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Rearrangement inequalities in the discrete setting and some applications

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We prove extended Hardy–Littlewood, Riesz and Pólya–Szegő inequalities in the discrete case. We also establish cases of equality in these inqualities and give some applications of our result.

1. Introduction

In the continuous setting, symmetrization inequalities have numerous applications in various domains: economics, probability, nonlinear optics, chemistry (see [1,2] for surveys on these matters)...An area where they have played a crucial role is the study of critical points and ground state solutions of the elliptic partial differential equation:

$$\Delta u + f(|x|, u) + \lambda u = 0. \tag{1}$$

The method relies on the following symmetrization inequalities

$$\int_{\mathbf{R}^n} |\nabla u^*|^2 \le \int_{\mathbf{R}^n} |\nabla u|^2 \tag{2}$$

$$\int_{\mathbf{R}^n} (u^*)^2 = \int_{\mathbf{R}^n} u^2 \tag{3}$$

$$\int_{\mathbf{R}^n} F(|x|, u^*(x))\,\mathrm{d}x \ge \int_{\mathbf{R}^n} F(|x|, u(x))\,\mathrm{d}x, \tag{4}$$

where

$$F(|x|, s) = \int_0^s f(|x|, t)\,\mathrm{d}t. \tag{5}$$

These key inequalities can be used to prove necessary and sufficient conditions for the existence of ground state solutions of (1), see [3]. Moreover, the establishment of cases of equality in (2) enables us to determine conditions under which every ground state inherits the symmetry properties of f, see [4]. Symmetry questions for solutions of elliptic problems have been of constant and ongoing interest as plentiful literature shows.

In the one dimensional case, a discretization of (1) leads to the finite-difference equation

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h_i^2} + f(|h_i|, u_i) + \lambda u_i = 0, \tag{6}$$

where $(u_i)_{i\in\mathbf{Z}}$ is a sequence, which is nonnegative if we seek for positive solutions, and h_i is the stepsize of the networking.

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A natural question is to see whether the approaches developed in [3,4] remain valid in the discrete setting. It is quite easy to check that these two methods still apply provided

$$\sum_{k\in\mathbf{Z}}(u_k^*)^2 = \sum_{k\in\mathbf{Z}}(u_k)^2 \tag{7}$$

$$\sum_{k\in\mathbf{Z}}(u_k^* - u_{k+1}^*)^2 \leq \sum_{k\in\mathbf{Z}}(u_k - u_{k+1})^2 \tag{8}$$

$$\sum_{k\in\mathbf{Z}}F(|k|, u_k^*) \geq \sum_{k\in\mathbf{Z}}F(|k|, u_k). \tag{9}$$

Let us point out that proving that solutions of the discretized elliptic problems inherit symmetry properties is relevant for the design of numerical schemes; it also implies that instead of solving numerically these elliptic problems on a full rectangle, it suffices to study them on a quarter of a rectangle which considerably cuts down the computational costs. Note also that symmetrization inequalities in the discrete case enable us to improve some asymptotic symmetry results obtained in [5]. Amongst other things, we prove that the logarithmic estimate obtained in [5] for the symmetry defect of the solution is not optimal.

While (7) is an immediate consequence of the definition of the discrete Schwarz symmetrization (see Proposition 2.2), (8) and (9) are trickier and need an in-depth study. Inspired by [6], we develop a self-contained approach which is very efficient for our purposes. More precisely, we first define the polarization u^H of an admissible sequence u (see Definition 2.1). Combinatorial arguments permit us to prove (8) with u^* replaced by u^H and the discrete Riesz inequality. We then construct an appropriate sequence $T^n u$, obtained by iteration of two particular polarizations, which converges to u^*. We also notice that in some cases, this sequence coincides with u^* from a certain range (Corollary 3.5). These two observations are extremely important for the construction of simple and explicit finite algorithms. A limiting process enables us to deduce (8) and (9) from the corresponding inequalities with u^* replaced by u^H. We also determine cases of equality in these inequalities.

To our knowledge there are few results available in literature concerning discrete symmetrization (see [7,8]). A lot of open challenging questions can be raised about discrete symmetrization, as for example, Hardy and Sobolev inequalities.

