Homework 6 – Calc Emphasizing Proofs

Professor: Hichem Hajaiej
Assistant Professor: Huyuan Chen
NYU SHANGHAI 25-10-2014

P1. (*a*) Prove, working directly from the definition, that if $f(x) = 1/x$, then $f'(a) = -1/a^2$ for $a \neq 0$.

(*b*) Prove that the tangent line to the graph of f at $(a, 1/a)$ does not intersect the graph of f, except at $(a, 1/a)$.

Proof: (*a*) For $a \neq 0$, we have that

$$f'(a) = \lim_{y\to 0} \frac{f(a+y) - f(a)}{y} = \lim_{y\to 0} \frac{\frac{1}{a+y} - \frac{1}{a}}{y} = \lim_{y\to 0} \frac{-1}{a(a+y)} = \frac{-1}{a^2}.$$

(*b*) Suppose that the tangent line to the graph of f at $(a, 1/a)$ intersect the graph of f, and the intersection point is $(x, f(x))$ and $x \neq a$. Then

$$\frac{f(x) - \frac{1}{a}}{x - a} = f'(a) = \frac{-1}{a^2}.$$

So

$$f(x) = \frac{2a - x}{a^2}.$$

Since $f(x) = \frac{1}{x}$, then $\frac{1}{x} = \frac{2a-x}{a^2}$ and then $x = a$, which contradicts $x \neq a$. Therefore, the tangent line to the graph of f at $(a, 1/a)$ does not intersect the graph of f, except at $(a, 1/a)$.

P2. (*a*) Prove that if $f(x) = 1/x^2$, then $f'(a) = -2/a^3$ for $a \neq 0$.

(*b*) Prove that the tangent line to f at $(a, 1/a^2)$ intersects f at one other point, which lies on the opposite side of the vertical axis.

Proof: (*a*) For $a \neq 0$, we have that

$$f'(a) = \lim_{y\to 0} \frac{f(a+y) - f(a)}{y} = \lim_{y\to 0} \frac{\frac{1}{(a+y)^2} - \frac{1}{a^2}}{y} = \lim_{y\to 0} \frac{-(2a+y)}{a^2(a+y)^2} = \frac{-2}{a^3}.$$

(*b*) Suppose that the tangent line to the graph of f at $(a, 1/a^2)$ intersect the graph of f, and the intersection point is $(x, f(x))$ and $x \neq a$. Then

$$\frac{f(x) - \frac{1}{a^2}}{x - a} = f'(a) = \frac{-2}{a^3}.$$

So

$$f(x) = \frac{1}{a^2} - \frac{2(x-a)}{a^3}.$$

Since $f(x) = \frac{1}{x^2}$, then $\frac{1}{x^2} = \frac{1}{a^2} - \frac{2(x-a)}{a^3}$ and then $x = -\frac{a}{2}$. Therefore, the tangent line to f at $(a, \frac{1}{a^2})$ intersects f at one other point $(-\frac{a}{2}, \frac{4}{a^2})$, which lies on the opposite side of the vertical axis.

P3. Suppose that $f(x) = x^3$.

(*a*) What is $f'(9)$, $f'(25)$, $f'(36)$?

(b) What is $f'(3^2)$, $f'(5^2)$, $f'(6^2)$?

(c) What is $f'(a^2)$, $f'(x^2)$?

(d) For $f(x) = x^3$, compare $f'(x^2)$ and $g'(x)$, where $g(x) = f(x^2)$.

Answer: Since $f(x) = x^3$, then $f'(x) = 3x^2$.

(a)

$$f'(9) = 3 \times 9^2 = 243.$$
$$f'(25) = 3 \times 25^2 = 1875.$$
$$f'(36) = 3 \times 36^2 = 3888.$$

(b)

$$f'(3^2)) = 3 \times (3^2)^2 = 243.$$
$$f'(5^2)) = 3 \times (5^2)^2 = 1875.$$
$$f'(6^2)) = 3 \times (6^2)^2 = 3888.$$

(c)

$$f'(a^2) = 3(a^2)^2 = 3a^4.$$
$$f'(x^2) = 3(x^2)^2 = 3x^4.$$

(d)

$$g(x) = f(x^2) = (x^2)^3 = x^6,$$

then $g'(x) = 6x^5$ and $f'(x^2) = 3x^4$.

P4. Let f be any polynomial function, we will see in the next chapter that f is differentiable. The tangent line to f at $(a, f(a))$ is the graph of $g(x) = f'(a)(x-a)+f(a)$. Thus $f(x)-g(x)$ is the polynomial function $d(x) = f(x) - f'(a)(x-a) - f(a)$. We have already seen that if $f(x) = x^2$, then $d(x) = (x-a)^2$, and if $f(x) = x^3$, then $d(x) = (x-a)^2(x+2a)$.

(a) Find $d(x)$ when $f(x) = x^4$, and show that it is divisible by $(x-a)^2$.

(b) There certainly seems to be some evidence that $d(x)$ is always divisible by $(x-a)^2$. Figure 22 provides an intuitive argument: usually, lines parallel to the tangent line will intersect the graph at two points, the tangent line intersects the graph only once near the point, so the intersection should be a double intersection. To give a rigorous proof, first note that

$$\frac{d(x)}{x-a} = \frac{f(x) - f(a)}{x-a} - f'(a).$$

Now answer the following questions. Why is $f(x) - f(a)$ divisible by $(x-a)$? Why is there a polynomial function h such that $h(x) = d(x)/(x-a)$ for $x \neq a$? Why is $\lim_{x \to a} h(x) = 0$? Why is $h(a) = 0$? Why does this solve the problem?

Answer: (a) When $f(x) = x^4$, we know that the tangent line to f at $(a, f(a))$ is the graph of $g(x) = f'(a)(x-a) + f(a) = 4a^3(x-a) + a^4$. Then

$$d(x) = f(x) - g(x) = x^4 - 4a^3(x-a) - a^4 = x^4 - 4a^3x + 3a^4 = (x-a)^2(x^2 + 2ax + 3a^2).$$

So $d(x)$ is divisible by $(x-a)^2$.

(b) Since f is a polynomial function, we may assume that

$$f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0,$$

where $b_i \in \mathbb{R}$ for $i = 0, 1, \cdots, n$. Then $f(a) = b_n a^n + b_{n-1}a^{n-1} + \cdots + b_1 a + b_0$ and

$$f(x) - f(a) = b_n(x^n - a^n) + b_{n-1}(x^{n-1} - a^{n-1}) + \cdots + b_1(x - a).$$

As we know that $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$, which is divisible by $(x - a)$ for any $n \in \mathbb{N}$. Then $f(x) - f(a)$ is divisible by $(x - a)$.

As a consequence, we know that $\frac{f(x)-f(a)}{x-a}$ is a polynomial function for $x \neq a$, then

$$h(x) = \frac{d(x)}{x-a} = \frac{f(x) - f(a)}{x - a} - f'(a)$$

is a polynomial function for $x \neq a$ and then h is continuous.

From the definition of h, we have that

$$\lim_{x \to a} h(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a) = f'(a) - f'(a) = 0.$$

Since h is continuous, then $h(a) = 0$. This means that $h(x)$ is divisible by $(x - a)$ and then $d(x) = h(x)(x - a)$ is divisible by $(x - a)^2$.