ii), there exists $\{r_n\}$ positive and convergent to 0 such that

$$\lim_{n\to\infty} r_n^N K(r_n) = +\infty.$$

Let $\rho : \mathbb{R}_+ \to \{\frac{r_n}{2}, n \in \mathbb{N}\}$ such that, denoting $\rho_\epsilon = \rho(\epsilon)$, for any $\epsilon > 0$, $\rho_\epsilon = r_n$ for some n and

$$\lim_{\epsilon\to 0^+} \rho_\epsilon = 0.$$

It is obvious that

$$\lim_{\epsilon\to 0^+} (2\rho_\epsilon)^N K(2\rho_\epsilon) = +\infty. \tag{3.8}$$

Let $\eta_\epsilon = \epsilon\rho_\epsilon^{\frac{N}{2}}$ and take a collection of disjoints cubes $Q_1, ..., Q_M$ of side ρ_ϵ such that

$$\Omega \subset \bigcup_{j=1}^{N} Q_j.$$

For any $x \in \Omega$, there exists a unique integer $j(x)$ in $\{1, ..., M\}$ such that $x \in Q_{j(x)}$. Let

$$P(f)(x) := \frac{1}{|Q_{j(x)}|} \int_{Q_{j(x)}} f(y)dy,$$

then P is linear and $P(f)$ is constant in Q_j, which we denote by $q_j(f)$. We define the linear operator R by

$$R(f) = \rho_\epsilon^{\frac{N}{2}} (q_1(f), ..., q_M(f)) \in \mathbb{R}^M$$

and

$$\|v\|_2 := (\sum_{j=1}^{M} |v_j|^2)^{\frac{1}{2}}, \quad v \in \mathbb{R}^M.$$

We observe that for any $f \in \mathcal{T}$,

$$\begin{aligned}\|P(f)\|^2_{L^2(\Omega)} &= \sum_{j=1}^{M} \int_{Q_j} |P(f)(x)|^2 dx = \rho_\epsilon^N \sum_{j=1}^{M} |q_j(f)|^2 \\ &= \|R(f)\|_2^2 = \int_\Omega |f(y)|^2 dy \\ &= \|f\|^2_{L^2(\Omega)} \le c_{12}^2.\end{aligned}$$

Therefore, there exist $b_1, ...b_I \in \mathbb{R}^M$ such that

$$R(\mathcal{T}) \subset \bigcup_{i=1}^{I} B_{\eta_\epsilon}(b_i),$$

where the balls $\{B_{\eta_\epsilon}\}$ are taken in $\mathbb{R}^M$. For any $x \in \Omega$, we set

$$\beta_j(x) = \rho_\epsilon^{-\frac{N}{2}} b_{i,j(x)},$$

where $b_{i,j(x)}$ is the $j(x)$th coordinates of b_i. Noticing that β_j is constant on Q_j, i.e. for $x \in Q_j$, it follows that

$$P(\beta_i)(x) = \rho_\epsilon^{-\frac{N}{2}} b_{i,j} = \beta_i(x)$$

and so $q_j(\beta_i) = \rho_\epsilon^{-\frac{N}{2}} b_{i,j}$. Thus $R(\beta_i) = b_i$. Furthermore, for any $f \in \mathcal{T}$,

$$\begin{aligned}
\|f - P(f)\|^2_{L^2(\Omega)} &= \sum_{j=1}^{M} \int_{Q_j} |f(x) - P(f)(x)|^2 dx \\
&= \sum_{j=1}^{M} \int_{Q_j} \frac{1}{|Q_j|^2} | \int_{Q_j} f(x) - f(y) dy|^2 dx \\
&\leq \frac{1}{\rho_\epsilon^{2N}} \sum_{j=1}^{M} \int_{Q_j} [\int_{Q_j} |f(x) - f(y)| dy]^2 dx
\end{aligned}$$

and for any fixed $j \in \{1, ..., M\}$, by Hölder inequality, we obtain that

$$\begin{aligned}
\frac{1}{\rho_\epsilon^{2N}} [\int_{Q_j} |f(x) - f(y)| dy]^2 &\leq \frac{1}{\rho_\epsilon^{2N}} |Q_j| \int_{Q_j} |f(x) - f(y)|^2 dy \\
&\leq \frac{1}{\rho_\epsilon^{N}} \frac{1}{K(2\rho_\epsilon)} \int_{Q_j} |f(x) - f(y)|^2 K(x-y) dy \\
&\leq \frac{1}{\rho_\epsilon^{N} K(2\rho_\epsilon)} \|f\|^2_X,
\end{aligned}$$

where $K(2\rho_\epsilon) = \inf_{|x|=2\rho_\epsilon} K(x)$. Therefore,

$$\|f - P(f)\|^2_{L^2(\Omega)} \leq \frac{1}{\rho_\epsilon^{N} K(2\rho_\epsilon)} \|f\|^2_X \sum_{j=1}^{M} |Q_j| \leq \frac{c_{18}}{\rho_\epsilon^{N} K(2\rho_\epsilon)}. \tag{3.9}$$