2. Definition and basic properties

Definition 2.1. If $u : \mathbf{Z} \to \mathbf{R}$ is bounded from above, the discrete Schwarz symmetrization of u is the unique function $u^* : \mathbf{Z} \to \mathbf{R}$ such that

(i) for every $k \geq 0$,

$$u^*(k) \geq u^*(-k) \geq u^*(k+1),$$

(ii) for every $t \in \mathbf{R}$,

$$\#\{k \in \mathbf{Z} \ : \ u^*(k) > t\} = \#\{k \in \mathbf{Z} \ : \ u(k) > t\}.$$

Equivalently, u^*

$$u^*(k) = \begin{cases} \sup\{t \in \mathbf{R} \ : \ \#\{l \in \mathbf{Z} \ : \ u(l) > t\} \leq 2|k| + 1\} & \text{if } k \leq 0 \\ \sup\{t \in \mathbf{R} \ : \ \#\{l \in \mathbf{Z} \ : \ u(l) > t\} \leq 2k\} & \text{if } k \geq 0. \end{cases}$$

The construction of u^* goes thus by taking for $u^*(0)$ the maximum value of u, for $u(1)$ the second largest value, for $u(-1)$ the third and so on.

Definition 2.2. A function $u : \mathbf{Z} \to \mathbf{R}$ is *admissible* if

$$\#\{l \in \mathbf{Z} \ : \ u(l) \geq t\} < \infty$$

for every $t > \inf_{l\in\mathbf{Z}} u(l)$.

Since

$$\inf\{t \in \mathbf{R} \ : \ \#\{l \in \mathbf{Z} \ : \ u(l) \geq t\} < \infty\} = \overline{\lim_{|k|\to\infty}} u(l),$$

u is admissible if and only if it is bounded and

$$\inf_{k\in\mathbf{Z}} u \geq \overline{\lim_{|l|\to\infty}} u(l).$$

Lemma 2.1. *For every* $t > \overline{\lim}_{|k|\to\infty} u(k)$,

$$\#\{k \in \mathbf{Z} \ : \ u^*(k) = t\} = \#\{k \in \mathbf{Z} \ : \ u(k) = t\}.$$

Proof. If $t > \overline{\lim}_{|k|\to\infty} u(k)$, then for ε sufficiently small, $\#\{k \in \mathbf{Z} \ : \ u^*(k) = t - \varepsilon\}$ is finite, so that

$$\#\{k \in \mathbf{Z} \ : \ u(k) = t\} = \lim_{\varepsilon\to 0} \#\{k \in \mathbf{Z} \ : \ u(k) > t - \varepsilon\} - \#\{k \in \mathbf{Z} \ : \ u(k) > t\}.$$

The same holds for u^*. One concludes by (ii) in Definition 2.1. □

Proposition 2.2 (*Cavalieri Principle*). *Let $f : \mathbf{R} \to \mathbf{R}$ and $u : \mathbf{Z} \to \mathbf{R}$ be admissible. If $f(t) = 0$ for $t \le \overline{\lim}_{l\to\infty} u(l)$, then*

$$\sum_{k\in\mathbf{Z}} f(u^*(k)) = \sum_{k\in\mathbf{Z}} f(u(k)).$$

Proof. Set $D = \{t \in u(\mathbf{Z}) \ : \ u(t) > \overline{\lim}_{|k|\to\infty} u(k)\}$. One has in the case $f(t) = 0$ for $t \le \overline{\lim}_{l\to\infty} u(l)$:

$$\sum_{k\in\mathbf{Z}} f(u(k)) = \sum_{t\in D} f(t)\#\{k \in \mathbf{Z} \ : \ u(k) = t\}.$$

The analogous formula for u^* combined with Lemma 2.1 yields the conclusion. □

3. Approximation by polarizations

In the continuous case, the approximation of symmetrizations by polarizations is a powerful tool to investigate about symmetrization inequalities, see [6,9]. In this section, we study the corresponding approximation for discrete symmetrization. In particular we give a very simple explicit approximation scheme.

3.1. Polarizations

Let us first define the polarizations in the discrete setting. The set of semifinite open intervals whose boundary is contained in $\mathbf{Z}/2$ is denoted by $\mathcal{H} = [a/2, +\infty[$, $a \in \mathbf{Z}$. For $H \in \mathcal{H}$, the reflexion with respect to ∂H is denoted by σ_H. Note that if $H \in \mathcal{H}$, $\sigma_H(\mathbf{Z}) = \mathbf{Z}$.

Definition 3.1. The *polarization* of $u : \mathbf{Z} \to \mathbf{R}$ with respect to $H \in \mathcal{H}$ is the function $u^H : \mathbf{Z} \to \mathbf{R}$ defined by

$$u^H(k) = \begin{cases} \max(u(k), u(\sigma_H(k))) & \text{if } k \in \mathbf{Z} \cap H, \\ \min(u(k), u(\sigma_H(k))) & \text{if } k \in \mathbf{Z} \setminus H. \end{cases}$$

Proposition 3.1. *Let $u : \mathbf{Z} \to \mathbf{R}$ and $v : \mathbf{Z} \to \mathbf{R}$ be admissible. If $uv \in \ell^1(\mathbf{Z})$ and $u^H v^H \in \ell^1(\mathbf{Z})$, then*

$$\sum_{k\in\mathbf{Z}} u(k)v(k) \le \sum_{k\in\mathbf{Z}} u^H(k)v^H(k). \tag{10}$$

If $v = v^H$ and one has an equality in (10), then $u^H(k) = u(k)$ and $u^H(\sigma_H(k)) = u(\sigma_H(k))$ for every $k \in H$ such that $v(k) > v(\sigma_H(k))$.

Proof. For every $k \in \mathbf{Z} \cap H$,

$$u(k)v(k) + u(\sigma_H(k))v(\sigma_H(k)) \le u^H(k)v^H(k) + u^H(\sigma_H(k))v^H(\sigma_H(k)). \tag{11}$$

Adding these inequalities and noting that $u^H(k)v^H(k) = u(k)v(k)$ for $k \in \mathbf{Z} \cap \partial H$, one obtains (10).

In the case of equality, in (10) we get an equality for every $k \in \mathbf{Z} \cap H$ in (11). By the assumption on v, this means that $u(\sigma_H(k)) \le u(k)$ for every $k \in \mathbf{Z} \cap H$, i.e. $u = u^H$. □

We shall use two particular polarizations. Let $H_+ =]0, +\infty[$ and $H_- =]-\infty, 1/2[$, so that

$$u^{H_+}(k) = \begin{cases} \max(u(k), u(-k)) & \text{if } k \ge 0 \\ \min(u(k), u(-k)) & \text{if } k \le 0 \end{cases} \tag{12}$$

and

$$u^{H_-}(k) = \begin{cases} \max(u(k), u(1-k)) & \text{if } k \le 0 \\ \min(u(k), u(1-k)) & \text{if } k \ge 1 \end{cases} \tag{13}$$

Remark: Following the same techniques and using Section 5 of [10], we can easily prove the extended Hardy–Littlewood inequalities in the discrete case.

3.2. Approximation the Discrete Schwarz symmetrization

Let us now show how u^* can be obtained by iterating polarizations. Given $u : \mathbf{Z} \to \mathbf{R}$, we define

$$Tu = (u^{H-})^{H+},$$

Iterating T, one obtains the sequence $u, u^{H-H+}, u^{H-H+H-H+}, \ldots c$. We shall prove that $T^n u$ goes to u^* as $n \to \infty$.

Theorem 3.2. *Let $u : \mathbf{Z} \to \mathbf{R}$ be admissible. For every $k \in \mathbf{Z}$,*

$$T^n u(k) \to u^*(k).$$

A first observation is that, for every $k \in Z$, $T^n u(k)$ is bounded uniformly in n.

Lemma 3.3. *The sequence $(T^n u)_{n\geq 0}$ is precompact in (X, d).*
If v is one of its cluster points, then for every $t \in \mathbf{R}$,

$$\#\{k \in \mathbf{Z} : v(k) > t\} = \#\{k \in \mathbf{Z} : u(k) > t\}.$$

Proof. This is obtained by induction on n. □

Recall that $X = \{u : \mathbf{Z} \to \mathbf{R}\}$ endowed with the metric

$$d(u, v) = \sum_{k\in\mathbf{Z}} \frac{|u(k) - v(k)|}{1 + 2^{|k|}|u(k) - v(k)|}$$

is a complete metric space, and that $d(u, u_n) \to 0$ if and only if $u_n(k) \to u(k)$ for every $k \in \mathbf{Z}$.