Consequently, for any f, there exists $j \in \{1, ...M\}$ such that $P(f) \in B_{\eta_\epsilon}(b_j)$ and then we derive that

$$\begin{aligned}
&\|f - \beta_j\|_{L^2(\Omega)} \\
&\quad \leq \|f - P(f)\|_{L^2(\Omega)} + \|P(f) - P(\beta_j)\|_{L^2(\Omega)} + \|P(\beta_j) - \beta_j\|_{L^2(\Omega)} \\
&\quad \leq \frac{c_{18}}{\rho_\epsilon^{N} K(2\rho_\epsilon)} + \frac{\|R(f) - R(\beta_j)\|_{L^2(\Omega)}}{\rho_\epsilon^{\frac{N}{2}}} \\
&\quad \leq \frac{c_{18}}{\rho_\epsilon^{N} K(2\rho_\epsilon)} + \frac{\eta_\epsilon}{\rho_\epsilon^{\frac{N}{2}}},
\end{aligned}$$

where by (3.8), $\frac{1}{(2\rho_\epsilon)^N K(2\rho_\epsilon)} \to 0$ as $\epsilon \to 0$ and $\frac{\eta_\epsilon}{\rho_\epsilon^{N/2}} = \epsilon$. As a consequence, $\mathcal{T}$ is pre-compact in $L^2(\Omega)$.

Now we are in the position to prove that $\mathcal{T}$ is pre-compact in $L^q(\Omega)$ with $q \in [1, 2^*(s_0))$. Since $L^2(\Omega) \subset L^q(\Omega)$ with $q \in [1, 2)$, then $\mathcal{T}$ is pre-compact in $L^q(\Omega)$. For $q \in (2, 2^*(s_0))$, there exists $s \in (0, s_0)$ such that $q < 2^*(s)$, then using Hölder inequality with $\theta = \frac{2(2^*(s)-q)}{q(2^*(s)-2)}$, we get that

$$\begin{aligned}\|f-\beta_j\|_{L^q(\Omega)} &= \left(\int_\Omega |f-\beta_j|^{\theta q}|f-\beta_j|^{q(1-\theta)}dx\right)^{\frac{1}{q}} \\ &\le \||f-\beta_j|\|_{L^2(\Omega)}^{\frac{\theta}{2}} \||f-\beta_j|\|_{L^{2^*(s)}(\Omega)}^{\frac{1}{q}-\frac{\theta}{2}} \\ &\le \left(\frac{c_{18}}{\rho_\epsilon^N K(2\rho_\epsilon)} + \frac{\eta_\epsilon}{\rho_\epsilon^{\frac{N}{2}}}\right)^{\frac{\theta}{2}} \|f\|_X^{\frac{1}{q}-\frac{\theta}{2}},\end{aligned}$$

thus, $\mathcal{T}$ is pre-compact in $L^q(\Omega)$ with $q \in (2, 2^*(s_0))$. The proof ends. □

Proof of Theorem 1.1. For Theorem 1.1 part (i), let $\{f_n\}$ be a sequence functions in X_0 such that

$$\|f_n\|_X \le 1, \quad \forall n \in \mathbb{N}.$$

By Theorem 3.1, inequality (1.11) follows by (3.5). We obtain that the sequence (f_n) is pre-compact in L^q with $q \in [1, 2^*(s_0))$, then the compactness in Theorem 1.1 follows. □

4. Existence of weak solution to (1.1)

We observe that problem (1.1) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J} : X_0 \to \mathbb{R}$ defined as follows

$$\mathcal{J}(u) = \frac{1}{2}\|u\|_{X_0}^2 - \frac{1}{p}\int_\Omega |u|^p dx.$$

Note the functional $\mathcal{J}$ is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$,

$$\langle \mathcal{J}'(u), \varphi\rangle = \int_Q \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)dxdy - \int_\Omega |u|^{p-2}u(x)\varphi(x)dx.$$

We will make use of Mountain Pass theorem to obtain the weak solution. In what follows, we check the structure condition of Mountain Pass theorem. It is obvious that $\mathcal{J}(0) = 0$.

Proposition 4.1. *Under the hypotheses of Theorem* 1.2, *there exist* $\rho > 0$ *and* $\beta > 0$ *such that* $\mathcal{J}(u) \ge \beta$, *for any* $u \in X_0$ *with* $\|u\|_{X_0} = \rho$.