Lemma 3.4. *The sequence $(T^n u)_{n\geq 0}$ is precompact in (X, d).*
If v is one of its cluster points, then for every $t \in \mathbf{R}$,

$$\#\{k \in \mathbf{Z} \ : \ v(k) > t\} = \#\{k \in \mathbf{Z} \ : \ u(k) > t\}.$$

Proof. First observe that for every $k \in \mathbf{Z}$, one obtains by induction on $n \geq 0$,

$$\inf_{|l|\leq|k|} u(l) \leq (T^n u)(k) \leq \sup_{|l|\geq|k|} u(l).$$

The precompactness follows then by a diagonal argument.

Let v be a cluster point of the sequence $(T^n u)$. Assume that $T^{n_j} u(k) \to v(k)$ for every k. One has

$$\#\{k \in \mathbf{Z} \ : \ u^H(k) > t\} = \#\{k \in \mathbf{Z} \ : \ u(k) > t\},$$

so that for every $n \geq 0$,

$$\#\{k \in \mathbf{Z} \ : \ (T^n u)(k) > t\} = \#\{k \in \mathbf{Z} \ : \ u(k) > t\},$$

hence, one obtains

$$\#\{k \in \mathbf{Z} \ : \ v(k) > t\} \leq \varliminf_{l\to\infty} \#\{k \in \mathbf{Z} \ : \ (T^{n_j})(k) > t\} = \#\{k \in \mathbf{Z} \ : \ u(k) > t\}.$$

For the converse inequality, let $M > 0$, and note that

$$\#\{k \in \mathbf{Z} \ : \ |k| \leq M \text{ and } u^H(k) \geq t + 1/M\} \geq \#\{k \in \mathbf{Z} \ : \ |k| \leq M \text{ and } u(k) \geq t + 1/M\}.$$

so that

$$\#\{k \in \mathbf{Z} \ : \ |k| \leq M \text{ and } (T^n u)(k) \geq t + 1/M\} \geq \#\{k \in \mathbf{Z} \ : \ |k| \leq M \text{ and } u(k) \geq t + 1/M\}.$$

Therefore

$$\begin{aligned}\#\{k \in \mathbf{Z} \ : \ |k| \leq M \text{ and } v(k) \geq t + 1/M\} &\geq \lim_{l\to\infty} \#\{k \in \mathbf{Z} \ : \ |k| \leq M \text{ and } (T^{n_j} u)(k) \geq t + 1/M\} \\ &\geq \#\{k \in \mathbf{Z} \ : \ |k| \leq M \text{ and } u(k) \geq t + 1/M\}.\end{aligned}$$

Letting $M \to \infty$, one concludes that

$$\#\{k \in \mathbf{Z} \ : \ v(k) > t\} \geq \#\{k \in \mathbf{Z} \ : \ u(k) > t\}. \quad □$$

Proof of Theorem 3.2. By Lemma 3.4, the sequence $(T^n u)_{n\geq 0}$ is precompact in (X, d). Assume that $T^{n_j} u(k) \to v(k)$ for every $k \in Z$. We are going to show that $v = u^*$. Condition (ii) is satisfied by Lemma 3.4.

First note that for $k \geq 0$,

$$(T^n u)(k) \geq T^n u(-k),$$

so that

$$v(k) \geq v(-k).$$

For $k \geq 0$ and $l \in \mathbf{Z}$, set

$$w_k(l) = \begin{cases} 1 & \text{if } l \leq k, \\ 0 & \text{if } l > k. \end{cases}$$

and note that $w^{H-} = w$. By Proposition 3.1,

$$\sum_{l\in\mathbf{Z}} (T^{n_j} u)^{H-}(l) w_k(l) \leq \sum_{l\in\mathbf{Z}} (T^{n_j+1} u)(l) w_k(l).$$

Letting $j \to \infty$, one obtains

$$\sum_{l\in\mathbf{Z}} v^{H-}(l) w_k(l) \leq \sum_{l\in\mathbf{Z}} v(l) w_k(l).$$

Since $\sigma_H(-k) = 1 + k$, one has $w_k(-k) > w_k(\sigma_H(-k))$. Therefore, by the necessary condition for the equality in Proposition 3.1, we conclude that

$$v(-k) \geq v(k+1)$$

for $k \geq 0$. Therefore v satisfies (i) in Definition 2.1. □

3.3. Improved convergences

Corollary 3.5. *Let $u : \mathbf{Z} \to \mathbf{R}$ be bounded from above. For every $k \in \mathbf{Z}$, there exists n_0 such that if $n \geq n_0$,*

$$T^n u(k) \leq u^*(k).$$

If moreover u is admissible or $u^*(k) > \overline{\lim}_{l\to\infty} u(l)$, then there exists n' such that if $n \geq n'$,

$$T^n u(k) = u^*(k).$$

Proof. If $u^*(k) > \overline{\lim}_{l\to\infty} u(l)$, for $\varepsilon < u^*(k) - \overline{\lim}_{l\to\infty} u(l)$, $u(\mathbf{Z}) \cap]u^*(k) - \varepsilon, +\infty[= (T^n u)(\mathbf{Z}) \cap]u^*(k) - \varepsilon, +\infty[$ is finite. By Theorem 3.2, $T^n u(k) = u^*(k)$ for large n.

If $u^*(k) \leq \overline{\lim}_{l\to\infty} u(l)$ and u is admissible, then $u(\mathbf{Z}) = (T^n u)(\mathbf{Z})$ is finite, and $(T^n)(u(k)) = u^*(k)$ for large n by Theorem 3.2. If u is not admissible, one still has that $u(\mathbf{Z}) \cap]u^*(k), +\infty[= (T^n u)(\mathbf{Z}) \cap]u^*(k), +\infty[$ is finite, whence by Theorem 3.2, when n is large, $(T^n)u(k) \leq u^*(k)$. □

Proposition 3.6. *Let $u : \mathbf{Z} \to \mathbf{R}^+$. If $u \in \ell^p(\mathbf{Z})$ and $1 \leq p < \infty$, then $T^n u \to u$ in $\ell^p(\mathbf{Z})$. If $u \in c_0(\mathbf{Z})$, then $T^n u \to u$ uniformly.*

Proof. If $u \in \ell^p(\mathbf{Z})$, let $\varepsilon > 0$ and choose $M > 0$ such that

$$\sum_{|k|>M} |u(z)|^p \leq \varepsilon.$$

One has then since $u(z) \geq 0$

$$\sum_{|k|>M} |T^n u(z)|^p \leq \varepsilon$$

and

$$\sum_{|k|>M} |u^*(z)|^p \leq \varepsilon.$$

By Theorem 3.2, for n sufficiently large,

$$\sum_{|k|\leq M} |(T^n u)(z) - u^*(z)|^p \leq \varepsilon$$

whence

$$\sum_{k\in\mathbf{Z}} |(T^n u)(z) - u^*(z)|^p \leq \varepsilon + 2^p\varepsilon.$$

If $u \in c_0(\mathbf{Z})$, for $\varepsilon > 0$, there exists $M > 0$ such that $|u(k)| \leq \varepsilon$ for $|k| > M$, whence, since u is nonnegative, $u^*(k) \leq \varepsilon$ and $(T^n u)(k) \leq \varepsilon$ for $|k| \geq M$. One concludes as previously. □

4. Polya–Szego and Riesz inequalities

Theorem 4.1. *Let $u : \mathbf{Z} \to \mathbf{R}$ be admissible. If $J : \mathbf{R} \to \bar{\mathbf{R}}$ is even and convex, then*

$$\sum_{k\in\mathbf{Z}} J\left(u^*(k) - u^*(k+1)\right) \leq \sum_{k\in\mathbf{Z}} J(u(k) - u(k+1)).$$

If J is strictly convex and if $u(k) = u(l) > \overline{\lim}_{m\to\infty} u(m)$ implies $k = l$, then one has equality if and only if $u = u \circ i$ for some isometry $i : \mathbf{Z} \to \mathbf{Z}$.