Proof. Let $u \in X_0$, then

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{2}\|u\|_{X_0}^2 - \frac{1}{p}\int_\Omega |u(x)|^p\,dx\\ &\geq \frac{1}{2}\|u\|_{X_0}^2 - c_{19}\|u\|_{X_0}^p\\ &= \frac{1}{2}\|u\|_{X_0}^2(1 - c_{19}\|u\|_{X_0}^{p-2}),\end{aligned}$$

where we used Theorem 1.1 and Corollary 3.1 for the inequality. We choose $\sigma > 0$ such that $1 - c_{19}\sigma^{\frac{p-2}{2}} = \frac{1}{2}$, since $p > 2$. Then for $\|u\|_{X_0}^2 = \sigma$, $1 - C\|u\|_{X_0}^{p-2} = \frac{1}{2}$, then we have

$$\mathcal{J}(u) \geq \frac{1}{4}\sigma.$$

The proof is complete. □

Proposition 4.2. *Under the hypotheses of Theorem* 1.2*, there exists* $e \in X_0$ *such that* $\|e\|_{X_0} > \rho$ *and* $\mathcal{J}(e) \leq 0$*, where* ρ *is given in Proposition* 4.1*.*

Proof. We fix a function $u_0 \in X_0$ with $\|u_0\| = 1$ in Ω. Since the space of $\{tu_0 : t \in \mathbb{R}\}$ is a subspace of X_0 with dimension 1 and all the norms are equivalent, then $\int_\Omega |u_0(x)|^p dx > 0$. Then there exists $t_0 > 0$ such that for $t \geq t_0$,

$$\begin{aligned}\mathcal{J}(tu_0) &= \frac{t^2}{2}\|u_0\|_{X_0}^2 - \frac{t^p}{p}\int_\Omega |u_0(x)|^p dx\\ &\leq c_{20}(t^2 - t^p) \leq 0.\end{aligned}$$

We choose $e = t_0u_0$. The proof is complete. □

We say that $\mathcal{J}$ has *P.S.* condition, if for any sequence $\{u_n\}$ in X_0 satisfying $\mathcal{J}(u_n) \to c$ and $\mathcal{J}'(u_n) \to 0$ as $n \to \infty$, there is a convergent subsequence, where $c \in \mathbb{R}$.

Proposition 4.3. *Under the hypotheses of Theorem* 1.2*,* $\mathcal{J}$ *has P.S. condition in* X_0*.*

Proof. Let $\{u_n\}$ be a *P.S.* sequence, then we need to show that there are a subsequence $\{u_{n_k}\}$ and u such that

$$u_{n_k} \to u \quad \text{in} \quad L^p(\Omega) \quad \text{as } k \to \infty.$$

For some $c_{21} > 0$, we have that

$$c_{21}\|u_n\|_{X_0} \geq \mathcal{J}'(u_n)u_n = \|u_n\|_{X_0}^2 - \int_\Omega |u_n|^p dx \tag{4.1}$$

and

$$c - 1 \leq \mathcal{J}(u_n) = \frac{1}{2}\|u_n\|_{X_0}^2 - \frac{1}{p}\int_\Omega |u_n|^p dx. \tag{4.2}$$

Then $p\times$(4.2)-(4.1) implies that

$$(\frac{p}{2} - 1)\|u_n\|_{X_0}^2 \leq c + c_{21}\|u_n\|_{X_0},$$

then u_n is uniformly bounded in X_0.

Thus, by Theorem 1.1 and Corollary 3.1, there exists a subsequence $\{u_{n_k}\}$ and u such that

$$u_{n_k} \rightharpoonup u \quad \text{in} \quad X_0,$$
$$u_{n_k} \to u \quad \text{a.e. in } \Omega \quad \text{and} \quad \text{in} \quad L^p(\Omega),$$

when $k \to \infty$. Together with $\lim_{k\to\infty} \mathcal{J}(u_{n_k}) = c$, we have that $\|u_{n_k}\|_{X_0} \to \|u\|_{X_0}$ as $k \to \infty$. Then we obtain that $u_{n_k} \to u$ in X_0 as $k \to \infty$. □

Proof of Theorem 1.2. By Proposition 4.1, Proposition 4.2 and Proposition 4.3, we may use Mountain Pass Theorem (for instance, [24, Theorem 6.1]; see also [1, 18]) to obtain that there exists a critical point $u \in X_0$ of $\mathcal{J}$ at some value $c \geq \beta > 0$. By $\beta > 0$, we have u is nontrivial. Therefore, (1.1) admits a nonnegative weak solution. The proof is complete. □

Remark 4.1. *Suppose that $s_0 \in (0,1)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying the following hypothesis:*

(f_1) *there exist $a_1, a_2 > 0$ and $q \in (2, 2^*(s_0))$ such that*

$$|f(x,t)| \leq a_1 + a_2|t|^{q-1} \quad a.e.\ x \in \Omega,\ t \in \mathbb{R};$$

(f_2) $\lim_{t\to 0} \frac{f(x,t)}{|t|} = 0$ uniformly in $x \in \Omega$;

(f_3) *there exist $\mu > 2$ and $r > 0$ such that a.e. $x \in \Omega, t \in \mathbb{R}, |t| \geq r$*

$$0 < \mu F(x,t) \leq t f(x,t),$$

where the function F is the primitive of f with respect to the variable t, that is

$$F(x,t) = \int_0^t f(x,\tau)d\tau.$$

*Then fractional elliptic problem (*1.1*) admits a nontrivial weak solution.*

Proof. Using the technique in the proof of Theorem 1 in [22] and Theorem 1.1 part (ii), we derive a nontrivial weak solution of (1.1) by Mountain Pass Theorem. □

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