Corollary 4.2. *For every $u : \mathbf{Z} \to \mathbf{R}$ bounded from above and for every $p \geq 1$,*

$$\sum_{k\in\mathbf{Z}} |u^*(k) - u^*(k+1)|^p \leq \sum_{k\in\mathbf{Z}} |u(k) - u(k+1)|^p$$

and

$$\sup_{k\in\mathbf{Z}} |u^*(k) - u^*(k+1)| \leq \sup_{k\in\mathbf{Z}} |u(k) - u(k+1)|.$$

4.1. Polarization inequalities

Definition 4.1. A function $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is supermodular if for every $x, y \in \mathbf{R}$ and $s, t > 0$,

$$F(x+s, y) + F(x, y+t) \leq F(x, y) + F(x+s, y+t).$$

It is strictly supermodular if the strict inequality holds.

Proposition 4.3. *Let $H \in \mathcal{H}$, $u : \mathbf{Z} \to \mathbf{R}$ and $v : \mathbf{Z} \to \mathbf{R}$. If F is supermodular and $w : \mathbf{N} \to \mathbf{R}$ is decreasing, then*

$$\sum_{k,l\in\mathbf{Z}} F\left(u(k), v(l)\right) w(|k-l|) \leq \sum_{k,l\in\mathbf{Z}} F\left(u^H(k), v^H(l)\right) w(|k-l|).$$

If one has equality, F is strictly supermodular, $w(1) > w(2)$, $u = v$ and $u(k) \neq u(\sigma_H(k))$ for $\check{k} \leq k \leq \bar{k}$, with $\check{k}, \bar{k} \in \mathbf{Z} \cap H$, then $u^H(k) = u(k)$ or $u^H(k) = u(\sigma_H(k))$ for $\check{k} \leq k \leq \bar{k}$.

Proof. Let $k, l \in \mathbf{Z}$ be such that $\partial H \not\subset (k, l)$ and set $k' = \sigma_H(k)$ and $l' = \sigma_H(l)$. Note that $|k-l| = |k'-l'| \leq |k-l'| = |k'-l|$. Therefore,

$$\begin{aligned} &F\left(u(k), v(l)\right) w(|k-l|) + F\left(u(k), v(l')\right) w(|k-l'|) + F\left(u(k'), v(l)\right) w(|k'-l|) + F\left(u(k'), v(l')\right) w(|k'-l'|) \\ &\quad = w(|k'-l|)\left(F\left(u(k), v(l)\right) + F\left(u(k), v(l')\right) + F\left(u(k'), v(l)\right) + F\left(u(k), v(l')\right)\right) \\ &\qquad + \left(w(|k-l|) - w(|k'-l|)\right)\left(F\left(u(k), v(k)\right) + F\left(u(k'), v(l')\right)\right) \end{aligned}$$

and:

$$\begin{aligned} &F\left(u(k), v(l)\right) + F\left(u(k), v(l')\right) + F\left(u(k'), v(l)\right) + F\left(u(k'), v(l')\right) \\ &\quad = F\left(u^H(k), v^H(l)\right) + F\left(u^H(k), v^H(l')\right) + F\left(u^H(k'), v^H(l)\right) + F\left(u^H(k), v^H(l')\right). \end{aligned}$$

Now we employ that F is supermodular and w is decreasing, and we get:

$$\begin{aligned} &\left(w(|k-l|) - w(|k'-l|)\right)\left(F\left(u(k), v(k)\right) + F\left(u(k'), v(l')\right)\right) \\ &\quad \leq (w(|k-l|) - w(|k'-l|))(F(u^H(k), v^H(k)) + F(u^H(k'), v^H(l'))). \end{aligned}$$

Thus concludes that

$$\begin{aligned} &F\left(u(k), v(l)\right) w(|k-l|) + F\left(u(k), v(l')\right) w(|k-l'|) + F\left(u(k'), v(l)\right) w(|k'-l|) + F\left(u(k'), v(l')\right) w(|k'-l'|) \\ &\quad \leq F\left(u^H(k), v^H(l)\right) w(|k-l|) + F\left(u^H(k), v(l')\right) w(|k-l'|) \\ &\qquad + F\left(u^H(k'), v^H(l)\right) w(|k'-l|) + F\left(u^H(k'), v(l')\right) w(|k'-l'|). \end{aligned} \tag{14}$$

Summing over k and l supplies the conclusion.

If there is an equality, then the equality holds for every k and l in the previous inequalities. In particular, when $\check{k} \le k \le l = k+1 \le \bar{k}$ then $|k-l| = 1$, $|k-l'| \ge 2$, so that $w(|k-l|) - w(|k-l'|) \ge 0$. Since F is strictly supermodular, we conclude that either $u^H(k) = u(k)$ and $u^H(k+1) = u(k+1)$ or $u^H(k) = u(\sigma_H(k))$ and $u^H(k+1) = u(\sigma_H(k+1))$. Since $u(k) \ne u(\sigma_H(k))$ this implies the conclusion. □

Proposition 4.4. *Let $H \in \mathcal{H}$ and $u : \mathbf{Z} \to \mathbf{R}$. If $J : \mathbf{R} \to \bar{\mathbf{R}}$ is even and convex and $J(0) = 0$, then*

$$\sum_{k\in\mathbf{Z}} J\left(u^H(k) - u^H(k+1)\right) \le \sum_{k\in\mathbf{Z}} J\left(u(k) - u(k+1)\right).$$

If J is strictly convex and $u(k) \ne u(\sigma_H(k))$ for $\check{k} \le k \le \bar{k}$, with $\check{k}, \bar{k} \in \mathbf{Z} \cap H$, then $u^H(k) = u(k)$ or $u^H(k) = u(\sigma_H(k))$ for $\check{k} \le k \le \bar{k}$.

Proof. Let $F(s,t) = -J(|s-t|)$ and

$$w(n) = \begin{cases} 1 & \text{if } |n| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $J(0) = 0$ and J is even,

$$\begin{aligned} \sum_{k,l\in\mathbf{Z}} F\left(u(k), u(l)\right) w(|k-l|) &= -\sum_{k\in\mathbf{Z}} \left[J\left(u(k) - u(k-1)\right) + J\left(u(k) - u(k+1)\right)\right] \\ &= -2\sum_{k\in\mathbf{Z}} J\left(u(k) - u(k+1)\right). \end{aligned}$$

Noting that F and w satisfy the assumptions of Proposition 4.3, one obtains the conclusion. □

4.2. Proofs of the symmetrization inequalities

Proof of Theorem 4.1. By Proposition 4.4, one has

$$\sum_{k\in\mathbf{Z}} J(T^n u(k) - T^n u(k+1)) \le \sum_{k\in\mathbf{Z}} J\left(u(k) - u(k+1)\right). \tag{15}$$

Let $M > 0$. By Corollary 3.5, when n is large, $|u^*(k) - u^*(k+1)| \le |T^n u(k) - T^n u(k+1)|$, so that

$$\sum_{|k|\le M} J\left(u^*(k) - u^*(k+1)\right) \le \sum_{|k|\le M} J(T^n u(k) - T^n u(k+1)).$$

Since J is nonnegative, this implies in view of (15) that

$$\sum_{|k|\le M} J\left(u^*(k) - u^*(k+1)\right) \le \sum_{k\in\mathbf{Z}} J\left(u(k) - u(k+1)\right).$$

Letting $M \to \infty$ yields the conclusion.

If one has an equality, then since $u^* = (u^H)^*$, one has

$$\sum_{k\in\mathbf{Z}} J\left(u^H(k) - u^H(k+1)\right) = \sum_{k\in\mathbf{Z}} J\left(u(k) - u(k+1)\right).$$

for every $H \in \mathcal{H}$. □

5. Applications: Symmetry of minimizers of some discretized variational problems over symmetric domains

Continuous variational problems arise directly from modeling physical systems. When one tries to numerically solve these problems, finite-dimensional approximations lead to discretized variational problems (see [15]). The steps followed in the discretization procedure are:

Step 1: First the functionals are discretized by replacing derivatives by finite differences or by restricting the function spaces to finite dimensional spaces. As in the continuous case, critical points satisfy the Euler–Lagrange equation, which is finite dimensional in this case.

Step 2: Next, the exact solutions of finite difference/finite element problems are analyzed.

Step 3: Finally, algorithms are designed to compute numerical approximations of solutions of the discretized problems.

Of course, the more one knows about the qualitative properties of these solutions, the easier and more efficient is the design of algorithms. Recently, McKenna and Reichel have proved a very interesting result in [5]. Indeed, they showed that critical points of a class of underlying discretized variational problems do not generally have the same symmetry property as the critical points of the corresponding continuous variational problems. They were able to prove that unlike in the continuous case, symmetries are not respected by positive critical points of (18) (see below).

The purpose of this paper is to show that critical points which are minimizers of some discretized variational problems are not spurious, i.e, solutions with no relation to solutions of the continuous case [11–13]. Moreover, in some cases, they do inherit the same symmetry properties of the functional and the domain.

Let us first summarize the main result obtained by McKenna and Reichel [5], and then explain our contribution. In [5], the following continuous variational problem was considered:

$$C[u] = \int_{\Omega} \frac{1}{2}|\nabla u|^2 - F(|x|, u)\,\mathrm{d}x. \tag{16}$$

Ω is an open bounded domain in $\mathbf{R}^n$. The corresponding discretized functional is:

$$D^h[u] = \sum_{x\in\Omega_h} \left(\frac{1}{2}|\nabla_h^+ u|^2 - F(|x|, u)\right) h^n. \tag{17}$$

Ω_h consists of the points of a regular mesh of step size h that belong to the bounded open set Ω in $\mathbf{R}^n$.
$\nabla u^+ = \left(D_1^+ u, \ldots, D_n^+ u\right)$ stands for the forward finite difference gradient

$$D_i^+ u = \frac{u(x + he_i) - u(x)}{h}; \qquad D_i^- u = \frac{u(x) - u(x - he_i)}{h}$$

where e_i is the unit coordinate vector in the direction x_i.

The function u is defined on points $x \in \Omega_h$. The functional $D^h[u]$ stands for the finite difference discretization of (16). The critical points of (D^h) satisfy:

$$-\sum_{k=1}^{m} D_i^-(D_i^+ u) = f(|x|, u) \quad \text{in } \Omega_h. \tag{18}$$

It is the finite difference counterpart for $-\Delta u = f(|x|, u)$ in Ω.

The first asymptotic symmetry result concerning positive solutions of (18) was obtained by McKenna and Reichel [5]. More precisely they proved that:

$$|u(-x_1, \ldots, x_n) - u(x_1, \ldots, x_n)| < \frac{c}{|\ln h|}$$

for $(x_1, \ldots, x_n) \in \Omega_h$, where they could obtain an explicit control over the constant c in terms of the size of Ω, the nonlinearity of f and L^∞-norm of u. They also exhibited examples of f where some critical points are far from being symmetric.

When $n = 1$ and Ω is an interval centered in zero, Corollary 4.2 and the remark following Proposition 3.1 imply that all minimizers of the discretized problem (D^h) are symmetric if $-F$ is strictly supermodular. Our work completes that of McKenna and Reichel [5], and helps us to understand better the difference between the behavior of minimizers and saddle points in some discretized variational problems. We have also given a complete answer to Question 4 stated in the last section of [5] without using the Steiner symmetrization concept.

However, compared to the vast literature on continuous variational problems, the theory of discretized variational problems is much smaller. It was the breakthrough paper of Choi and McKenna [14] that gave a new energy to this rather new field. In that paper, the authors design a very interesting and tricky algorithm for the computation of mountain pass critical points of discretized functionals. Nevertheless, their approach does not seem to be applicable to our case. Note that Step 3 is in general very subtle since precise information about critical points is needed. Our work will certainly have an immediate impact for the design of numerical schemes and we now know that the sparsity of the literature is due to the fact that there are usually many unstable critical points of the discretized problems. Minimizers are stable, making the analysis of Step 3 less hard for these kind of points. We will get the whole picture once we obtain some good a priori bounds for such points.

Finally, let us point out that the symmetry of the minimizers in the entire space $\mathbf{R}^n$ is guaranteed if one proves (8) and (9) for $k \in \mathbf{Z}^n$. These very challenging anisotropic inequalities are under intensive study.

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